

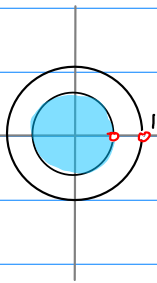
# Laurent Series and z-Transform Examples case 2.A

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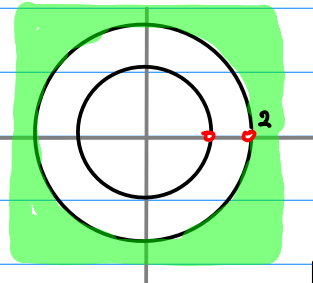
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I



$$a_n = \begin{cases} [1 - 2^{n-1}] & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

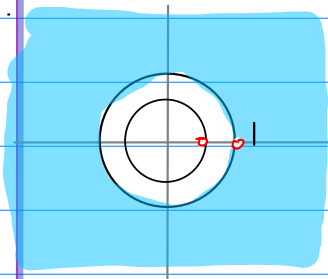
$$f(z) = \sum_{n=1}^{\infty} [1 - 2^{n-1}] z^n$$



$$x_n = \begin{cases} [1 - 2^{n-1}] & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

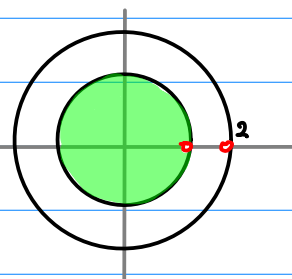
$$X(z) = \sum_{n=1}^{\infty} [1 - 2^{n-1}] z^{-n}$$

II



$$a_n = \begin{cases} 0 & (n > 0) \\ [2^{n-1} - 1] & (n \leq 0) \end{cases}$$

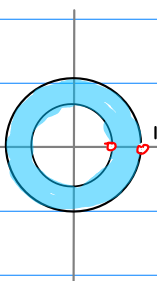
$$f(z) = \sum_{n=0}^{-\infty} [2^{n-1} - 1] z^n$$



$$x_n = \begin{cases} 0 & (n > 0) \\ [2^{n-1} - 1] & (n \leq 0) \end{cases}$$

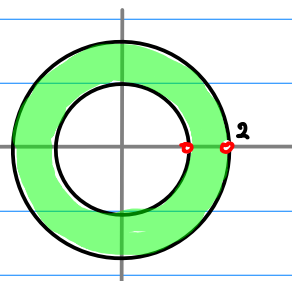
$$X(z) = \sum_{n=0}^{-\infty} [2^{n-1} - 1] z^{-n}$$

III



$$a_n = \begin{cases} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

$$f(z) = \sum_{n=1}^{\infty} 1 \cdot z^n + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^n$$

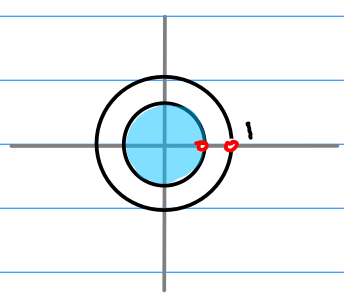


$$x_n = \begin{cases} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{cases}$$

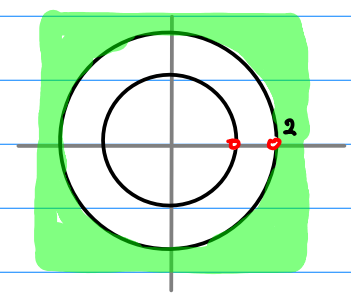
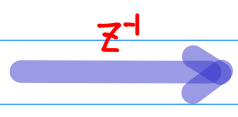
$$X(z) = \sum_{n=1}^{\infty} 1 \cdot z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n}$$

2.A

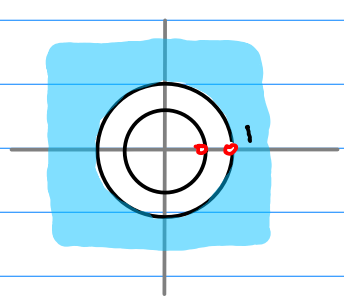
$$f(z) = \frac{-0.5 z^2}{(z-1)(z-0.5)} \xrightarrow{z^{-1}} X(z) = \frac{-1}{(z-1)(z-2)}$$



