Hybrid CORDIC 2.A
Sine/Cosine Generator

20170705

Wilson ROM based Sine/Cosine Generation

[24] Fu & Willson Sine / Cosine Generation

ROM-based

For high resolution, ROM size grows exponentially

quater-wave symmetry

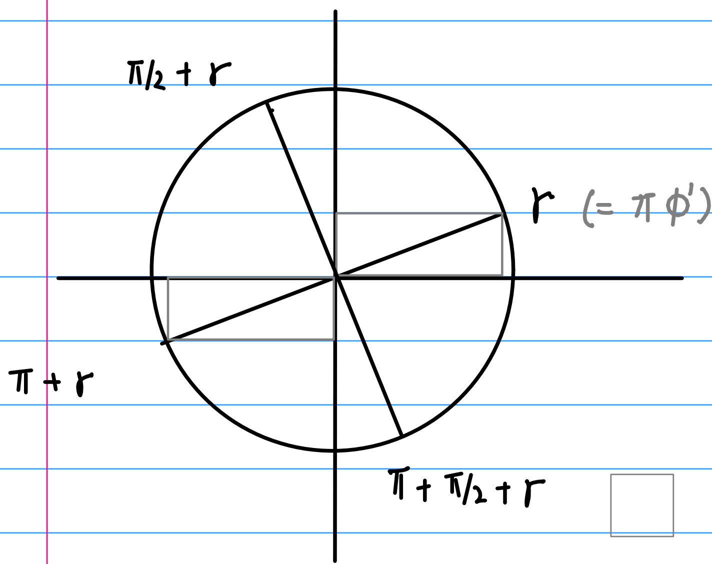
$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

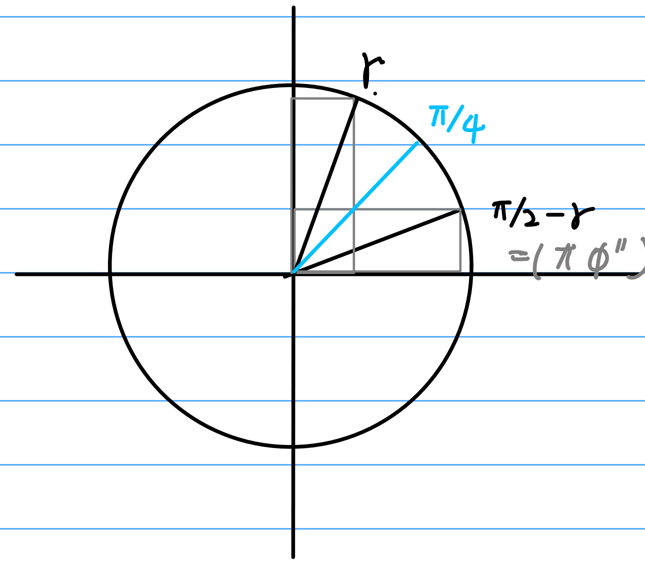
$$\phi \quad [0, 2\pi] \longrightarrow [0, \tfrac{\pi}{4}]$$

conditionally interchanging inputs X_0 & Y_0
Conditionally interchanging and negating outputs X & Y

$$X = X_0 \cos\phi - Y_0 \sin\phi$$
$$Y = Y_0 \cos\phi + X_0 \sin\phi$$

Madisetti VLSI arch





for frequency synthesis

argument: signed normalized by π angle [-1, 1]

binary representation of a radian angle required

$[-1, 1] \longrightarrow [0, \pi/4] \longrightarrow$ sine/cosine generator

$\quad\;\; \phi \qquad\qquad\qquad\; \theta$

$\pi\phi \longleftarrow$

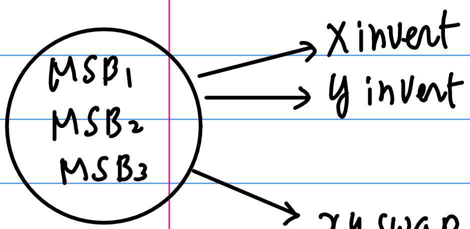
① a phase accumulator $\quad \phi \quad [-1, 1]$
② a radian converter $\quad \phi \to \theta$
③ a sine/cosine generator $\quad \sin\theta, \ \cos\theta$
④ an output stage $\quad \sin\theta, \ \cos\theta$

$\qquad\qquad\qquad\qquad\quad \downarrow \qquad\quad \downarrow$

$\qquad\qquad\qquad\qquad \sin\pi\phi \quad \cos\pi\phi$

output stage $\sin\theta \to \sin\pi\phi$ $[-\pi, +\pi]$
$\cos\theta \to \cos\pi\phi$

