First Order ODEs (H.1)

20150625

1차 미분 방정식 ch 2.

ⓐ Separable eq. 변수분리 2.2

ⓑ Linear eq 선형방정식 2.3

ⓒ Exact eq 완전방정식. 2.4

↑

1st order differential eq

Solution : $y(x)$: y는 x의 함수

y' ~~y''~~ ~~y'''~~ 1차 미분 방정식

y, x

ODE Ordinary Differential Equation 상미분 방정식

PDE Partial Differential Equation 편미분 방정식

$$\frac{df}{dx} \qquad f(x) \ \cdots\cdots \quad \text{2차원 그림}$$

$$\underbrace{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}}_{\text{partial derivative.}} \qquad f(x, y) \ \cdots \quad \text{3차원 그림}$$

$$\frac{dy}{dx} = 6y^2 x = (6x)(y^2)$$

$$y' = \frac{3x^2+4x-4}{2y-4} = (3x^2+4x-4)\left(\frac{1}{2y-4}\right)$$

$$y' = \frac{xy^3}{\sqrt{1+x^2}} = \left(\frac{x}{\sqrt{1+x^2}}\right)(y^3)$$

$$y' = e^{-y}(2x-4) = (2x-4)(e^{-y})$$

$$\frac{dr}{d\theta} = \frac{r^2}{\theta}$$ r(θ) 함수를 θ에 대하여 미분

$$\underline{\frac{d}{d\theta} r(\theta) = \frac{r^2(\theta)}{\theta}} \Rightarrow \underline{r(\theta)\,?}$$

$$\frac{dr}{d\theta} = \frac{r^2}{\theta} = \left(\frac{1}{\theta}\right)(r^2)$$

$y(t)$?

$$\frac{dy}{dt} = e^{y-t}\sec(y)(1+t^2) = \left(e^{-t}(1+t^2)\right)\left(e^{y}\sec(y)\right)$$

1차 미분 방정식 $\underbrace{y'}_{x에 대해서\ y를\ 미분} = \underbrace{g(x, y)}_{x,\ y의\ 수식}$ $y(x) = ?$

$$y' = g_1(x) \times g_2(y)$$

$\underline{y'}$ (y 식) = (x 식)

(y 식) $\boxed{\frac{dy}{dx}\ dx}$ = (x 식) dx

↑

접선의 기울기

(y 식) dy = (x 식) dx

$\int$ (y 식) dy = $\int$ (x 식) dx

1차 미분 방정식 $\underbrace{y'}_{x\text{에 대해서 } y\text{를 미분}} = \underbrace{g(x, y)}_{x,\ y\text{의 수식}}$ $y(x)$?

Linear Eq

$$\underbrace{a_1(x)}_{x\text{의 수식 or 상수}} \boxed{y'} + \underbrace{a_2(x)}_{x\text{의 수식 or 상수}} \boxed{y} = \underbrace{g(x)}_{x\text{의 수식}}$$

$$1 \cdot \boxed{y'} + \underbrace{p(x)}_{x\text{의 수식 or 상수}} \boxed{y} = \underbrace{g(x)}_{x\text{의 수식}}$$

1. $\boxed{y'} + p(x)\boxed{y} = g(x)$

$$\frac{dv}{dt} = 9.8 - 0.196v \qquad v(t) = ?$$

$$1\ \frac{dv}{dt} + 0.196v = 9.8$$

$$\cos(x)\boxed{y'} + \sin(x)\boxed{y} = 2\cos^3(x)\sin(x) - 1$$

$$\boxed{y'} + \frac{\sin(x)}{\cos(x)}\boxed{y} = 2\cos^2(x)\sin(x) - \frac{1}{\cos(x)}$$