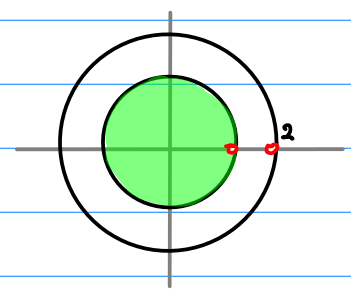
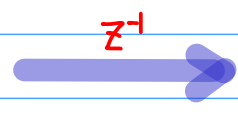
$$\sum_{n=1}^{\infty} [1 - 2^{n-1}] z^n$$



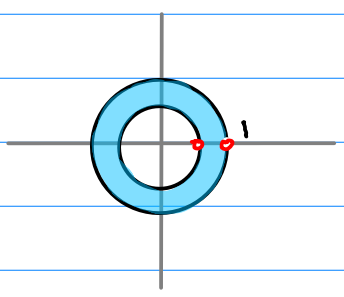
$$\sum_{n=1}^{\infty} [1 - 2^{n-1}] z^{-n}$$



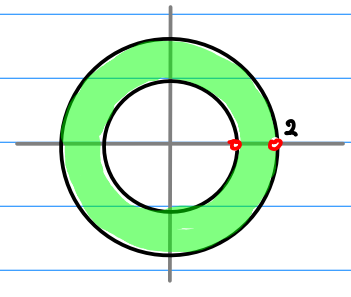
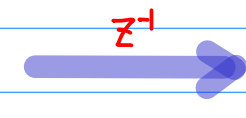
$$\sum_{n=0}^{-\infty} [2^{n-1} - 1] z^n$$



$$\sum_{n=0}^{-\infty} [2^{n-1} - 1] z^{-n}$$

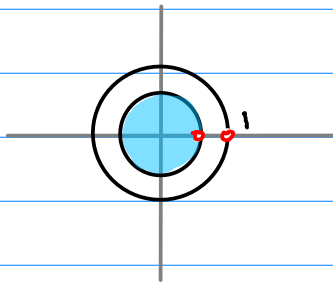


$$\sum_{n=1}^{\infty} 1 \cdot z^n + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^n$$

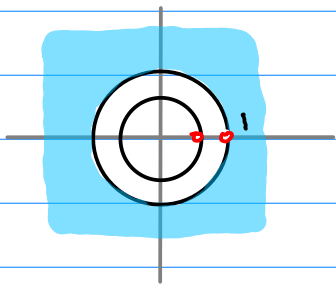


$$\sum_{n=1}^{\infty} 1 \cdot z^{-n} + \sum_{n=0}^{-\infty} 2^{n-1} \cdot z^{-n}$$

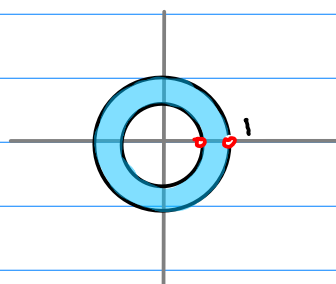
$$f(z) = \frac{-0.5z^2}{(z-1)(z-0.5)} = \frac{-z}{z-1} + \frac{0.5z}{z-0.5}$$



$$\begin{aligned} & + \frac{\left(\frac{z}{1}\right)}{1-\left(\frac{z}{1}\right)} - \frac{\left(\frac{z}{1}\right)}{1-\left(\frac{2z}{1}\right)} \\ & = + \sum_{n=0}^{\infty} \left(\frac{z}{1}\right) \left(\frac{z}{1}\right)^n - \sum_{n=0}^{\infty} \left(\frac{z}{1}\right) \left(\frac{2z}{1}\right)^n \\ & = + \sum_{n=0}^{\infty} z^{n+1} - \sum_{n=0}^{\infty} 2^{n+1} z^{n+1} \\ & = \sum_{n=1}^{\infty} [1 - 2^{n-1}] z^n \end{aligned}$$

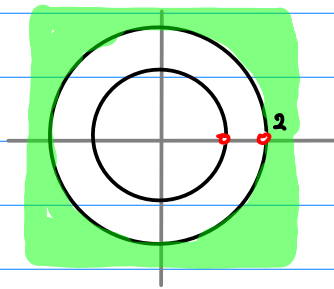


$$\begin{aligned} & - \frac{(1)}{1-\left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{2z}\right)} \\ & = - \sum_{n=0}^{\infty} (1) \left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \left(\frac{1}{2z}\right)^n \\ & = \sum_{n=0}^{\infty} [2^{n-1} - 1] z^{-n} \\ & = \sum_{n=0}^{\infty} [2^{n-1} - 1] z^n \end{aligned}$$