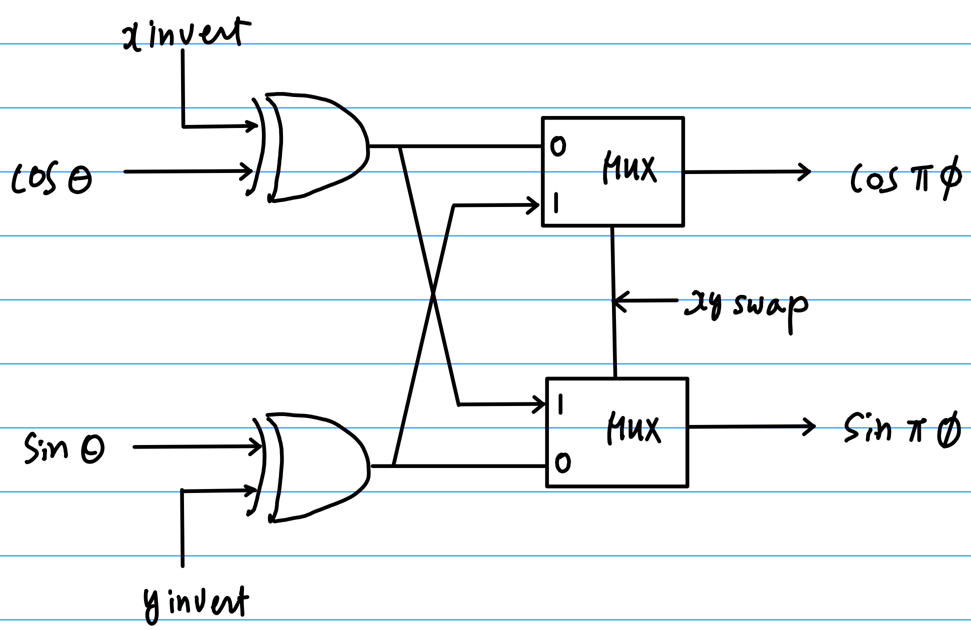
negation / interchange



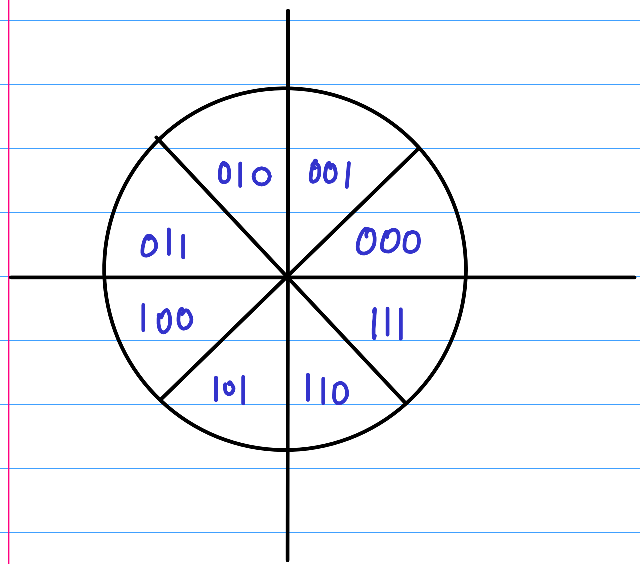
the negation of $\cos\theta = X_{N+1}$
$\sin\theta = Y_{N+1}$

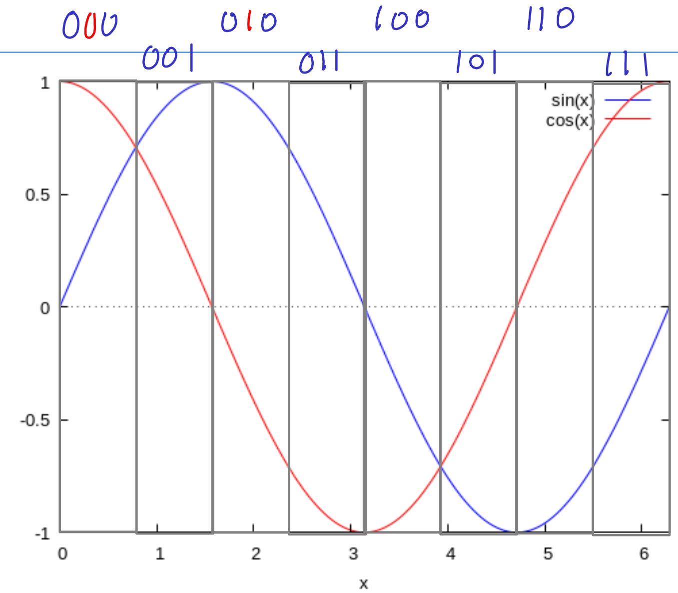
Interchange

negate before swap



Madisetti & Willson, DDS Freq synthesizer





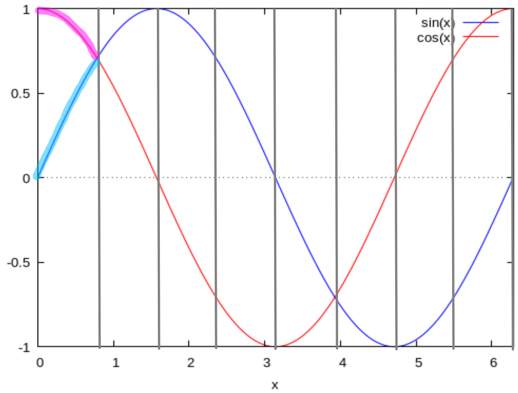
	cos	sin			
	xinv	yinv	swap	$\cos\pi\theta$	$\sin\pi\phi$
0 0 0	0	0	0	$\cos\theta$	$\sin\theta$
0 0 1	0	0	1	$\sin\theta$	$\cos\theta$
0 1 0	0	1	1	$-\sin\theta$	$\cos\theta$
0 1 1	1	0	0	$-\cos\theta$	$\sin\theta$
1 0 0	1	1	0	$-\cos\theta$	$-\sin\theta$
1 0 1	1	1	1	$-\sin\theta$	$-\cos\theta$
1 1 0	1	0	1	$\sin\theta$	$-\cos\theta$
1 1 1	0	1	0	$\cos\theta$	$-\sin\theta$

