Vector Calculus (H.1)
Curl

20160111

Curl (Circulation Density)

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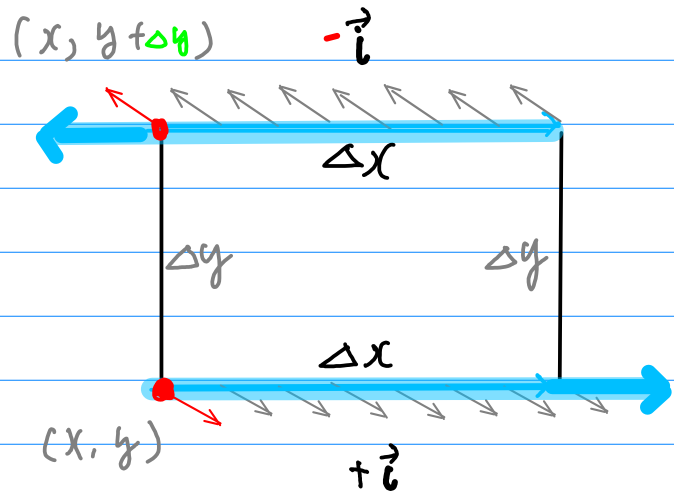
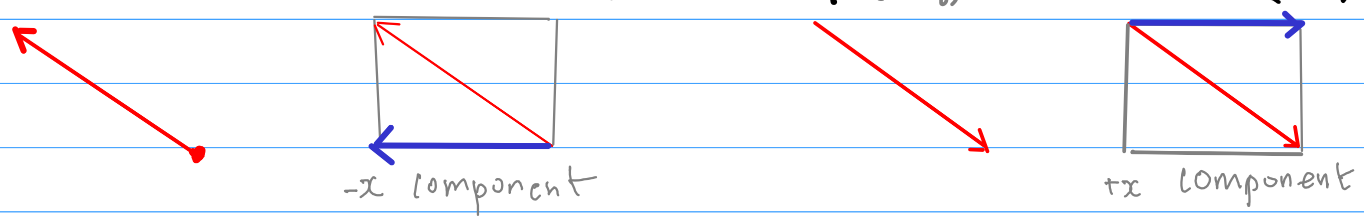
of a vector field $\vec{F} = M\vec{i} + N\vec{j}$

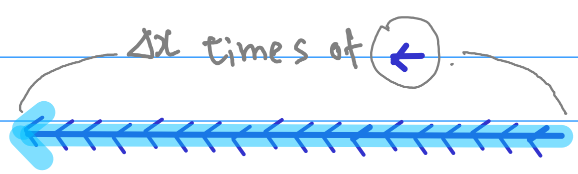
$$\text{curl } \vec{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

$\boxed{\vec{F}(x, y+\Delta y)\cdot(-\vec{i})\,\Delta x}$ $\boxed{\vec{F}(x, y)\cdot(+\vec{i})\,\Delta x}$

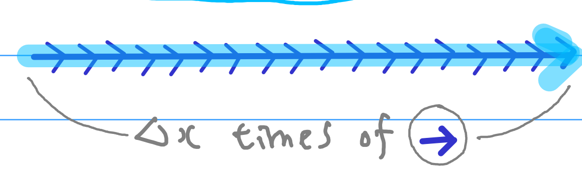


$\vec{F}(x, y+\Delta y)$ $\vec{F}(x, y+\Delta y)\cdot(-\vec{i})$ $\vec{F}(x, y)$ $\vec{F}(x, y)\cdot(+\vec{i})$

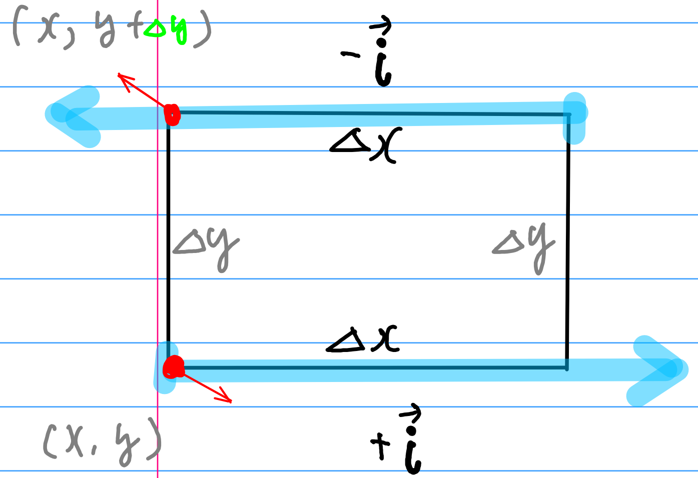




$\vec{F}(x, y+\Delta y)\cdot(-\vec{i})\,\Delta x$ $\vec{F}(x, y)\cdot(+\vec{i})\,\Delta x$



$\boxed{\text{Consider the } +\vec{i} \text{ component of } \vec{F} \text{ only} \Rightarrow M(x,y)}$



$$\vec{F}(x, y+\Delta y) \cdot (-\vec{i})\,\Delta x$$

$$\Rightarrow -M(x, y+\Delta y)\,\Delta x$$

$$\Rightarrow M(x, y)\,\Delta x$$

$$\vec{F}(x, y) \cdot (+\vec{i})\,\Delta x$$

$$\vec{F}(x, y+\Delta y) \cdot (-\vec{i})\,\Delta x + \vec{F}(x, y) \cdot (+\vec{i})\,\Delta x$$

