Minimum Mean Squared Error

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February 7, 2020

Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Definition

$$W(t) = X(t) + N(t)$$

$$\varepsilon(t) = X(t+t_0) - Y(t)$$

$$E\left[\varepsilon^2(t)\right] = E\left[|X(t+t_0) - Y(t)|^2\right]$$

$$= E\left[X^2(t+t_0) - 2Y(t)X(t+t_0) + Y^2(t)\right]$$

$$= R_{XX}(0) - 2R_{YX}(t_0) + R_{YY}(0)$$

Definition

$$R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)d\omega$$

$$R_{YY}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{WW}(\omega)|H(\omega)|^2 d\omega$$

$$E\left[\varepsilon^2(t)\right] = -2R_{YX}(t_0) + R_{XX}(0) + R_{YY}(0)$$

$$= -2R_{YX}(t_0) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[S_{XX}(\omega) + S_{WW}(\omega)|H(\omega)|^2\right] d\omega$$

Definition

$$R_{YX}(t_0) = E\left[Y(t)X(t+t_0)\right]$$

$$= E\left[X(t+t_0)\int_{-\infty}^{\infty} h(\xi)W(t-\xi)d\xi\right]$$

$$= \int_{-\infty}^{\infty} R_{WX}(t_0+\xi)h(\xi)d\xi$$

Definition

$$R_{YX}(t_0) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{WX}(\omega) e^{j\omega(t_0+\xi)} d\omega h(\xi) d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{WX}(\omega) e^{j\omega t_0} \left\{ \int_{-\infty}^{\infty} h(\xi) e^{j\omega\xi} d\xi \right\} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{WX}(\omega) H(-\omega) e^{j\omega t_0} d\omega$$

Definition

$$E\left[\varepsilon^2(t)\right] = E\left[\left|X(t+t_0) - Y(t)\right|^2\right]$$

$$= -2R_{YX}(t_0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[S_{XX}(\omega) + S_{WW}(\omega)|H(\omega)|^2\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{WX}(\omega)H(-\omega)e^{j\omega t_0}\,d\omega$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[S_{XX}(\omega) + S_{WW}(\omega)|H(\omega)|^2\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{S_{WX}(\omega)H(-\omega)e^{j\omega t_0} + S_{XX}(\omega) + S_{WW}(\omega)|H(\omega)|^2\right\}d\omega$$

Definition

$$H(\omega) = A(\omega)e^{jB(\omega)}$$

$$S_{WX}(\omega) = C(\omega)e^{jD(\omega)}$$

$$H(-\omega) = H^*(\omega)$$

$$E\left[\varepsilon^2(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{S_{XX}(\omega) + S_{WW}(\omega)A^2(\omega)\right\}d\omega$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{2C(\omega)A(\omega)e^{j[\omega t_0 + D(\omega) - B(\omega)]}\right\}d\omega$$

Definition

$$B(\omega) = \omega t_0 + D(\omega)$$

$$e^{j[\omega t_0 + D(\omega) - B(\omega)]} = e^{j0} = 1$$

$$E\left[\varepsilon^2(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{S_{XX}(\omega) - 2C(\omega)A(\omega) + S_{WW}(\omega)A^2(\omega)\right\} d\omega$$

Definition

$$S_{ww}(\omega)\left[A(\omega)-\frac{C(\omega)}{S_{WW}(\omega)}\right]^2$$

$$=S_{ww}(\omega)A^2(\omega)-2C(\omega)A(\omega)+\frac{C^2(\omega)}{S_{WW}(\omega)}$$

$$\left\{S_{XX}(\omega)-2C(\omega)A(\omega)+S_{WW}(\omega)A^2(\omega)\right\}$$

$$=\left\{S_{XX}(\omega)-\frac{C^2(\omega)}{S_{WW}(\omega)}+S_{ww}(\omega)\left[A(\omega)-\frac{C(\omega)}{S_{WW}(\omega)}\right]^2\right\}$$

Definition

$$\left[A(\omega) - \frac{C(\omega)}{S_{WW}(\omega)}\right] = 0$$

$$H_{opt}(\omega) = \frac{S_{WX}(\omega)}{S_{WW}(\omega)} e^{j\omega t_0}$$

$$S_{WW}(\omega) = S_{XX}(\omega) + S_{NN}(\omega)$$

$$H_{opt}(\omega) = \frac{S_{XX}(\omega)}{S_{XX}(\omega) + S_{NN}(\omega)} e^{j\omega t_0}$$

Definition

$$E_{min}\left[\varepsilon^2(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{S_{XX}(\omega)S_{WW}(\omega)+|S_{WX}(\omega)|^2}{S_{WW}(\omega)}\right\}d\omega$$

$$E_{min}\left[\varepsilon^2(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{S_{XX}(\omega)S_{NN}(\omega)}{S_{XX}(\omega)+S_{NN}(\omega)}\right\}d\omega$$

Definition

$$E_{min}\left[\varepsilon^2(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\varepsilon\varepsilon}(\omega)d\omega$$

$$S_{\varepsilon\varepsilon}(\omega) = S_{XX}(\omega) - 2S_{XX}(\omega)H(-\omega)e^{j\omega t_0}$$
$$+\left[S_{XX}(\omega) + S_{NN}(\omega)\right]\left|H(\omega)\right|^2$$

$$S_{\varepsilon\varepsilon}(\omega) = S_{XX}(\omega)T_1(\omega) + S_{NN}(\omega)\left|H(\omega)\right|^2$$
$$T_1(\omega) = 1 - H(\omega)e^{-j\omega t_0} - H(-\omega)e^{j\omega t_0} + \left|H(\omega)\right|^2$$
$$S_{\varepsilon\varepsilon}(\omega) = S_{XX}(\omega)\left|1 - H(-\omega)e^{j\omega t_0}\right|^2 + S_{NN}(\omega)\left|H(\omega)\right|^2$$