Madisetti & Willson, DDS Freq synthesizer

0

$\cos\theta$

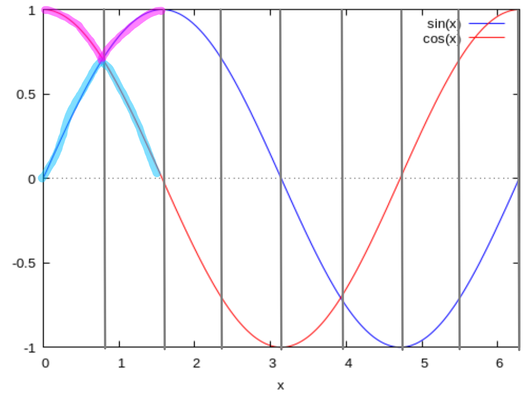
$\sin\theta$



1

$\sin\theta$

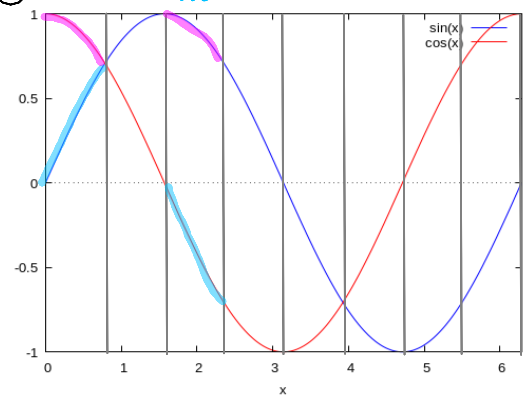
$\cos\theta$



2

$-\sin\theta$

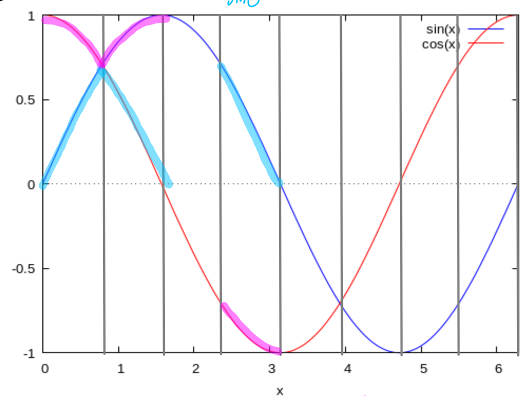
$\cos\theta$



3

$-\cos\theta$

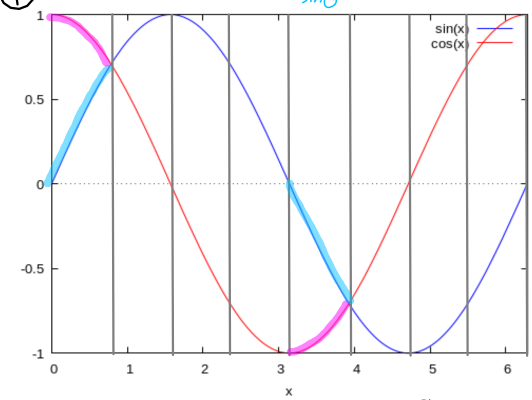
$\sin\theta$



4

$-\cos\theta$

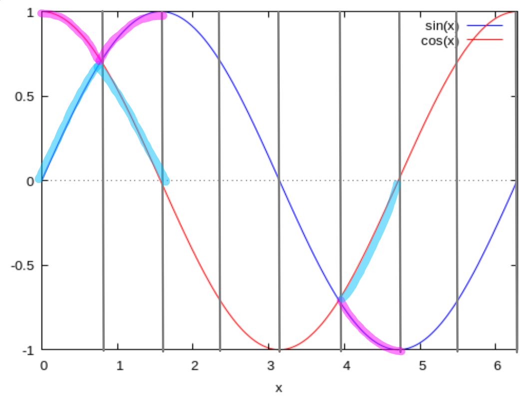
$-\sin\theta$



5

$-\sin\theta$

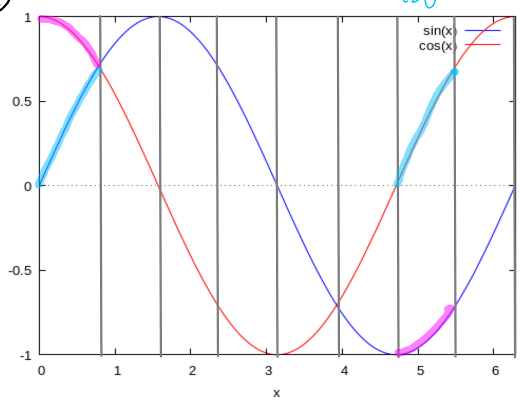
$-\cos\theta$



6

$\sin\theta$

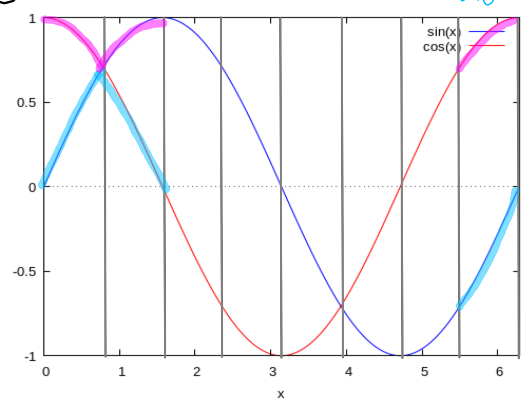
$-\cos\theta$



7

$\cos\theta$

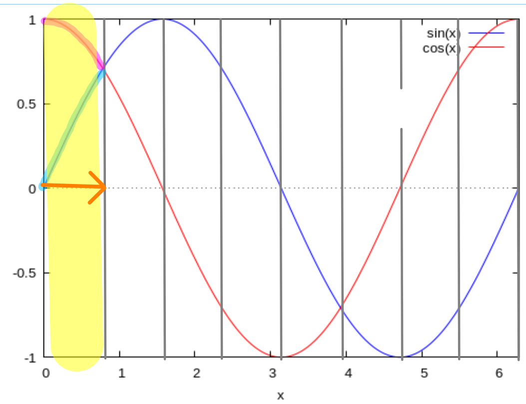
$-\sin\theta$



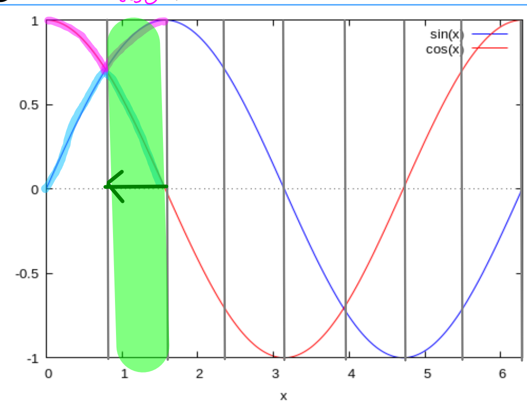
$\begin{cases} \cos\phi \\ \sin\phi \end{cases}$