$$t\boxed{y'} + 2\boxed{y} = t^2 - t + 1$$

$$\boxed{y'} + \frac{2}{t}\boxed{y} = t - 1 + \frac{1}{t}$$

$$t\boxed{y'} - 2\boxed{y} = t^5\sin(2t) - t^3 + 4t^4$$

$$1\ \boxed{y'} - \frac{2}{t}\boxed{y} = t^4\sin(2t) - t^2 + 4t^3$$

$$2\boxed{y'} - \boxed{y} = 4\sin(3t) \qquad y(t)\ ?$$

$$1\ \boxed{y'} - \frac{1}{2}\boxed{y} = 2\sin(3t)$$

homogeneous eq

$$y' + p(x)\, y = 0 \qquad \text{sol } \; y_h$$

$$\Leftrightarrow \quad y_h' + p(x)\, y_h = 0$$

non-homogeneous eq

$$y' + p(x)\, y = Q(x) \qquad \text{sol } \; y_p$$

$$\Leftrightarrow \quad y_p' + p(x)\, y_p = Q(x)$$

$$y_h' + p(x)\, y_h = 0$$

$$y_p' + p(x)\, y_p = Q(x)$$

$$(y_p + y_h)' + p(x)(y_p + y_h) = Q(x)$$

$$\Leftrightarrow \quad y' + p(x)\, y = Q(x) \qquad \text{sol } \; (y_p + y_h)$$

homogeneous eq

$y' + p(x)\, y = 0$ sol $y_h = c \cdot y_1$

$= c \cdot e^{-\int p(x)dx}$

non-homogeneous eq

$y' + p(x)\, y = Q(x)$ sol $y_p = u \cdot y_1$

$= u(x)\, e^{-\int p(x)dx}$

1st Order Linear Equations

y_p 구하기

$$\boxed{1\cdot y' + \boxed{P(x)}\, y = Q(x)}$$

$$y_h = c\cdot e^{-\int \boxed{P(x)}dx}$$

$$y_p = u(x)\, e^{-\int \boxed{P(x)}dx}$$

$$\boxed{e^{+\int \boxed{P(x)}dx}}$$

Integrating Factor

- const. c X
- sign X

$$\boxed{e^{+\int \boxed{P(x)}dx}}\ \boxed{\left(1\cdot y' + \boxed{P(x)}\, y\right) = \left(Q(x)\right)}\ \boxed{e^{+\int \boxed{P(x)}dx}}$$

$$\left(\boxed{e^{+\int \boxed{P(x)}dx}\cdot}\ \boxed{y_p}\right)' = Q(x)\, e^{+\int \boxed{P(x)}dx}$$

$$e^{+\int \boxed{P(x)}dx}\cdot y_p = \int Q(x)\, e^{+\int \boxed{P(x)}dx}\, dx$$

$$y_p = e^{-\int \boxed{P(x)}dx}\cdot \int Q(x)\, e^{+\int \boxed{P(x)}dx}\, dx$$

$$y = y_h + y_p$$

$$y = \underbrace{c\cdot e^{-\int \boxed{P(x)}dx}}_{y_h} + \underbrace{e^{-\int \boxed{P(x)}dx}\cdot \int Q(x)\, e^{+\int \boxed{P(x)}dx}\, dx}_{y_p}$$

Calculus 1

Review : Exponential

Review : Logarithmic

Derivatives : Trig Derivatives

Differential Equation

First Order : Linear Equation

First Order Differential Equations (1P.pdf) .....
Linear Equation (2A.pdf)

cf) Partial Derivatives (9.4 Zill & Wright)