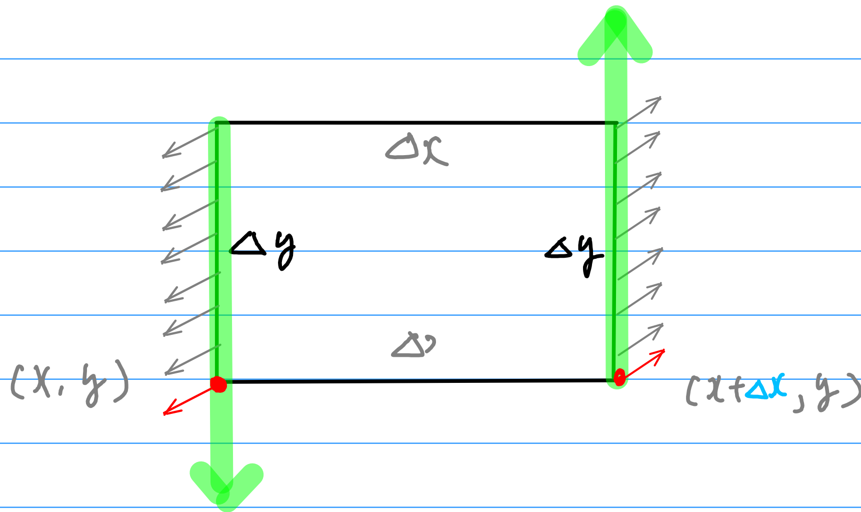
$$\Rightarrow -M(x, y+\Delta y)\,\Delta x + M(x, y)\,\Delta x$$

$$\Rightarrow -\big(M(x, y+\Delta y) - M(x, y)\big)\,\Delta x$$

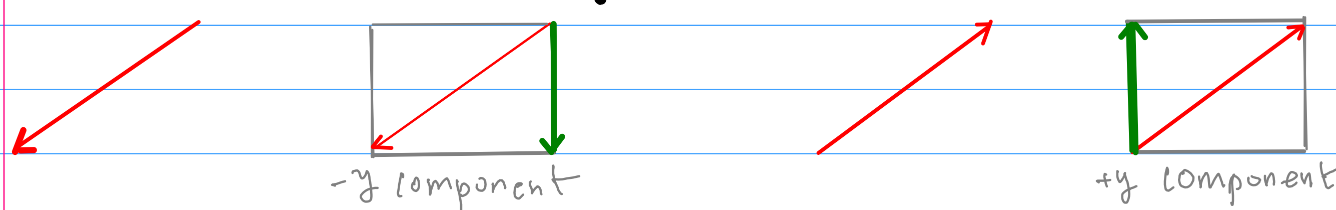
$$\Rightarrow \boxed{-\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x}$$

$$\boxed{\frac{\partial M}{\partial y} = \lim_{\Delta y \to 0} \frac{M(x, y+\Delta y) - M(x, y)}{\Delta y}}$$

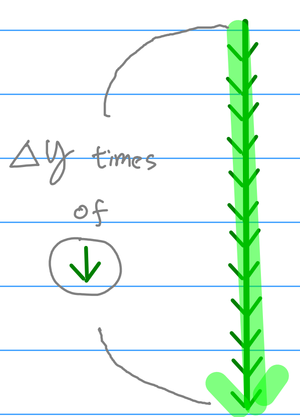
$\boxed{\vec{F}(x,y)\cdot(-\vec{j})\,\Delta y}$ $\boxed{\vec{F}(x+\Delta x, y)\cdot(+\vec{j})\,\Delta y}$

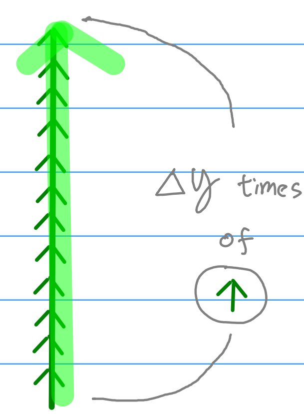


$\vec{F}(x,y)$ $\vec{F}(x,y)\cdot(-\vec{j})$ $\vec{F}(x+\Delta x, y)$ $\vec{F}(x+\Delta x, y)\cdot(+\vec{j})$



$\vec{F}(x,y)\cdot(-\vec{j})\,\Delta y$ $\vec{F}(x+\Delta x, y)\cdot(+\vec{j})\,\Delta y$

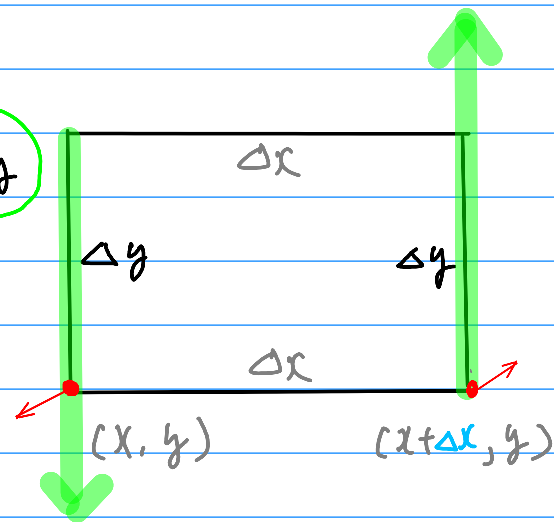




Consider the $+\vec{j}$ component of $\vec{F}$ only $\Rightarrow$ $N(x,y)$

$\vec{F}(x,y)\cdot(-\vec{j})\,\Delta y$

$-N(x,y)\,\Delta y$



$\vec{F}(x+\Delta x,y)\cdot(+\vec{j})\,\Delta y$

$N(x+\Delta x,y)\,\Delta y$

$$\vec{F}(x+\Delta x,y)\cdot(+\vec{j})\,\Delta y + \vec{F}(x,y)\cdot(-\vec{j})\,\Delta y$$

$$\Rightarrow N(x+\Delta x,y)\,\Delta y - N(x,y)\,\Delta y$$

$$\Rightarrow \big(N(x+\Delta x,y) - N(x,y)\big)\,\Delta y$$

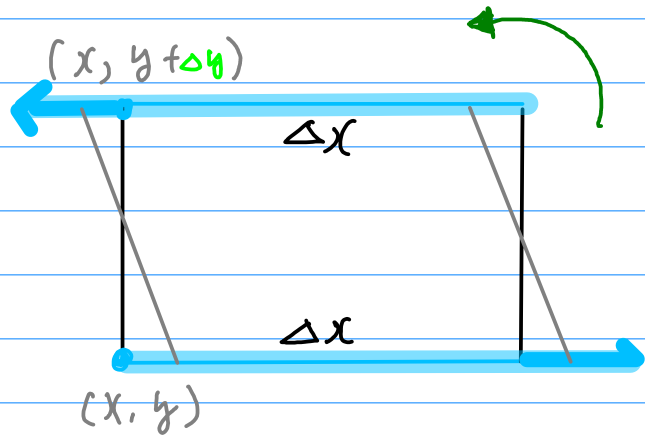
$$\Rightarrow \boxed{\left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y}$$

$$\boxed{\frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \frac{N(x+\Delta x,y) - N(x,y)}{\Delta x}}$$