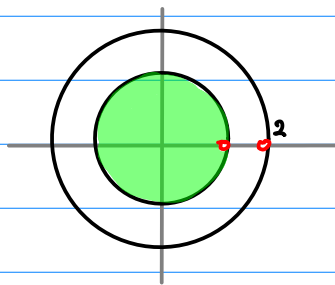
$$\begin{aligned} & + \frac{\left(\frac{z}{1}\right)}{1-\left(\frac{z}{1}\right)} + \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{2z}\right)} \\ & = + \sum_{n=0}^{\infty} \left(\frac{z}{1}\right) \left(\frac{z}{1}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \left(\frac{1}{2z}\right)^n \\ & = + \sum_{n=0}^{\infty} z^{n+1} + \sum_{n=0}^{\infty} 2^{-n-1} z^{-n} \\ & = + \sum_{n=1}^{\infty} z^n + \sum_{n=0}^{\infty} 2^{n-1} z^n \end{aligned}$$

$$X(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{(z-1)} - \frac{1}{(z-2)}$$



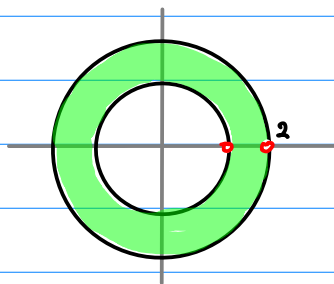
$$\sum_{n=-\infty}^{\infty} [1 - 2^{n-1}] z^{-n}$$

$$\begin{aligned} & + \frac{\left(\frac{1}{z}\right)}{1 - \left(\frac{1}{z}\right)} - \frac{\left(\frac{1}{z}\right)}{1 - \left(\frac{2}{z}\right)} \\ & = + \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \left(\frac{2}{z}\right)^n \\ & = \sum_{n=0}^{\infty} [1 - 2^n] z^{-n-1} \\ & = \sum_{n=-1}^{\infty} \left[1 - \left(\frac{1}{2}\right)^{n+1}\right] z^n \end{aligned}$$



$$\sum_{n=0}^{\infty} [2^{n-1} - 1] z^{-n}$$

$$\begin{aligned} & - \frac{\left(\frac{1}{z}\right)}{1 - \left(\frac{2}{z}\right)} + \frac{\left(\frac{1}{z}\right)}{1 - \left(\frac{2}{z}\right)} \\ & = - \sum_{n=0}^{\infty} (1) \left(\frac{2}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \left(\frac{2}{z}\right)^n \\ & = - \sum_{n=0}^{\infty} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^{-n} \\ & = \sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\right)^{n+1} - 1\right] z^{-n} \end{aligned}$$