Exact Equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\boxed{y(x) = ?} \quad \cdots \text{ 1 variable}$$

$$f(x,y) \quad \cdots \quad \text{2 variables}$$

$$\boxed{\text{Exact}} \longrightarrow f(x,y) \text{ exists}$$

total differential

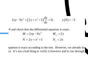
$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial}{\partial y}(M) = \frac{\partial}{\partial x}(N)$$

$$f(x,y) \qquad df = 0 \qquad \boxed{f(x,y) = c}$$



$$y(x) \leftarrow \text{Curve} \leftarrow \text{Surface} \cap \text{평면}$$

$$(2xy - 9x^2) + (2y + x^2 + 1)\frac{dy}{dx} = 0$$

$$(2xy - 9x^2)\,dx + (2y + x^2 + 1)\,dy = 0$$

$$\boxed{M = \frac{\partial f}{\partial x}}\,dx + \boxed{N = \frac{\partial f}{\partial y}}\,dy = 0$$

$$M\,dx + N\,dy = 0$$

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0$$

* Exact?

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$2x = 2x \quad \therefore \text{ exact}$$

$$f(x,y) = \int \frac{\partial f}{\partial x}\,dx + c$$

variable y may be included

$$= \int \frac{\partial f}{\partial x}\,dx + g(y)$$

$$= \int (2xy - 9x^2)\,dx + g(y)$$

$$f(x,y) = x^2y - 3x^3 + g(y) \quad ?$$

$$\frac{\partial f}{\partial y} = \cancel{x^2} + g'(y) = (2y + \cancel{x^2} + 1)$$

$$g'(y) = 2y + 1 \qquad g'(y) = y^2 + y$$

$$f(x,y) = x^2y - 3x^3 + y^2 + y$$

$$df = 0$$

$$f(x,y) = x^2y - 3x^3 + y^2 + y = C$$

$$\underline{y(x)}$$

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0} \Longrightarrow \boxed{df = 0}$$

$\boxed{y(x) = ?}$ … 1 variable $\quad f(x,y) = c$

$f(x,y)$ … 2 variables $\quad \boxed{f(x,\ y(x))}$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$M(x,y) = \dfrac{\partial f}{\partial x}$, $\quad N(x,y) = \dfrac{\partial f}{\partial y}$

① Verify "exactness"

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

② $M(x,y) = \dfrac{\partial f}{\partial x}$

$$f(x,y) = \int M(x,y)\,dx + g(y)$$

③ $N(x,y) = \dfrac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right] + g'(y)$$

$$= N(x,y)$$

④ $f(x,y) = c$, $\quad z = z$

f의 surface $\cap$ 평면 $\Rightarrow$ $y(x)$

Non-Exact Case

Zill (Ex 4)

$$xy\,dx + (2x^2+3y^2-20)\,dy = 0$$

$$\frac{\partial}{\partial y}(xy) = x \neq \frac{\partial}{\partial x}(2x^2+3y^2-20) = 4x$$

non-exact eq

$\boxed{y^3} \leftarrow$ Integrating Factor $\mu(x)$ $\mu(y)$

$$\boxed{xy\cdot y^3}\,dx + \boxed{(2x^2+3y^2-20)y^3}\,dy = 0$$

$$xy^4\,dx + (2x^2y^3+3y^5-20y^3)\,dy = 0$$

$$\frac{\partial}{\partial y}(xy^4) = 4xy^3 = \frac{\partial}{\partial x}(2x^2y^3+3y^5-20y^3) = 4xy^3$$

Now exact!

Non-Exact Case : Finding Integrating Factor

Exact $\longrightarrow$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ $\quad M_y = N_x$

$$M_y - N_x = 0$$

Non-Exact $\longrightarrow$ $M_y - N_x \neq 0$

$\dfrac{M_y - N_x}{N}$: x의 수식 ✓ $\qquad I.F. = e^{\int \frac{M_y - N_x}{N} dx}$

$-\dfrac{M_y - N_x}{M}$: y의 수식 ✓ $\qquad I.F. = e^{-\int \frac{M_y - N_x}{M} dy}$