Madisetti & Willson, DDS Freq synthesizer

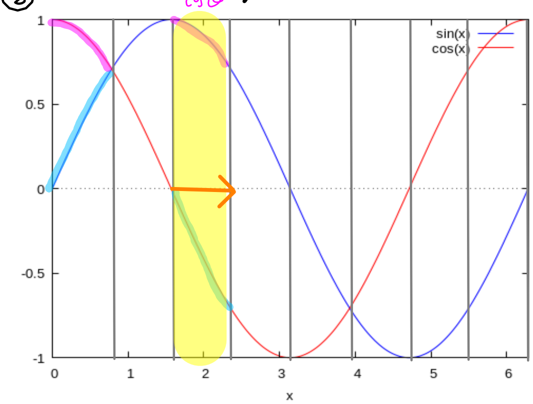
0
$\cos\theta$
$\sin\theta$



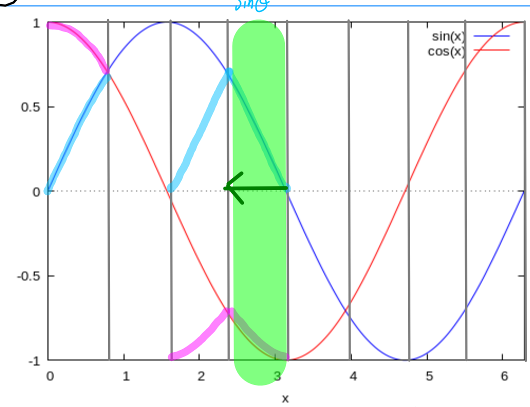
1
$\sin\theta$
$\cos\theta$



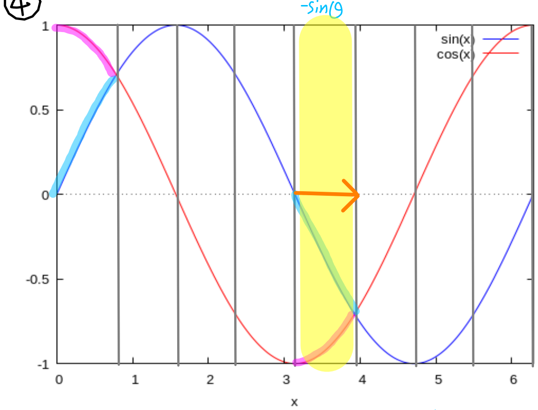
2
$-\sin\theta$
$\cos\theta$



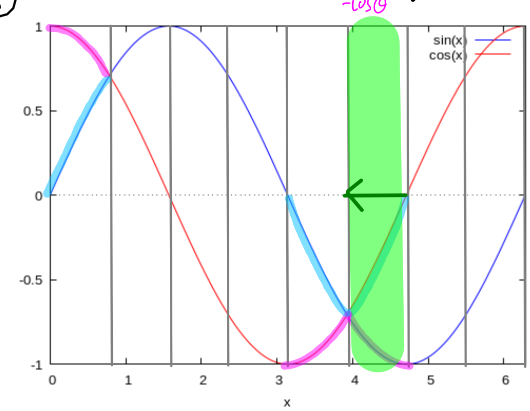
3
$-\cos\theta$
$\sin\theta$



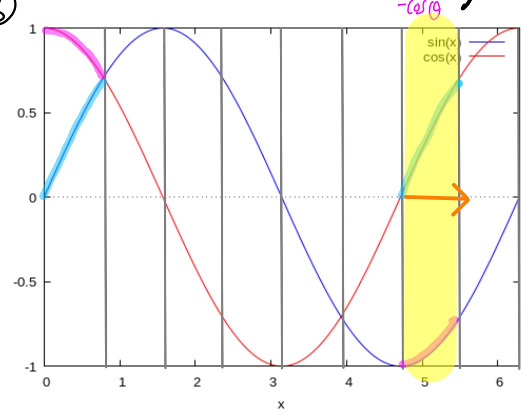
4
$-\cos\theta$
$-\sin\theta$



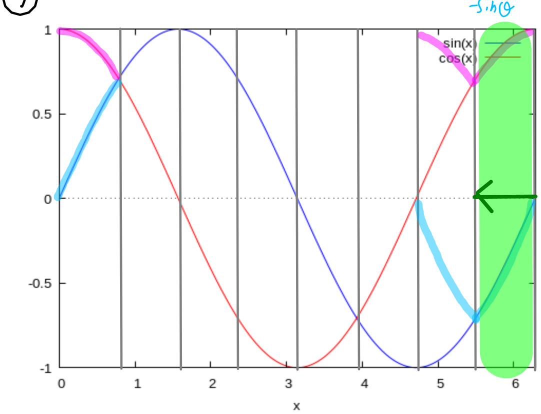
5
$-\sin\theta$
$-\cos\theta$



6
$\sin\theta$
$-\cos\theta$

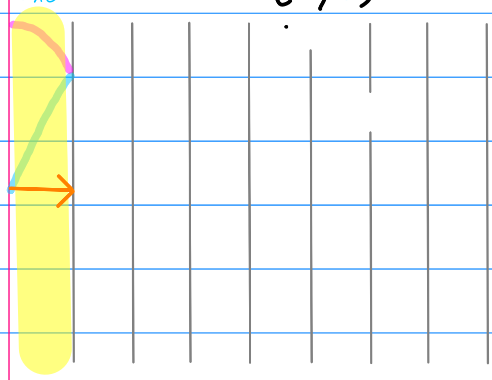


7
$\cos\theta$
$-\sin\theta$

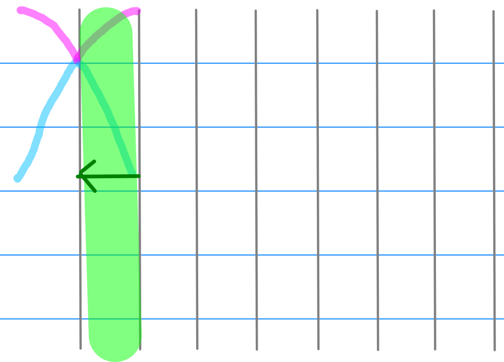


sin ϕ

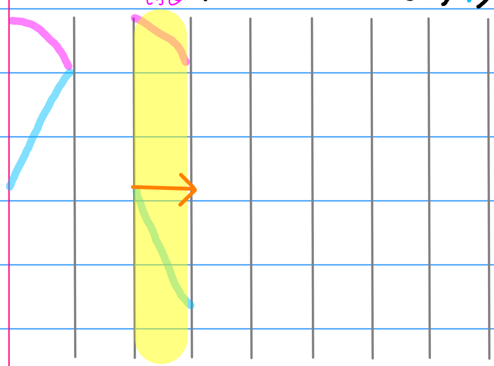
0 $\cos\theta$ $\sin\theta$ $+$ $+$ $(0, 0)$



