

# Minimum Mean Squared Error

Young W Lim

February 7, 2020

Copyright (c) 2018 Young W. Lim. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

This work is licensed under a Creative Commons "Attribution-NonCommercial-ShareAlike 3.0 Unported" license.



Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles, Jr. and B. Shi



# Wiener Filters (1)

$N$  Gaussian random variables

## Definition

$$W(t) = X(t) + N(t)$$

$$\varepsilon(t) = X(t + t_0) - Y(t)$$

$$\begin{aligned} E[\varepsilon^2(t)] &= E[|X(t + t_0) - Y(t)|^2] \\ &= E[X^2(t + t_0) - 2Y(t)X(t + t_0) + Y^2(t)] \\ &= R_{XX}(0) - 2R_{YX}(t_0) + R_{YY}(0) \end{aligned}$$

# Wiener Filters (2)

$N$  Gaussian random variables

## Definition

$$R_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) d\omega$$

$$R_{YY}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{WW}(\omega) |H(\omega)|^2 d\omega$$

$$E[\varepsilon^2(t)] = -2R_{YX}(t_0) + R_{XX}(0) + R_{YY}(0)$$

$$= -2R_{YX}(t_0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} [S_{XX}(\omega) + S_{WW}(\omega) |H(\omega)|^2] d\omega$$

# Wiener Filters (3)

$N$  Gaussian random variables

## Definition

$$\begin{aligned}R_{YX}(t_0) &= E[Y(t)X(t+t_0)] \\ &= E\left[X(t+t_0)\int_{-\infty}^{\infty}h(\xi)W(t-\xi)d\xi\right] \\ &= \int_{-\infty}^{\infty}R_{WX}(t_0+\xi)h(\xi)d\xi\end{aligned}$$

# Wiener Filters (4)

$N$  Gaussian random variables

## Definition

$$\begin{aligned} R_{YX}(t_0) &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{WX}(\omega) e^{j\omega(t_0+\xi)} d\omega h(\xi) d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{WX}(\omega) e^{j\omega t_0} \left\{ \int_{-\infty}^{\infty} h(\xi) e^{j\omega\xi} d\xi \right\} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{WX}(\omega) H(-\omega) e^{j\omega t_0} d\omega \end{aligned}$$

# Wiener Filters (5)

$N$  Gaussian random variables

## Definition

$$\begin{aligned} E[\varepsilon^2(t)] &= E[|X(t+t_0) - Y(t)|^2] \\ &= -2R_{YX}(t_0) + \frac{1}{2\pi} \int_{-\infty}^{\infty} [S_{XX}(\omega) + S_{WW}(\omega)|H(\omega)|^2] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{WX}(\omega)H(-\omega)e^{j\omega t_0} d\omega \\ &\quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} [S_{XX}(\omega) + S_{WW}(\omega)|H(\omega)|^2] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \{S_{WX}(\omega)H(-\omega)e^{j\omega t_0} + S_{XX}(\omega) + S_{WW}(\omega)|H(\omega)|^2\} d\omega \end{aligned}$$



# Wiener Filters (6)

$N$  Gaussian random variables

## Definition

$$H(\omega) = A(\omega)e^{jB(\omega)}$$

$$S_{WX}(\omega) = C(\omega)e^{jD(\omega)}$$

$$H(-\omega) = H^*(\omega)$$

$$E[\varepsilon^2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{S_{XX}(\omega) + S_{WW}(\omega)A^2(\omega)\} d\omega \\ - \frac{1}{2\pi} \int_{-\infty}^{\infty} \{2C(\omega)A(\omega)e^{j[\omega t_0 + D(\omega) - B(\omega)]}\} d\omega$$

# Wiener Filters (7)

$N$  Gaussian random variables

## Definition

$$B(\omega) = \omega t_0 + D(\omega)$$

$$e^{j[\omega t_0 + D(\omega) - B(\omega)]} = e^{j0} = 1$$

$$E[\epsilon^2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{S_{XX}(\omega) - 2C(\omega)A(\omega) + S_{WW}(\omega)A^2(\omega)\} d\omega$$

# Wiener Filters (8)

$N$  Gaussian random variables

## Definition

$$\begin{aligned} & S_{ww}(\omega) \left[ A(\omega) - \frac{C(\omega)}{S_{WW}(\omega)} \right]^2 \\ &= S_{ww}(\omega) A^2(\omega) - 2C(\omega)A(\omega) + \frac{C^2(\omega)}{S_{WW}(\omega)} \\ & \quad \{ S_{XX}(\omega) - 2C(\omega)A(\omega) + S_{WW}(\omega)A^2(\omega) \} \\ &= \left\{ S_{XX}(\omega) - \frac{C^2(\omega)}{S_{WW}(\omega)} + S_{ww}(\omega) \left[ A(\omega) - \frac{C(\omega)}{S_{WW}(\omega)} \right]^2 \right\} \end{aligned}$$

# Wiener Filters (9)

$N$  Gaussian random variables

## Definition

$$\left[ A(\omega) - \frac{C(\omega)}{S_{WW}(\omega)} \right] = 0$$

$$H_{opt}(\omega) = \frac{S_{WX}(\omega)}{S_{WW}(\omega)} e^{j\omega t_0}$$

$$S_{WW}(\omega) = S_{XX}(\omega) + S_{NN}(\omega)$$

$$H_{opt}(\omega) = \frac{S_{XX}(\omega)}{S_{XX}(\omega) + S_{NN}(\omega)} e^{j\omega t_0}$$

# Minimum Mean Squared Error

$N$  Gaussian random variables

## Definition

$$E_{min} [\varepsilon^2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{S_{XX}(\omega)S_{WW}(\omega) + |S_{WX}(\omega)|^2}{S_{WW}(\omega)} \right\} d\omega$$

$$E_{min} [\varepsilon^2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{S_{XX}(\omega)S_{NN}(\omega)}{S_{XX}(\omega) + S_{NN}(\omega)} \right\} d\omega$$

## Definition

$$E_{min} [\varepsilon^2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{\varepsilon\varepsilon}(\omega) d\omega$$

$$S_{\varepsilon\varepsilon}(\omega) = S_{XX}(\omega) - 2S_{XX}(\omega)H(-\omega)e^{j\omega t_0} \\ + [S_{XX}(\omega) + S_{NN}(\omega)] |H(\omega)|^2$$

$$S_{\varepsilon\varepsilon}(\omega) = S_{XX}(\omega) T_1(\omega) + S_{NN}(\omega) |H(\omega)|^2$$

$$T_1(\omega) = 1 - H(\omega)e^{-j\omega t_0} - H(-\omega)e^{j\omega t_0} + |H(\omega)|^2$$

$$S_{\varepsilon\varepsilon}(\omega) = S_{XX}(\omega) |1 - H(-\omega)e^{j\omega t_0}|^2 + S_{NN}(\omega) |H(\omega)|^2$$