Circulation Density along $\vec{i}$ axis

$$\vec{F}(x, y+\Delta y) \cdot (-\vec{i})\, \Delta x$$

the $-\vec{i}$ direction component of $\vec{F}$ multiplied by Δx



$$\vec{F}(x, y) \cdot (+\vec{i})\, \Delta x$$

the $+\vec{i}$ direction component of $\vec{F}$ multiplied by Δx

$$\vec{F}(x, y+\Delta y) \cdot (-\vec{i})\, \Delta x + \vec{F}(x, y) \cdot (+\vec{i})\, \Delta x$$

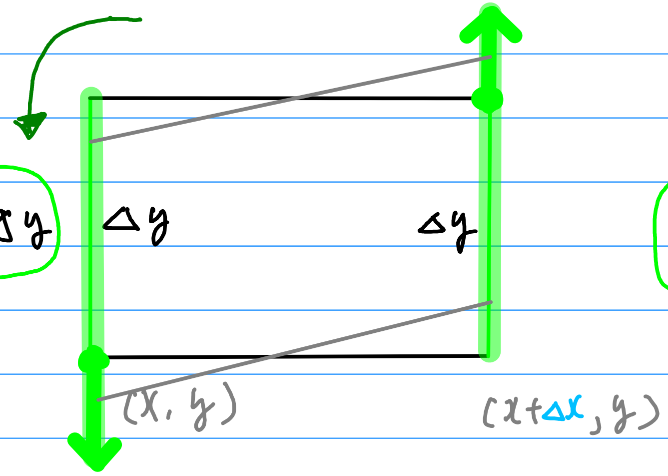
$$\Rightarrow \boxed{-\left(\frac{\partial M}{\partial y} \Delta y\right) \Delta x}$$

$$\vec{F}(x, y) = M(x, y)\,\vec{i} + N(x, y)\,\vec{j}$$

Velocity field of a fluid flow in a plane

Circulation Density along $\vec{j}$ axis

$\vec{F}(x,y)\cdot(-\vec{j})\,\Delta y$



$\vec{F}(x+\Delta x, y)\cdot(+\vec{j})\,\Delta y$

the $-\vec{j}$ direction component of $\vec{F}$ multiplied by Δy

the $-\vec{j}$ direction component of $\vec{F}$ multiplied by Δy

$$\vec{F}(x+\Delta x, y)\cdot(+\vec{j})\,\Delta y + \vec{F}(x,y)\cdot(-\vec{j})\,\Delta y$$

$$\Rightarrow \boxed{\left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y}$$

$$\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}$$

Velocity field of a fluid flow in a plane

$$\leftarrow \quad \vec{F}(x, y+\Delta y)\cdot(-\vec{i})\,\Delta x = -M(x, y+\Delta y)\,\Delta x$$

$$\rightarrow \quad \vec{F}(x, y)\cdot(+\vec{i})\,\Delta x = +M(x, y)\,\Delta x$$

scalar

$$\uparrow \quad \vec{F}(x+\Delta x, y)\cdot(+\vec{j})\,\Delta y = +N(x+\Delta x, y)\,\Delta y$$

$$\downarrow \quad \vec{F}(x, y)\cdot(-\vec{j})\,\Delta y = -N(x, y)\,\Delta y$$

$$-\Big(M(x, y+\Delta y) - M(x, y)\Big)\Delta x \approx \left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

$$\Big(N(x+\Delta x, y) - N(x, y)\Big)\Delta y \approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$$

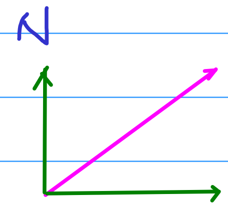
$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\underbrace{\Delta x\,\Delta y}_{R}$$

$$\frac{\text{Circulation around rectangle}}{\text{rectangle area}} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