Integrating Factor

- sign x
- const c x

$$\boxed{\mu(x) = e^{\int \frac{M_y - N_x}{N} dx} \Leftarrow \frac{M_y - N_x}{N} \text{ ..... } x\text{의 수식}}$$

$$\boxed{\mu(y) = e^{\int \frac{-M_y + N_x}{M} dy} \Leftarrow -\frac{M_y - N_x}{M} \text{ ... } y\text{의 수식}}$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = M_y \qquad \frac{\partial N}{\partial x} = N_x$$

difference

x 또는
y 수식

$$xy\,dx + (2x^2+3y^2-20)\,dy = 0$$

$$\frac{\partial}{\partial y}(xy) = x \neq \frac{\partial}{\partial x}(2x^2+3y^2-20) = 4x$$

non-exact eq

$$\frac{\partial M}{\partial y} = M_y = x \qquad \frac{\partial N}{\partial x} = N_x = 4x$$

x only expression

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx}$$

$$\mu(x) = e^{\int \frac{-3x}{(2x^2+3y^2-20)}\,dx}$$

$$xy\,dx + (2x^2+3y^2-20)\,dy = 0$$

$$\frac{\partial}{\partial y}(xy) = x \neq \frac{\partial}{\partial x}(2x^2+3y^2-20) = 4x$$

non-exact eq

$$\frac{\partial M}{\partial y} = M_y = x \qquad \frac{\partial N}{\partial x} = N_x = 4x$$

x only expression

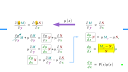
$$\mu(y) = e^{\int \frac{-M_y + N_x}{M}\,dy} \Leftarrow -M + N \cdots$$ y의 수식

$$\mu(y) = e^{\int \frac{3\cancel{x}}{\cancel{x}y}\,dy} = e^{3\int \frac{1}{y}\,dy} = e^{\ln|y|^3}$$

$$= |y|^3$$

$$\mu(y) = y^3,\ -y^{-3} \Rightarrow \boxed{y^3}$$

$$xy \cdot y^3\,dx + (2x^2+3y^2-20)\,y^3\,dy = 0$$



integrating factor 공식 유도 과정

$(f \cdot g)' = f'g + fg'$

② $\boxed{\dfrac{\partial(\mu M)}{\partial y} = \dfrac{\partial(\mu N)}{\partial x}} \xleftarrow[\text{y가 없는 수식}]{\mu(x)}$ ① $\boxed{\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}}$

$$\cancelto{0}{\frac{\partial \mu}{\partial y}} M + \mu \frac{\partial M}{\partial y} = \boxed{\frac{\partial \mu}{\partial x}} N + \mu \frac{\partial N}{\partial x}$$

$$\mu \frac{\partial M}{\partial y} = \boxed{\frac{d\mu}{dx}} N + \mu \frac{\partial N}{\partial x}$$

$$\boxed{\frac{d\mu}{dx}} N = \mu \frac{\partial M}{\partial y} - \mu \frac{\partial N}{\partial x}$$

$$\boxed{\frac{d\mu}{dx}} N = \mu M_y - \mu N_x$$

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu$$

$$\frac{d\mu}{dx} = P(x)\mu(x)$$

만일 □에 y가 없는 수식이면

$$\frac{d\mu}{dx} - P(x)\mu = 0$$

$$\mu(x) = c e^{\int P(x)\,dx}$$

$$\mu(x) = \cancel{c}\, e^{\int \left(\frac{M_y - N_x}{N}\right) dx}$$

Substitution Method

$$\frac{dy}{dx} = g(x, y)$$

$\left(\frac{y}{x}\right)$ 항상 이런 형태가 있다.