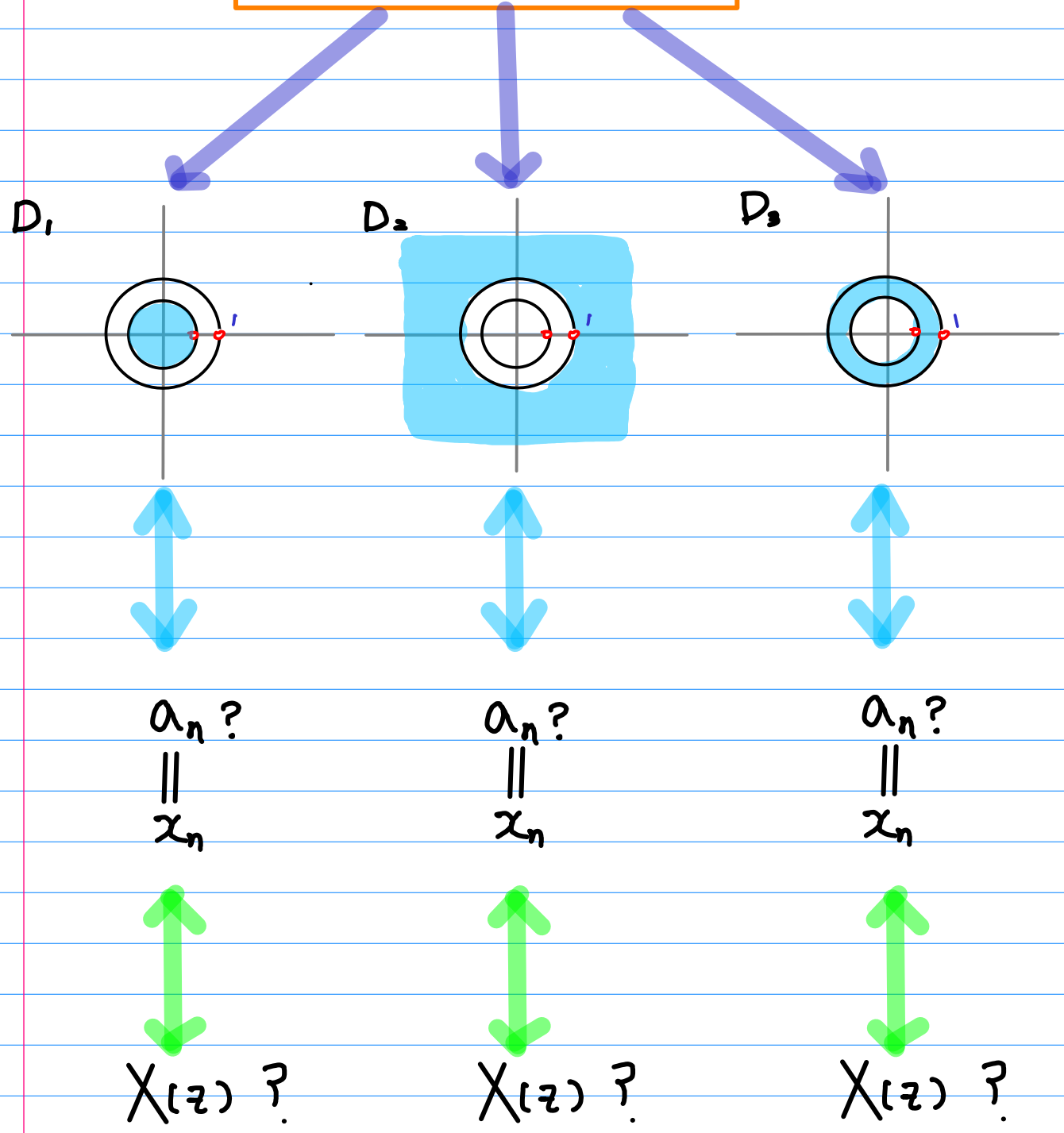


$$+ \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{\infty} 2^{n-1} z^{-n}$$

$$\begin{aligned} & + \frac{\left(\frac{1}{z}\right)}{1 - \left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{z}\right)}{1 - \left(\frac{2}{z}\right)} \\ & = + \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \left(\frac{2}{z}\right)^n \\ & = + \sum_{n=0}^{\infty} \left(\frac{1}{z}\right) \left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^{-n} \\ & = + \sum_{n=1}^{\infty} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^{-n} \end{aligned}$$

L.S. first

$$f(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$



$$X(z) = ?$$

$$f(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$\frac{z}{(z-1)(z-0.5)} = \frac{z}{z-1} - \frac{1}{z-0.5}$$

$$= \frac{2z-1-z+1}{(z-1)(z-0.5)}$$

$$\frac{-0.5z^2}{(z-1)(z-0.5)} = -0.5 \left( \frac{2z}{z-1} - \frac{z}{z-0.5} \right) = \frac{-z}{z-1} + \frac{0.5z}{z-0.5}$$

$$\frac{-z}{z-1} = \begin{cases} \frac{\left(\frac{z}{1}\right)}{1-\left(\frac{z}{1}\right)} = \sum_{n=0}^{\infty} z \cdot z^n = \sum_{n=0}^{\infty} 1 \cdot z^{n+1} & \left|\frac{z}{1}\right| < 1 \\ \frac{-1}{1-\left(\frac{1}{z}\right)} = -\sum_{n=0}^{\infty} 1 \cdot z^{-n} = -\sum_{n=0}^{\infty} 1 \cdot z^{-n} & \left|\frac{1}{z}\right| < 1 \end{cases}$$

$$\frac{0.5z}{z-0.5} = \begin{cases} \frac{-\left(\frac{z}{1}\right)}{1-\left(\frac{2z}{1}\right)} = -\sum_{n=0}^{\infty} \left(\frac{z}{1}\right)\left(\frac{2z}{1}\right)^n = -\sum_{n=0}^{\infty} 2^n z^{n+1} & \left|\frac{2z}{1}\right| < 1 \\ \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{1}{2z}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} z^{-n} & \left|\frac{1}{2z}\right| < 1 \end{cases}$$