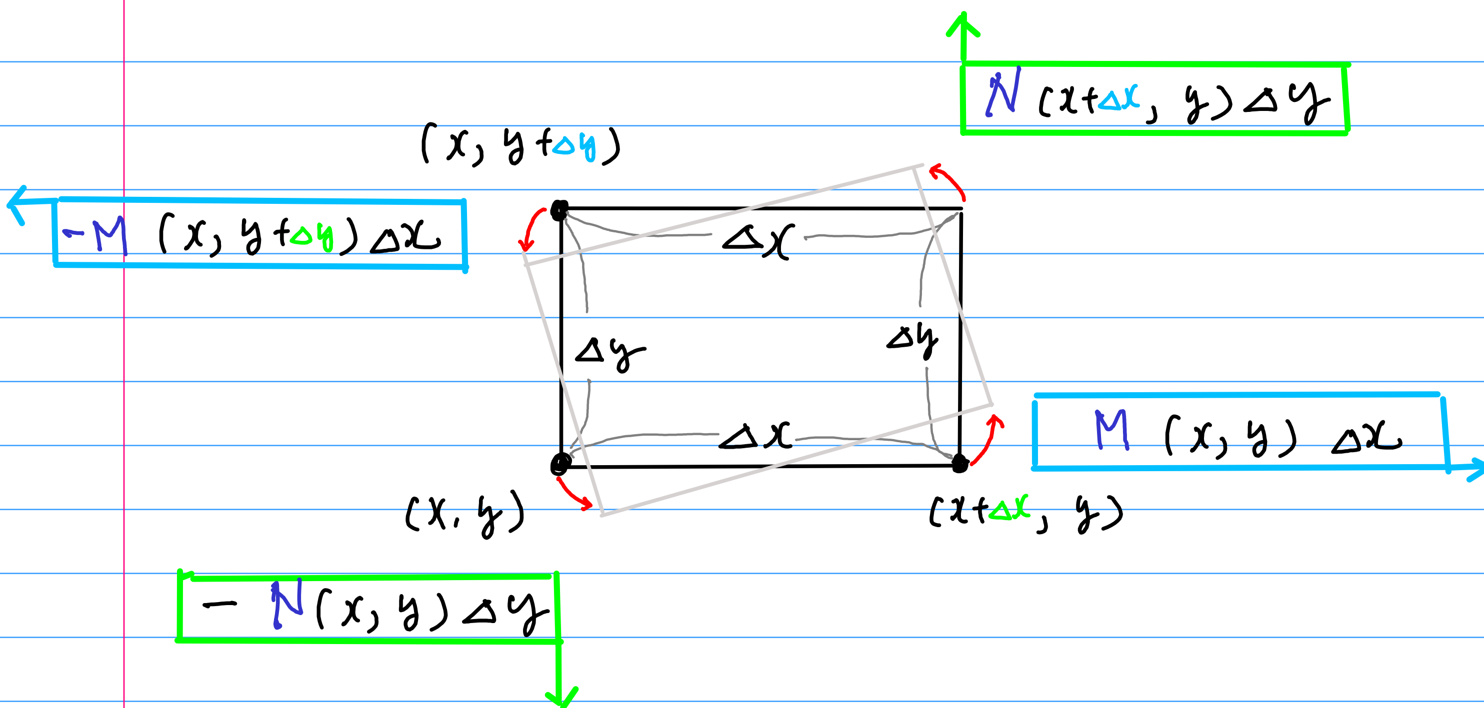
Curl (Circulation Density)

of a vector field $\vec{F} = M\vec{i} + N\vec{j}$

$$\operatorname{curl}\vec{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$



$$\vec{F} = M\vec{i} + N\vec{j} \Longrightarrow \text{curl } \vec{F} = \underbrace{\underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\text{circulation density}} \vec{k}}_{\text{curl}}$$

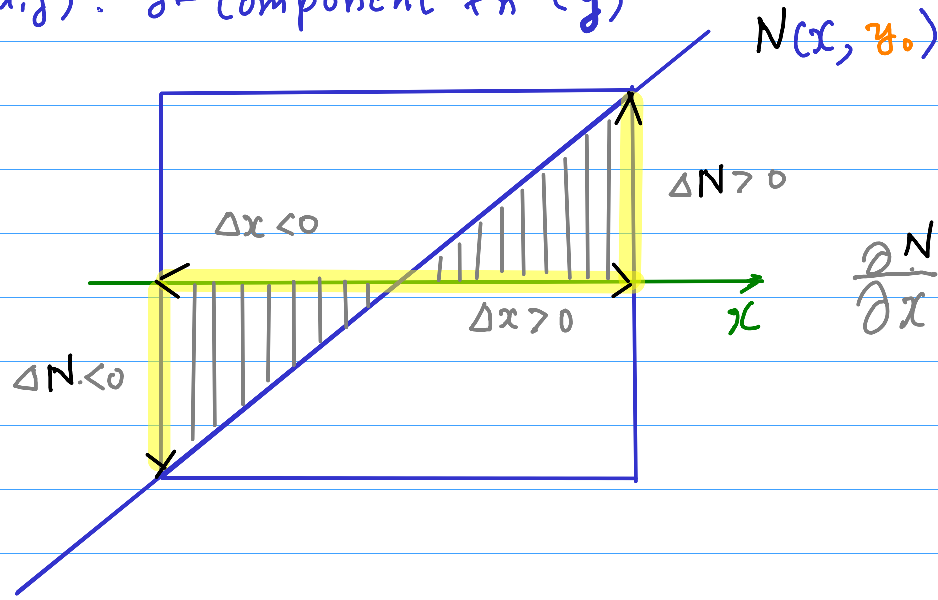


$$-\left[M(x, y+\Delta y) - M(x,y)\right]\Delta x \approx -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

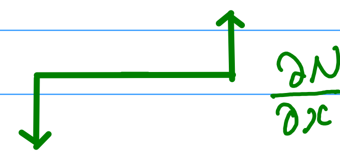
$$\left[N(x+\Delta x, y) - N(x,y)\right]\Delta y \approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$$

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y$$

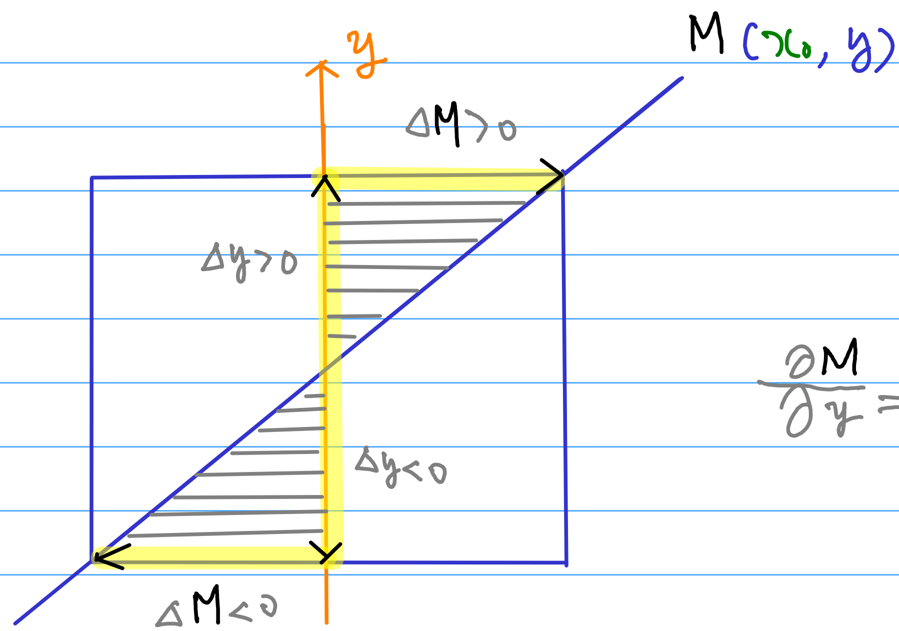
⑥ $N(x,y)$: y- Component fn $(\vec{j})$



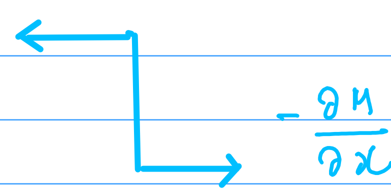
$$\frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x}$$