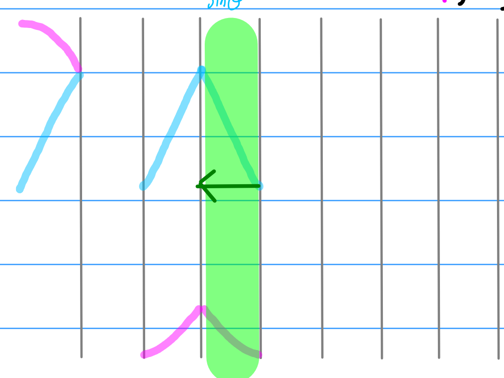
1 $\sin\theta$ $\cos\theta$) $+$ $+$ $(0, 0)$



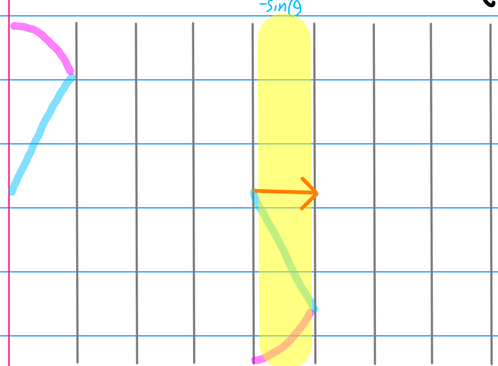
2 $-\sin\theta$ $\cos\theta$) $+$ $-$ $(0, 1)$



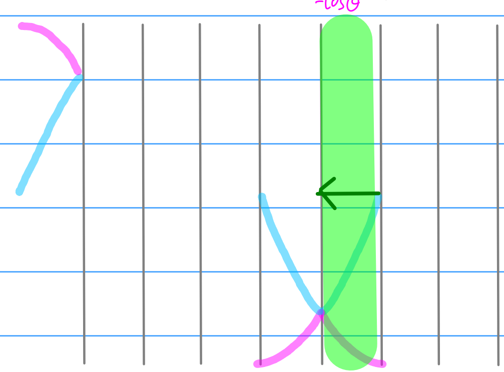
3 $-\cos\theta$ $\sin\theta$ $-$ $+$ $(1, 0)$



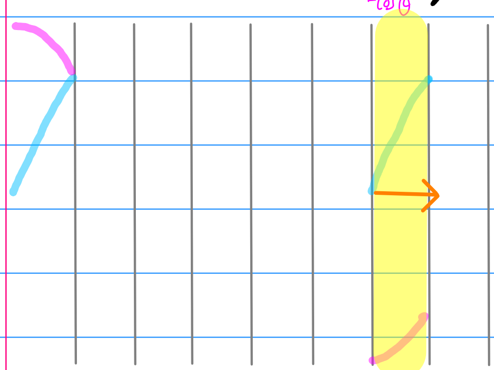
4 $-\cos\theta$ $-\sin\theta$ $-$ $-$ $(1, 1)$



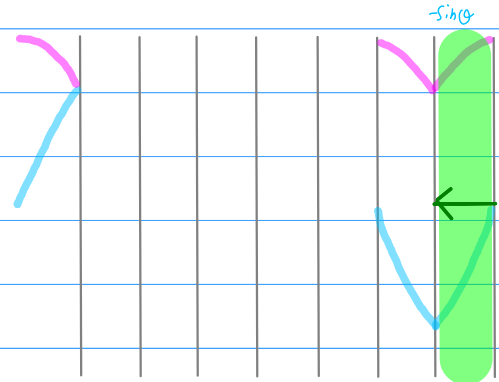
5 $-\sin\theta$ $-\cos\theta$) $-$ $-$ $(1, 1)$



6 $\sin\theta$ $-\cos\theta$) $-$ $+$ $(1, 0)$



7 $\cos\theta$ $-\sin\theta$ $+$ $-$ $(0, 1)$



	Xinv	Yinv	swap	$\cos \pi \phi$	$\sin \pi \phi$
0 0 0	0	0	0	$\cos\theta$	$\sin\theta$
0 0 1	0	0	1	$\sin\theta$	$\cos\theta$
0 1 0	0	1	1	$-\sin\theta$	$\cos\theta$
0 1 1	1	0	0	$-\cos\theta$	$\sin\theta$
1 0 0	1	1	0	$-\cos\theta$	$-\sin\theta$
1 0 1	1	1	1	$-\sin\theta$	$-\cos\theta$
1 1 0	1	0	1	$\sin\theta$	$-\cos\theta$
1 1 1	0	1	0	$\cos\theta$	$-\sin\theta$

0 0	0 0
0 1	1 0
1 1	1 1
1 0	0 1

0 0 0 0

0 1 1 0

1 1 1 1

1 0 0 1

Madisetti & Willson, DDS Freq synthesizer

$$\theta = \sum_{k=1}^{N} b_k \theta_k$$

b_k sign + N bit — (N+1) bit fractional b

$b_k \in \{0, 1\}$

$\theta_k = 2^{-k}$

θ is constrained to be positive $\boxed{b_0 = 0}$

$$\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

$r_k \in \{-1, +1\}$ signed digits

ϕ_0 constant

⊕ subrotation by 2^{-k}

2 equal ⊕ half rotations by 2^{-k-1}

⊖ subrotation

2 equal opposite half rotations by $\pm 2^{-k-1}$

Binary Representation

$b_k = 1$: rotation by 2^{-k}

$b_k = 0$: zero rotation

k-th rotation

- fixed rotation by 2^{-k-1}
- $\begin{cases} \text{pos rotation} \leftarrow b_k = 1 \\ \text{neg rotation} \leftarrow b_k = 0 \end{cases}$