$= u \qquad y = ux$

Substitute

ex) $y = ux$

$$\boxed{y = h(x,u)} \Leftarrow \boxed{u = \Phi(x)}$$

$$\Downarrow$$

$$\boxed{\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}}$$

$$\frac{d}{dx}y = \frac{d}{dx}h(x,u)$$

$\frac{\partial h}{\partial x} = u \quad \frac{\partial h}{\partial u} = x \quad \frac{du}{dx} = u' \quad \Rightarrow \quad u + xu'$

$$g(x, h(x,u)) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$g(x, h(x,u)) = h_x(x,y) + h_u(x,u)\frac{du}{dx}$$

$$\frac{dy}{dx} = g(x, y)$$

* substitute y

and y'

both y & y'

$\left(\frac{y}{x}\right) \to u$

$y \to ux$

$y' \to u'x + u$

Paul's online math note ex1)

$$x y y' + 4x^2 + y^2 = 0$$

divide by x^2

$$\left(\frac{y}{x}\right) y' + 4 + \left(\frac{y}{x}\right)^2 = 0$$

u $\qquad$ u

$$u = \frac{y}{x} = \frac{y(x)}{x}$$

$$\frac{x\text{의 수식}}{x\text{의 수식}}$$

Substitute both y and y'

$$y = xu$$

$$y' = u + xu'$$

$$\left(\frac{y}{x}\right) y' + 4 + \left(\frac{y}{x}\right)^2 = 0$$

$$\left(\frac{xu}{x}\right)\cdot\left(u + xu'\right) + 4 + \left(\frac{xu}{x}\right)^2 = 0$$

$$u\left(u + xu'\right) + 4 + u^2 = 0$$

$$2u^2 + xuu' + 4 = 0$$

$$2u^2 + xuu' + 4 = 0$$

$$\frac{xu'u}{u} = \frac{-2u^2 - 4}{u}$$

$$xu' = -\frac{4+2u^2}{u}$$

$$\left(\frac{u}{4+2u^2}\right) u' = -\frac{1}{x}$$

$$u' = \frac{du}{dx}$$

$$\frac{u}{4+2u^2} \cdot \frac{du}{dx} dx = -\frac{1}{x} dx$$

$$\frac{u}{4+2u^2} du = -\frac{1}{x} dx$$

$$\frac{1}{4} \int \frac{4u}{4+2u^2} du = \int -\frac{1}{x} dx$$

$$\ln\left(4+2v^2\right)^{\frac{1}{4}} = \ln\left(x\right)^{-1} + c$$

sides and do a little rewriting

$$\left(4+2v^2\right)^{\frac{1}{4}} = \mathbf{e}^{\ln(x)^{-1}+c} = \mathbf{e}^c\mathbf{e}^{\ln(x)^{-1}} = \frac{c}{x}$$

$$\frac{1}{4} \ln|4+2u^2| = -\ln|x| + C$$

$$\ln|4+2u^2|^{\frac{1}{4}} = \ln|x|^{-1} + C$$

$$|4+2u^2|^{\frac{1}{4}} = e^{\ln|x|^{-1}+C} = e^{\ln|x|^{-1}} e^{C}$$

$$(4+2u^2)^{\frac{1}{4}} = \frac{1}{x} C$$

$$(4+2u^2)^{\frac{1}{4}} = \frac{1}{x}C$$

$$(4+2u^2) = \left(\frac{c}{x}\right)^4$$

$u = \frac{y}{x}$

$$4+2\left(\frac{y}{x}\right)^2 = \left(\frac{c}{x}\right)^4$$

$$2\left(\frac{y}{x}\right)^2 = \left(\frac{C_1}{x^4} - 4\right)$$

$$\frac{y^2}{x^2} = \frac{1}{2}\left(\frac{C_1 - 4x^4}{x^4}\right)$$

$$y^2 = \frac{1}{2}x^2\left(\frac{C_1 - 4x^4}{x^4}\right)$$

$$= \frac{C_1 - 4x^4}{2x^2}$$

Paul's online math note ex2)