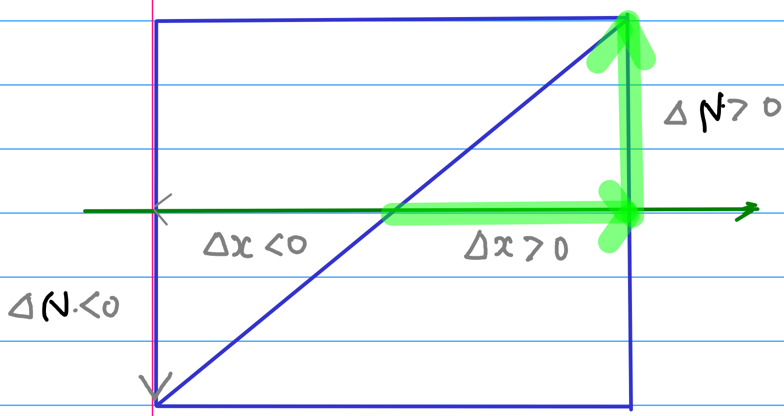
⑥ $M(x,y)$: x component fn $(\vec{i})$



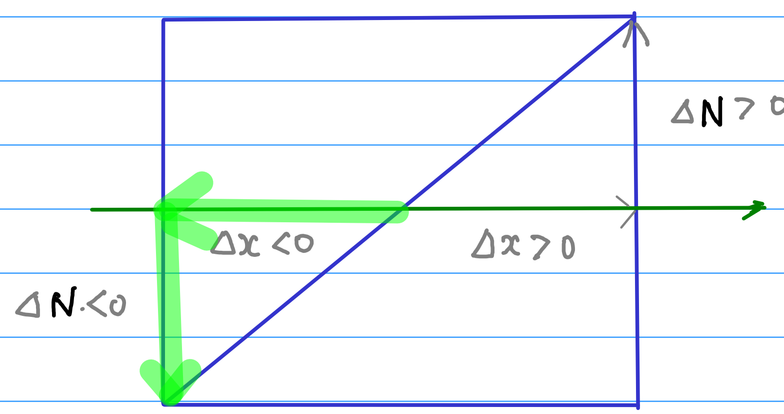
$$\frac{\partial M}{\partial y} = \lim_{\Delta x \to 0} \frac{\Delta M}{\Delta y}$$



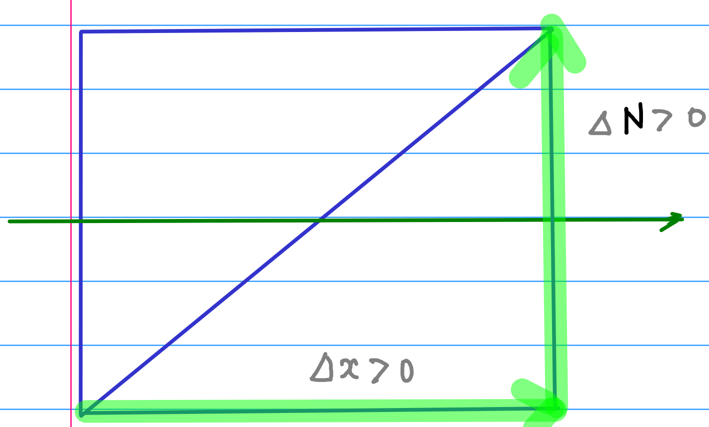
All the same $\frac{\partial N}{\partial x}$



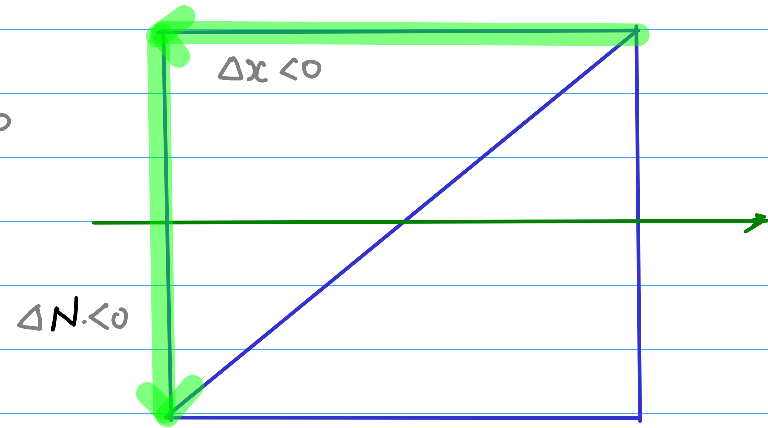
$$\frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x}$$