Combining all the fixed rotations

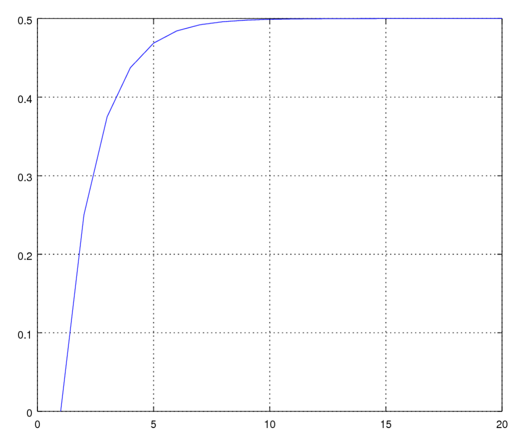
→ initial fixed rotation

	b_1 2^{-1}	b_2 2^{-2}	b_3 2^{-3}		b_N 2^{-N}
fixed $\Rightarrow$	$+2^{-2}$	$+2^{-3}$	$+2^{-4}$		$+2^{-N-1}$
	$(b_1=1)$ $+2^{-2}$	$(b_2=1)$ $+2^{-3}$	$(b_3=1)$ $+2^{-4}$		$(b_N=1)$ $+2^{-N-1}$
	$(b_1=0)$ -2^{-2}	$(b_2=0)$ -2^{-3}	$(b_3=0)$ -2^{-4}		$(b_N=0)$ -2^{-N-1}

initial fixed rotation

$$\phi_0 = \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{N+1}}$$

$$= \frac{\frac{1}{2^2}\left(1 - \frac{1}{2^N}\right)}{\left(1 - \frac{1}{2}\right)} = \frac{1}{2}\left(1 - \frac{1}{2^N}\right) = \frac{1}{2} - \frac{1}{2^{N+1}}$$



the rotation after recoding

— a fixed initial rotation ϕ_0

a sequence of $\oplus / \ominus$ rotations

$b_k = 1$ $\quad + 2^{-k-1}$ rotation

$b_k = 0$ $\quad - 2^{-k-1}$ rotation

$r_k = (2b_{k-1} - 1)$

$2 \cdot 1 - 1 = +1$ $\qquad b_{k-1} = 1 \longrightarrow r_k = +1$

$2 \cdot 0 - 1 = -1$ $\qquad b_{k-1} = 0 \longrightarrow r_k = -1$

The recoding need not be explicitly performed

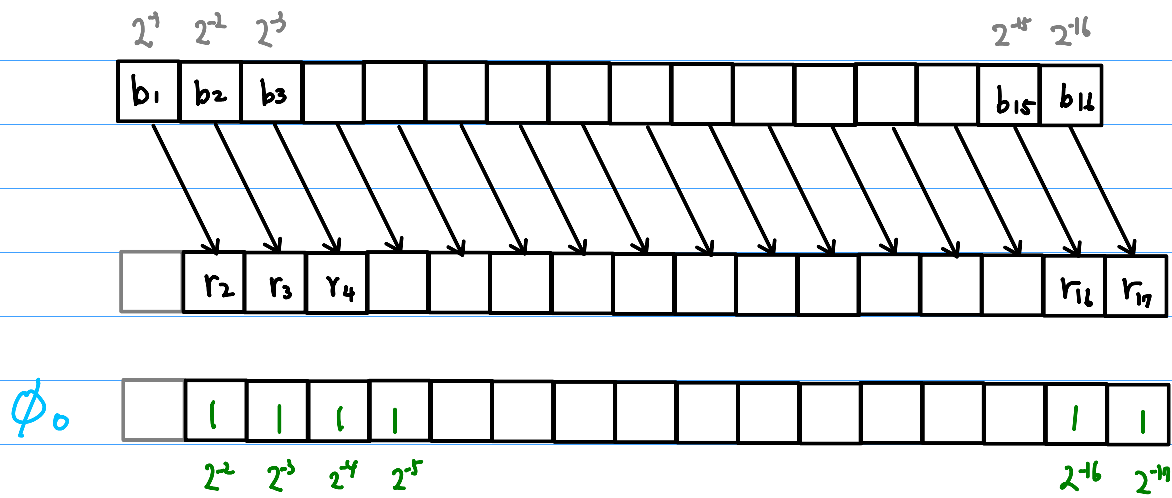
Simply replacing $b_k = 0$ with -1

This recoding maintains

a constant scaling factor K

$$\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

Binary Representation $\{b_k\}$



Signed Digit Recoding $\{r_k\}$

The Scaling K.
The initial rotation ϕ_0

rotation starting point
$$(X_0, Y_0) = (K\cos\phi_0, K\sin\phi)$$

— fixed
— no error buildup

— rotation direction
immediately obtained from the binary representation
→ no need for comparison

the subangles $\theta_k = 2^{-k}$ used in recoding
the subangles $\theta_k = \tan^{-1}(2^{-k})$ used in CORDIC

$\tan\theta_k$ multipliers used
in the first few subrotation stages
cannot be implemented
as a simple shift-and-add operations

→ ROM implementation

reduced chip area
higher operating speed.

Architecture

1. phase accumulator $\phi \in [-1, +1]$
2. radian converter $\phi \to \theta \in [0, \frac{\pi}{4}]$
3. sine/cosine generator $\sin(\theta)$ $\cos(\theta)$
4. output stage $\sin(\pi\phi)$ $\cos(\pi\phi)$