$$f(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

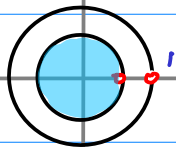
$$X(z) = \frac{-1}{(z-1)(z-2)}$$

L.S. first

Ⓘ - 1

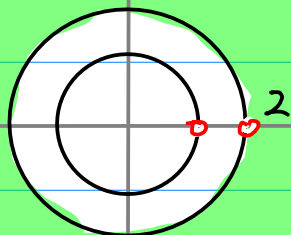
Ⓘ  $D_1 \quad |z| < 0.5 \quad \left[ \left| \frac{z}{1} \right| < 1, \quad \left| \frac{2z}{1} \right| < 1 \right]$

$p = 0.5$   
 $p = 1$   
 $|z| < \frac{1}{2}$



$$\begin{aligned} f(z) &= \frac{-z}{z-1} + \frac{0.5z}{z-0.5} \\ &= \frac{\left(\frac{z}{1}\right)}{1-\left(\frac{z}{1}\right)} + \frac{-\left(\frac{z}{1}\right)}{1-\left(\frac{2z}{1}\right)} \\ &= \sum_{n=0}^{\infty} 1 \cdot z^{n+1} - \sum_{n=0}^{\infty} 2^n z^{n+1} \\ &= \sum_{n=0}^{\infty} [1 - 2^n] z^{n+1} \\ &= \sum_{n=1}^{\infty} [1 - 2^{n-1}] z^n \quad |z| < \frac{1}{2} \end{aligned}$$

$p =$   
 $p =$   
 $|z| > 1$



$$\begin{aligned} X(z) = f(z^{-1}) &= \sum_{n=1}^{\infty} [1 - 2^{n-1}] z^{-n} \quad |z| > 2 \\ &= \sum_{n=1}^{\infty} 1 \cdot z^{-n} - \sum_{n=1}^{\infty} 2^{n-1} \cdot z^{-n} \\ &= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} - \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)} \\ &= \frac{1}{z-1} - \frac{1}{z-2} \\ &= \frac{z-2-z+1}{(z-1)(z-2)} \\ &= \frac{-1}{(z-1)(z-2)} \end{aligned}$$

$$f(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$f(z^{-1}) = \frac{-0.5z^{-2}}{(z^{-1}-1)(z^{-1}-0.5)} = \frac{-0.5}{(1-z)(1-0.5z)} = \frac{-1}{(z-1)(z-2)} = X(z)$$



II

$D_2$

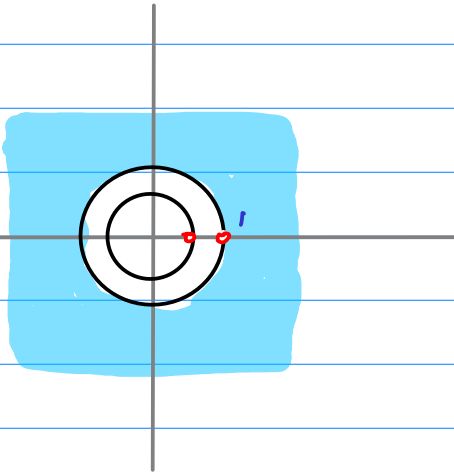
$1 < |z|$

$\left[ \left| \frac{1}{z} \right| < 1, \left| \frac{1}{2z} \right| < 1 \right]$

$p = 0.5$

$p = 1$

$|z| > \frac{1}{2}$



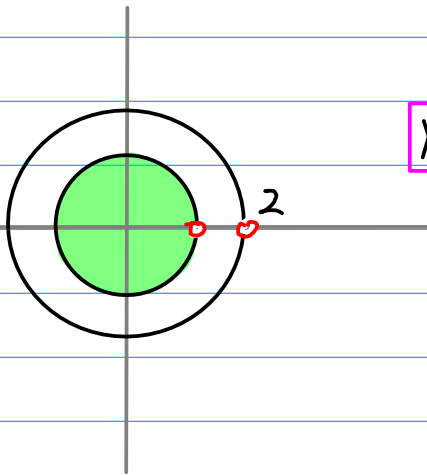
$$f(z) = \frac{-z}{z-1} + \frac{0.5z}{z-0.5}$$