$$\frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x}$$

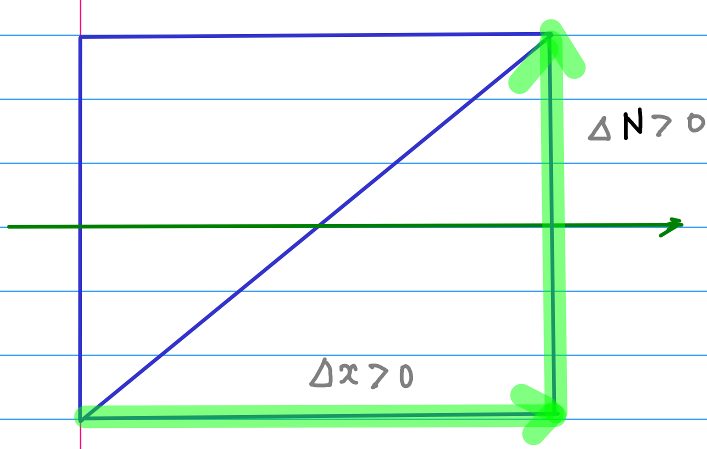


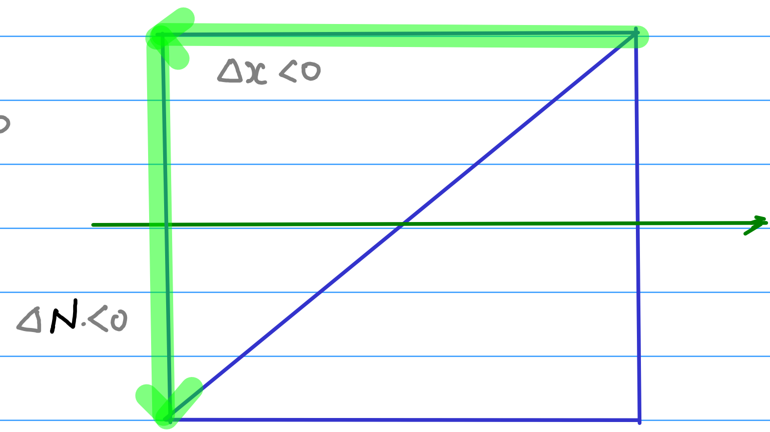
$$\frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x}$$



$$\frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x}$$

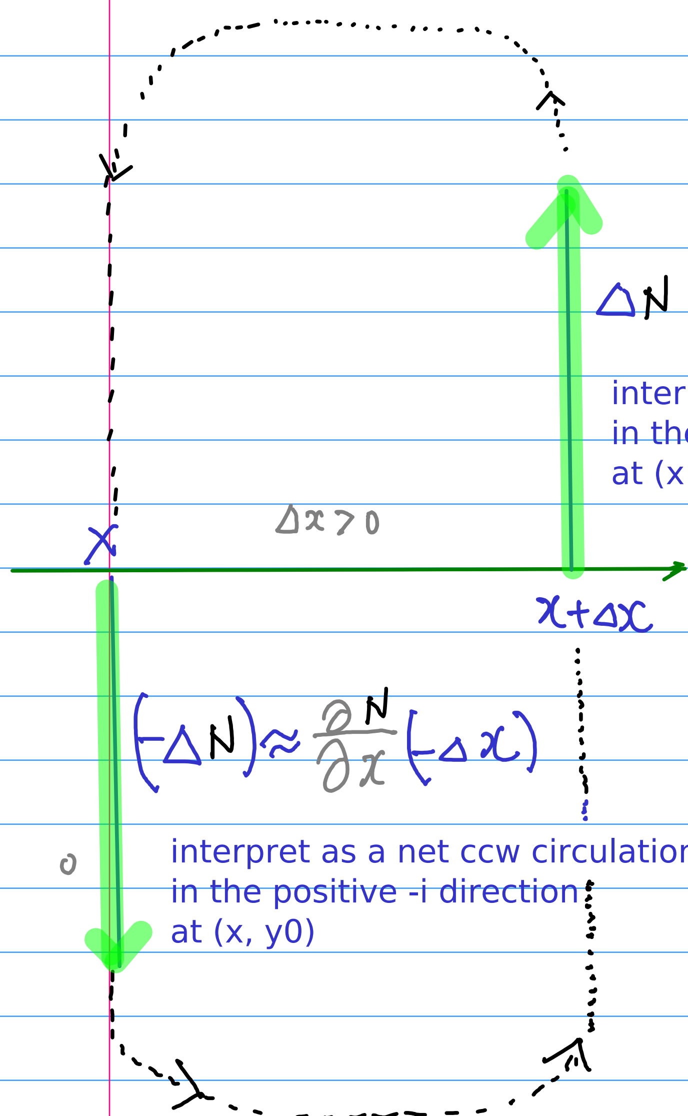
Circulation Interpretation





$$\frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x}$$

$$\frac{\partial N}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x}$$



$$\Delta N \approx \frac{\partial N}{\partial x} \Delta x$$

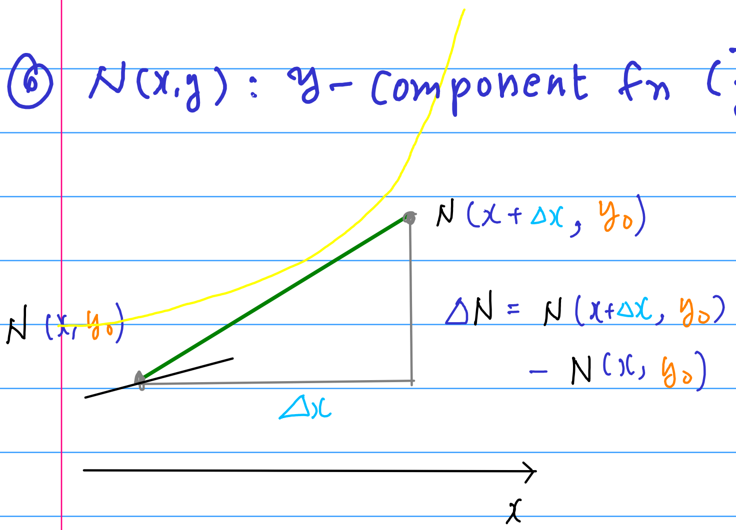
interpret as a net ccw circulation
in the positive +i direction
at (x+dx, y0)

$$(-\Delta N) \approx \frac{\partial N}{\partial x} (-\Delta x)$$

interpret as a net ccw circulation
in the positive -i direction
at (x, y0)