Overflowing 2's complement accumulator

normalized by π angle ϕ

need radian angle $\theta \in [0, \frac{\pi}{4}]$

$0 < \theta < 1$ rad

N-bit binary representation of θ

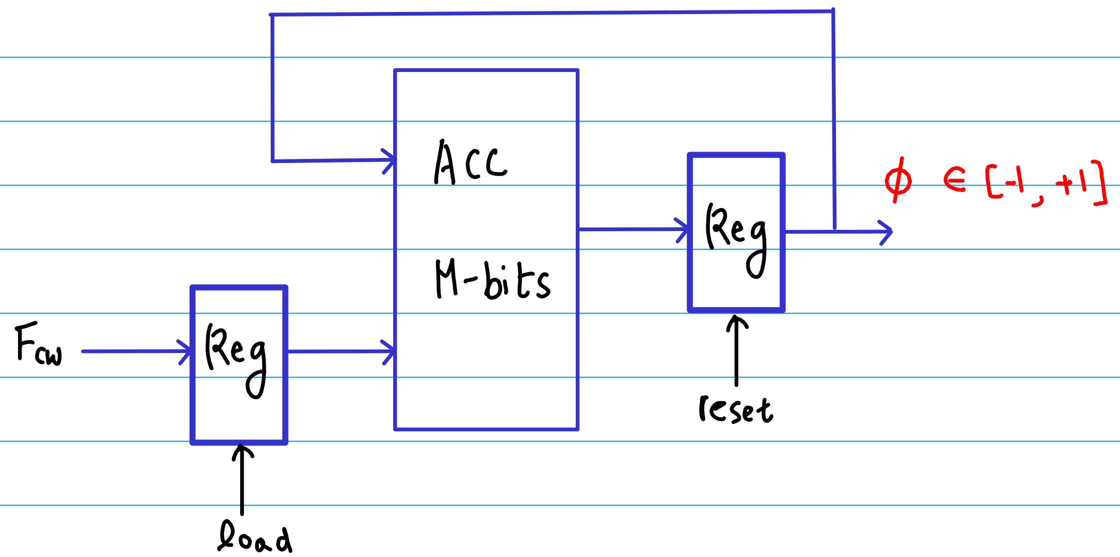
controls the direction of subrotation

N-bit precision of $\cos\theta$ & $\sin\theta$

Output stage

$$\theta \rightarrow \pi\phi$$
$$\sin\theta \rightarrow \sin\pi\phi$$
$$\cos\theta \rightarrow \cos\pi\phi$$

phase accumulator



M-bit adder

repeatedly increments the phase angle

by F_{cw} at each clock cycle

frequency control word

at time n, $\phi = n F_{cw} / 2^M$

$$\cos\phi = \cos(n F_{cw} / 2^M)$$
$$\sin\phi = \sin(n F_{cw} / 2^M)$$

Radian Converter

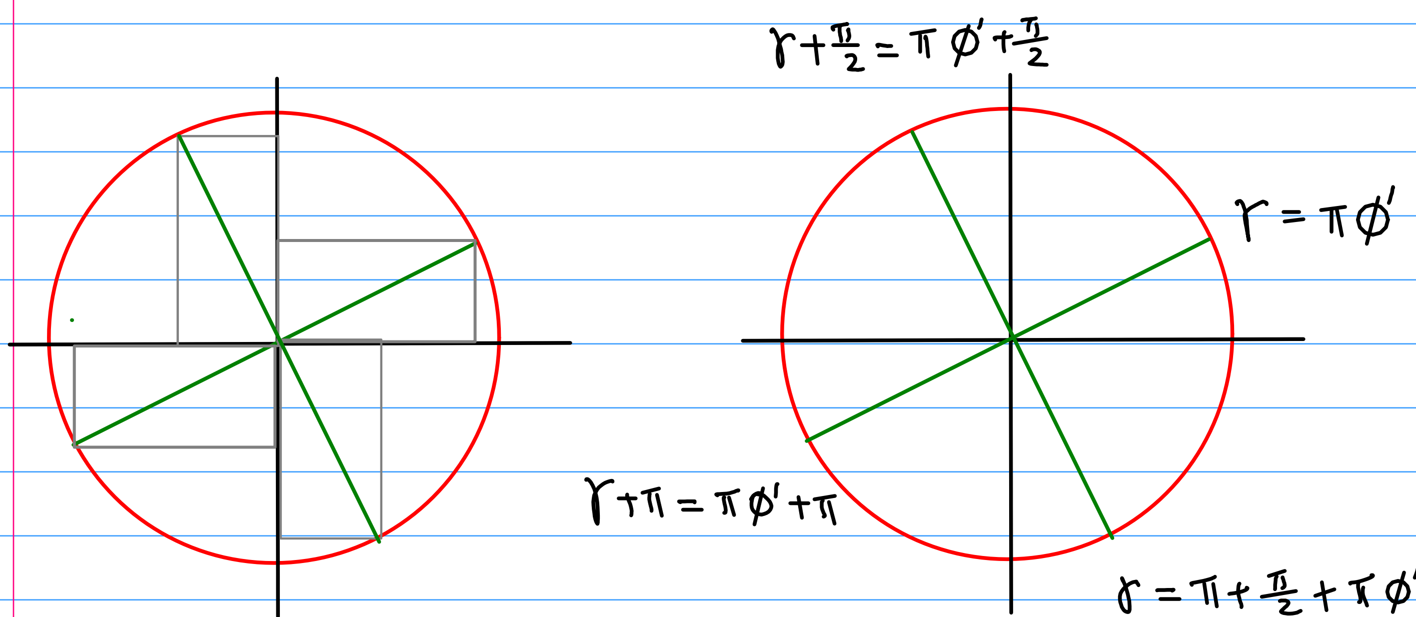
normalized angle ϕ

ϕ	MSB1	MSB2	MSB3				...		

$\xleftrightarrow{\text{Quadrant of } \pi\phi}$ (MSB1, MSB2) $\xleftrightarrow{\text{U. half / L. half}}$ (MSB3)

$\longleftrightarrow$ (MSB1–MSB3) Stored to interchange / negate

ϕ'	0	0	MSB3				...		