$$xy' = y(\ln x - \ln y)$$

$$\boxed{x > 0}$$

$$xy' = y \ln\left(\frac{x}{y}\right)$$

$$y' = \left(\frac{y}{x}\right) \ln\left(\frac{x}{y}\right)$$

$$\frac{y}{x} = u \qquad y = xu$$

$$y' = u + xu'$$

$$(u + xu') = u \ln\left(\frac{1}{u}\right)$$

$$xu' = u \ln\left(\frac{1}{u}\right) - u$$

$$\int \frac{1}{u\left(\ln\left(\frac{1}{u}\right) - 1\right)} \frac{du}{dx} dx = \int \frac{1}{x} dx$$

$$u$$

$$\int \frac{1}{u\left(\ln\left(\frac{1}{u}\right) - 1\right)} du = \int \frac{1}{x} dx$$

$$t = \ln\left(\frac{1}{u}\right) - 1$$

$$dt = \left(\frac{1}{\frac{1}{u}}\right)\left(\frac{1}{u}\right)' du$$

$$dt = u \cdot (-u^{-2}) du$$

$$dt = -\frac{1}{u} du$$

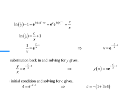
$$\int \frac{-1}{\left(\ln\left(\frac{1}{u}\right) - 1\right)} \left(\frac{-1}{u} du\right)$$

$$-\int \frac{1}{\left(\ln\left(\frac{1}{u}\right) - 1\right)} dt = \int \frac{1}{x} dx$$

$$dt = u \, du$$

$$\ln\left(\ln\left(\frac{1}{u}\right) - 1\right) = -(\ln x + c)$$

$$\ln\left(\ln\left(\frac{1}{u}\right) - 1\right) = c - \ln x$$



$$\ln\left(\ln\left(\frac{1}{u}\right)-1\right) = C - \ln x$$

$$\ln\left(\frac{1}{u}\right)-1 = e^{C-\ln x}$$

$$= c_1 e^{\ln\frac{1}{x}}$$

$$= \frac{c_1}{x}$$

$$\ln\left(\frac{1}{u}\right)-1 = \frac{c_1}{x}$$

$$\ln\left(\frac{1}{u}\right) = \frac{c_1}{x} + 1$$

$$\frac{1}{u} = e^{\frac{c_1}{x}+1}$$

$$u = e^{-\frac{c_1}{x}-1}$$

$$\frac{y}{x} = e^{-\frac{c_1}{x}-1}$$

$$\boxed{y = x \cdot e^{-\frac{c_1}{x}-1}}$$

* $\int \tan x \, dx$ → substitution $u = \cos x$

$\int t^2 \sin(2t)\, dt$ →

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

$u = \cos x$

$du = -\sin x \, dx$

$$= \int \frac{-1}{u} du = -\ln|u| = \ln|u|^{-1}$$

$$= \ln\left|\frac{1}{\cos x}\right| = \ln|\sec x|$$

integrating factor $\mu(x)$

→ multiplied to the original differential eq

$$y' + p(x)\, y = Q(x)$$

$$\mu(x)\left(y' + p(x)\, y\right) = \left(Q(x)\right)\mu(x)$$

40p Ex 6)

$\frac{2r}{39}$

Chain Rule

$$f(g(x)) \xrightarrow{\frac{d}{dx}} f'(g)\, g'(x)$$

$$\longleftarrow$$

substitution rule

Product Rule

$$(fg)' = f'g + fg'$$

$$(fg)' - fg' = f'g$$

$$fg - \int fg'\,dx = \int f'g\,dx$$

Pauls online math note
Calculus 1

Integral :

Computing Indefinite Integrals
Substitution Rule
More about substitution rule

a ~~b~~
c ~~d~~
e ~~f~~
g ~~h~~

a
c
e

Differential Equation
1st Order

Exact Equation
Substitution

다 좋고

$f(x,y) = C$

range

Graph X

상수 구하기

initial condi

IVP

Substitution Rule

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

$g(x) = u$

Chain rule

$$\frac{d}{dx} f(g(x)) = \frac{d}{dg} f(g) \; \frac{d}{dx} g(x)$$