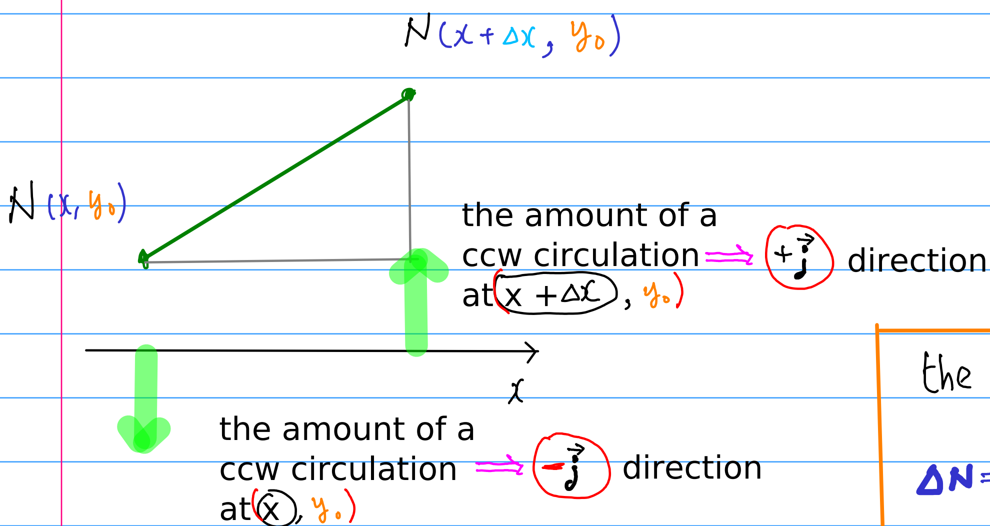
ΔN(x, y) Interpretation

⑥ $N(x,y)$: y-component fn ($\vec{j}$)



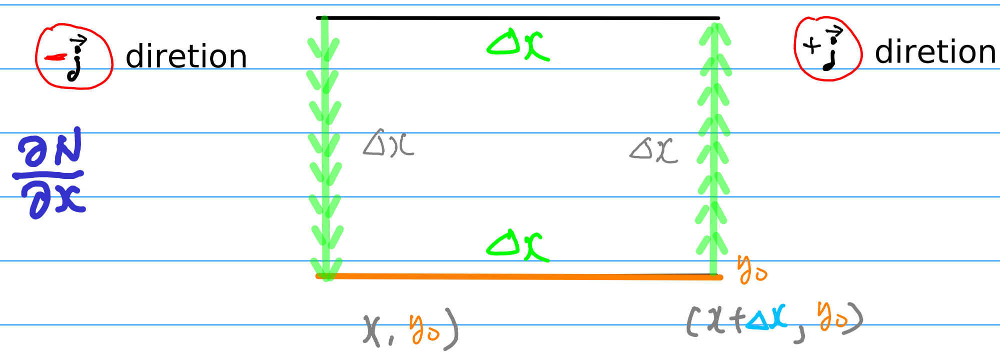
the slope of a tangent:

$$\lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x} = \frac{\partial N}{\partial x}$$



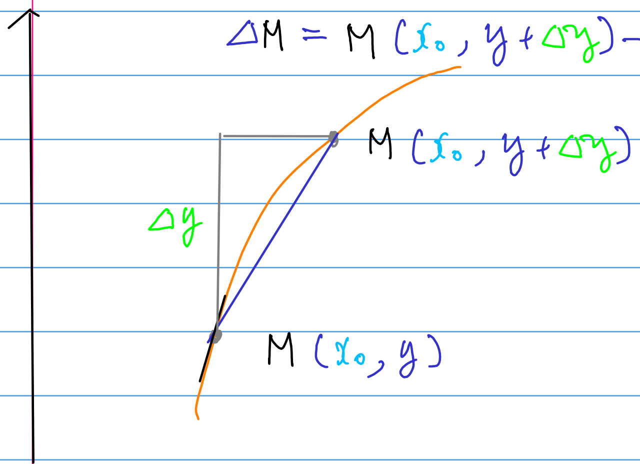
the ccw circulation:

$$\Delta N = N(x+\Delta x, y_0) - N(x, y_0)$$



$\Delta M(x, y)$ Interpretation

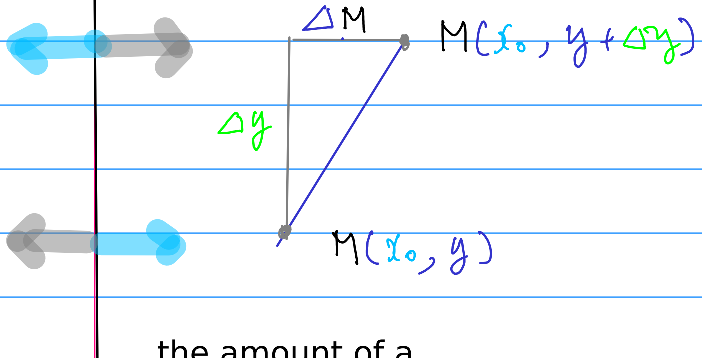
$$\Delta M = M(x_0, y+\Delta y) - M(x_0, y)$$



the slope of a tangent :

$$\lim_{\Delta y \to 0} \frac{\Delta M}{\Delta y} = \frac{\partial M}{\partial y}$$

the amount of a ccw circulation at $(x_0, y+\Delta y)$ $\Rightarrow$ $-\vec{j}$ direction