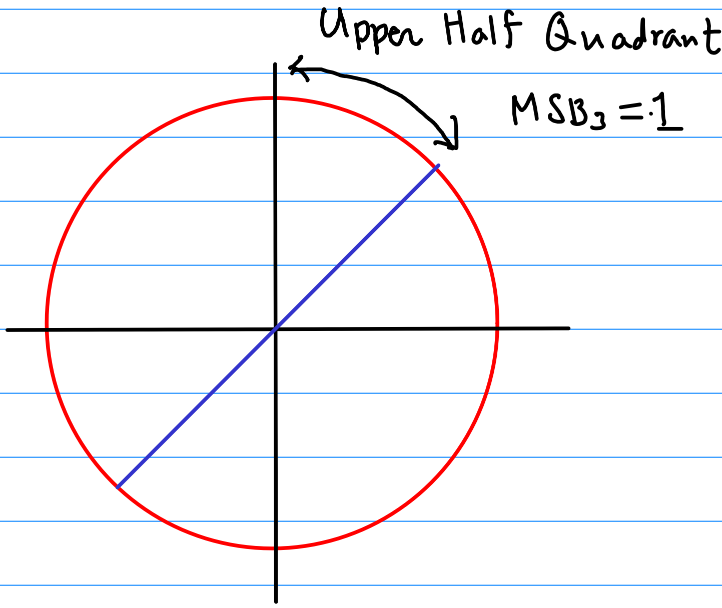


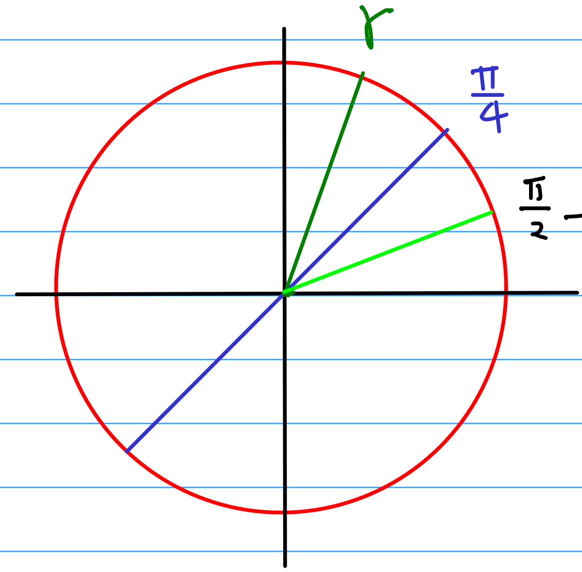
$\phi \rightarrow \phi' \rightarrow$

$\pi\phi'$	$+\ 0 \cdot \frac{\pi}{2}$	00
$\pi\phi'$	$+\ 1 \cdot \frac{\pi}{2}$	01
$\pi\phi'$	$+\ 2 \cdot \frac{\pi}{2}$	10
$\pi\phi'$	$+\ 3 \cdot \frac{\pi}{2}$	11

ϕ'

0	0	MSB3					...	





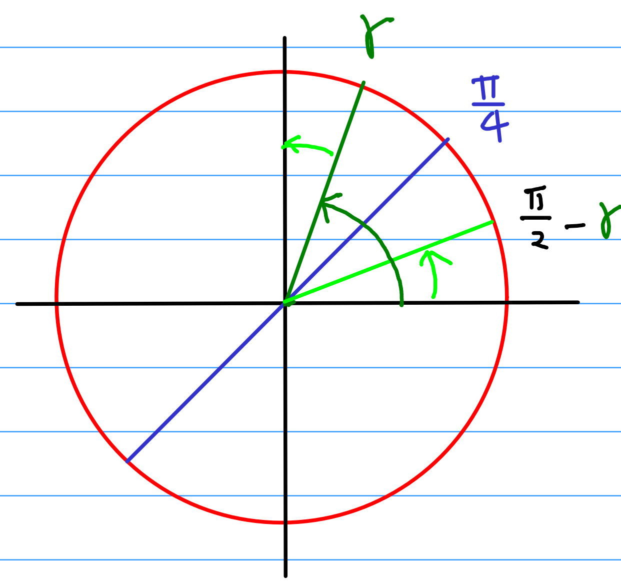
$\gamma > \frac{\pi}{4}$: Upper Half $(MSB_3 = 1)$

$\gamma < \frac{\pi}{4}$: Lower Half $(MSB_3 = 0)$

$$\cos\gamma = \sin\left(\frac{\pi}{2} - \gamma\right)$$

$$\sin\gamma = \cos\left(\frac{\pi}{2} - \gamma\right)$$

$\gamma > \frac{\pi}{4} \qquad \frac{\pi}{2} - \gamma < \frac{\pi}{4}.$



ϕ'	0	0	MSB3					$\cdots$	

MSB3 = 1 $\qquad \phi' > \frac{\pi}{4}$

$$\phi'' = \frac{\pi}{2} - \phi'$$

0.5	1	0	0					$\cdots$	
$\ominus$ ϕ'	0	0	MSB3					$\cdots$	

$$\begin{cases} MSB3 = 0 & \phi'' = \phi' \\ MSB_3 = 1 & \phi'' = 0.5 - \phi' \end{cases}$$

$\theta = \pi \phi''$ (Handwired Multiplier)

$$0 < \theta < \frac{\pi}{4}$$

Sine / Cosine Generator

Subrotation

$$X_{k+1} = X_k - (r_k \tan\theta_k) Y_k$$

$$Y_{k+1} = Y_k + (r_k \tan\theta_k) X_k$$

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

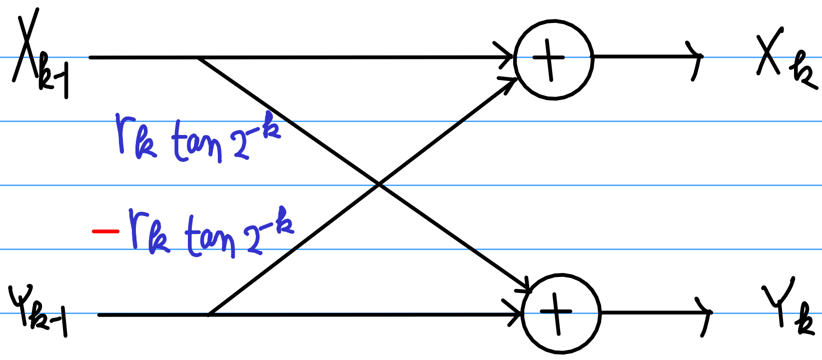
$$= \cos\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$\theta = \sigma_0\theta_0 + \sigma_1\theta_1 + \cdots + \sigma_N\theta_N$$

$$\sigma_k = \{-1, 0, +1\}$$

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = K \begin{bmatrix} 1 & -\tan\sigma_N\theta_N \\ \tan\sigma_N\theta_N & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan\sigma_0\theta_0 \\ \tan\sigma_0\theta_0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$K = \cos\sigma_0\theta_0 \cdot \cos\sigma_1\theta_1 \cdots \cos\sigma_N\theta_N$$



r_k or b_{k-1}