$$= \frac{-1}{1 - (\frac{1}{z})} + \frac{(\frac{1}{2})}{1 - (\frac{1}{2z})}$$

$$= -\sum_{n=0}^{\infty} 1 \cdot (\frac{1}{z})^n + \sum_{n=0}^{\infty} (\frac{1}{2}) (\frac{1}{2z})^n$$

$$= \sum_{n=0}^{\infty} [2^{-n-1} - 1] z^{-n}$$

$$= \sum_{n=0}^{-\infty} [2^{n-1} - 1] z^n \quad |z| > 1$$



$$X(z) = f(z^{-1}) = \sum_{n=0}^{-\infty} [2^{n-1} - 1] z^{-n} \quad |z| > 2$$

$$= \sum_{n=0}^{\infty} 2^{-n-1} \cdot z^n - \sum_{n=0}^{\infty} 1 \cdot z^n$$

$$= \frac{(\frac{1}{2})}{1 - (\frac{z}{2})} - \frac{1}{1 - (\frac{z}{1})}$$

$$= \frac{-1}{z-2} + \frac{1}{z-1}$$

$$= \frac{-z+1+z-2}{(z-1)(z-2)}$$

$$= \frac{-1}{(z-1)(z-2)}$$

$$f(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

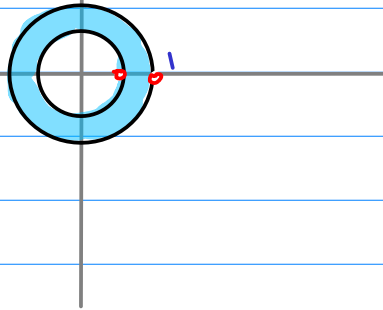
$$f(z^{-1}) = \frac{-0.5z^{-2}}{(z^{-1}-1)(z^{-1}-0.5)} = \frac{-0.5}{(1-z)(1-0.5z)} = \frac{-1}{(z-1)(z-2)} = X(z)$$



$D_3 \quad 0.5 < |z| < 1$

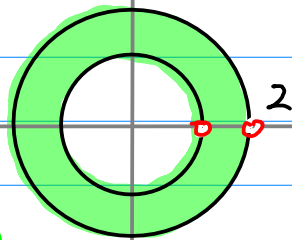
$\left[ \left| \frac{z}{1} \right| < 1, \quad \left| \frac{1}{2z} \right| < 1 \right]$

$p=1$   
 $p=0.5$   
 $\frac{1}{2} < |z| < 1$



$$\begin{aligned}
 f(z) &= \frac{-z}{z-1} + \frac{0.5z}{z-0.5} \\
 &= \frac{\left(\frac{z}{1}\right)}{1-\left(\frac{z}{1}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)} \\
 &= \sum_{n=0}^{\infty} z \cdot z^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right) \left(\frac{1}{2z}\right)^n \\
 &= \sum_{n=0}^{\infty} 1 \cdot z^{n+1} + \sum_{n=0}^{\infty} 2^{-n-1} z^{-n}
 \end{aligned}$$

$$= \sum_{n=1}^{\infty} 1 \cdot z^n + \sum_{n=0}^{\infty} 2^{n-1} \cdot z^{-n} \quad |z| < \frac{1}{2}$$



$p=1$   
 $p=2$   
 $1 < |z| < 2$

$$X(z) = f(z^{-1}) = \sum_{n=1}^{\infty} 1 \cdot z^{-n} + \sum_{n=0}^{\infty} 2^{n-1} \cdot z^{-n} \quad |z| > 2$$

$$\begin{aligned}
 &= \sum_{n=1}^{\infty} 1 \cdot z^{-n} + \sum_{n=0}^{\infty} 2^{n-1} \cdot z^{-n} \\
 &= \frac{\left(\frac{1}{z}\right)}{1-\left(\frac{1}{z}\right)} + \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)} \\
 &= \frac{1}{z-1} - \frac{1}{z-2}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{z-2-z+1}{(z-1)(z-2)} \\
 &= \frac{-1}{(z-1)(z-2)}
 \end{aligned}$$

$$f(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$f(z^{-1}) = \frac{-0.5z^{-2}}{(z^{-1}-1)(z^{-1}-0.5)} = \frac{-0.5}{(1-z)(1-0.5z)} = \frac{-1}{(z-1)(z-2)} = X(z)$$

$$X(z) = \frac{-0.5z^2}{(z-1)(z-0.5)}$$