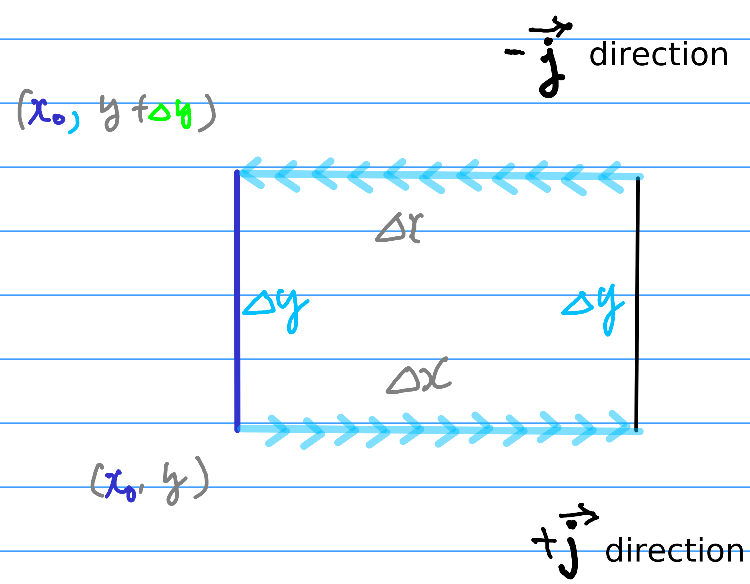


the ccw circulation :

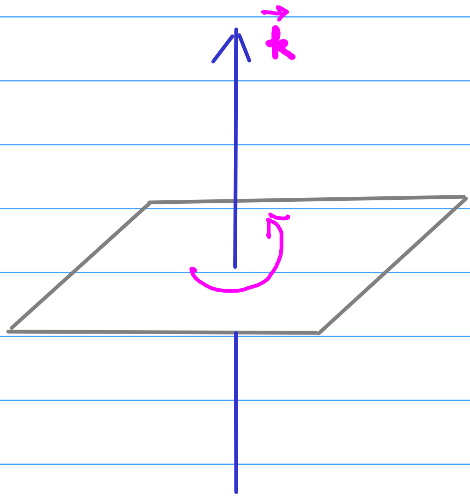
$$\Delta M = M(x_0, y+\Delta y) - M(x_0, y)$$

the amount of a ccw circulation at (x_0, y) $\Rightarrow$ $+\vec{j}$ direction

$$-\frac{\partial M}{\partial y}$$



Curl



$$\text{curl}\ \vec{F} = \underbrace{\underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\text{circulation density}} \vec{k}}_{\text{curl}}$$

$$\text{Curl}\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix} = \nabla \times \vec{F}$$

$$\boxed{\vec{F}(x,y) = M(x,y)\,\vec{i} + N(x,y)\,\vec{j}}$$

Velocity field of a fluid flow in a plane

$$f(x, y) = x^2 + xy + y^2$$

$$\frac{\partial f}{\partial x} = 2x + y = M(x, y)$$

$$\frac{\partial f}{\partial y} = x + 2y = N(x, y)$$

$$df = (2x + y)\,dx + (x + 2y)\,dy$$
$$= M(x, y)\,dx + N(x, y)\,dy$$

total differential

$$\left(\frac{\partial M}{\partial y} = 2 = \frac{\partial N}{\partial y} \right)$$

$$\vec{F}(x, y) = M(x, y)\,\vec{i} + N(x, y)\,\vec{j}$$

gradient field

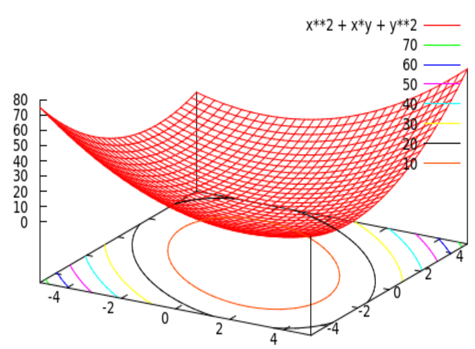
conservative field

$$\left(\frac{\partial M}{\partial x} = 2 = \frac{\partial N}{\partial x} \right)$$

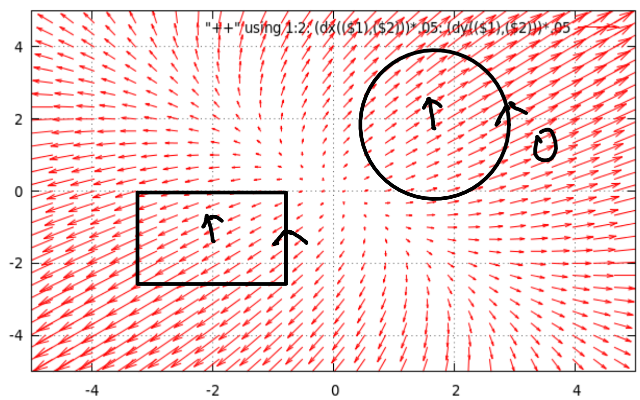
$$\text{curl } \vec{F} = \nabla \times \vec{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

not rotating

$$f(x, y) = x^2 + xy + y^2$$



$$\vec{F}(x, y) = M(x, y)\,\vec{i} + N(x, y)\,\vec{j}$$



$f(x, y)$ ✗

$$\frac{\partial f}{\partial x} = -y = M(x, y)$$

$$\frac{\partial f}{\partial y} = x = N(x, y)$$

no such f

$$\cancel{df} = (2x + y)\,dx + (x + 2y)\,dy$$
$$= M(x, y)\,dx + N(x, y)\,dy$$

total differential ✗

$$\left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$$

$$\vec{F}(x, y) = M(x, y)\,\vec{i} + N(x, y)\,\vec{j}$$

gradient field

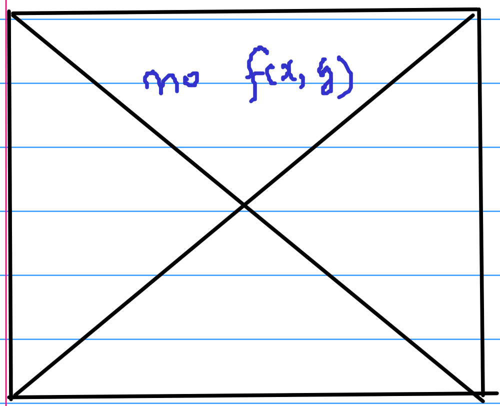
conservative field ✗

$$\left(\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right)$$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2$$

ccw rotating

$f(x, y) = x^2 + xy + y^2$



$$\vec{F}(x, y) = M(x, y)\,\vec{i} + N(x, y)\,\vec{j}$$