

CORDIC Background (4A)

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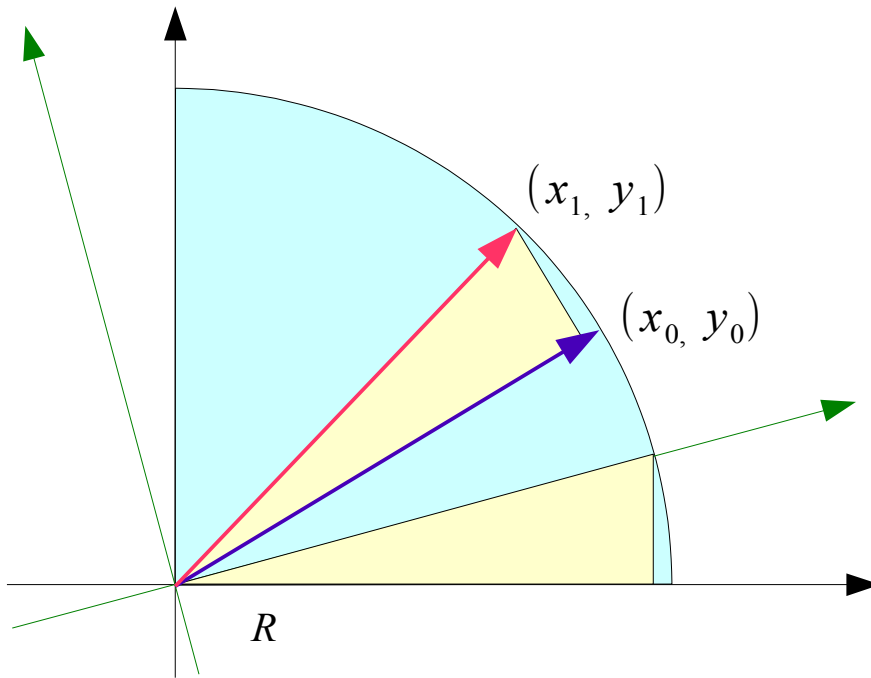
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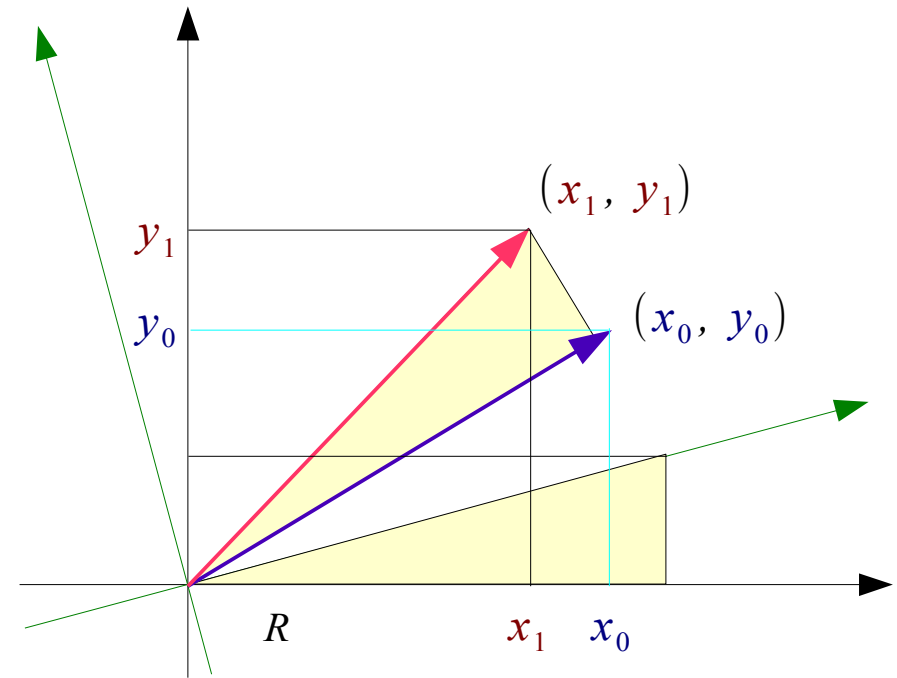
CORDIC Background

J. P. Deschamps, G. A. Bioul, G.D. Sutter, Synthesis of Arithmetic Circuits

Vector Rotation (1)

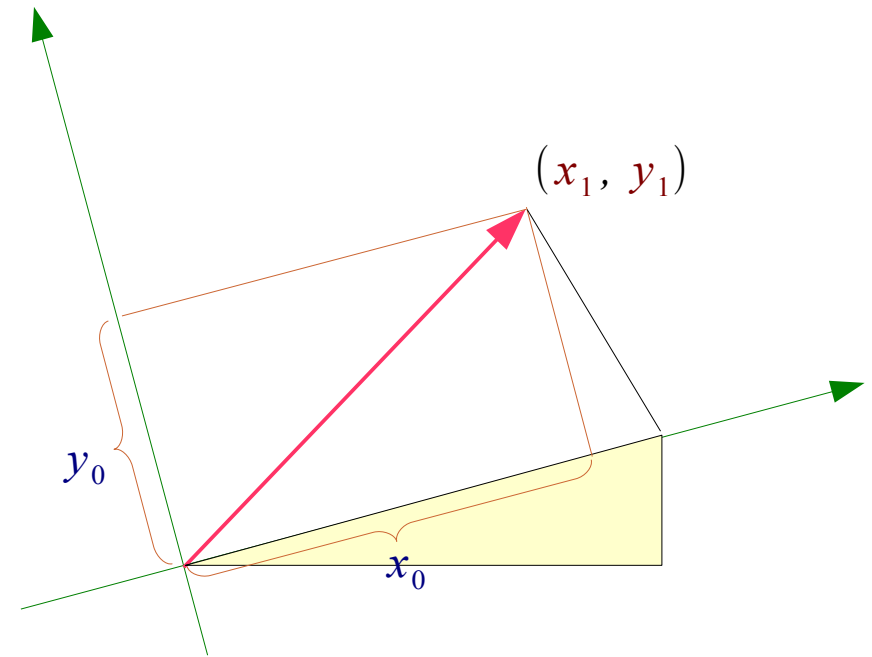
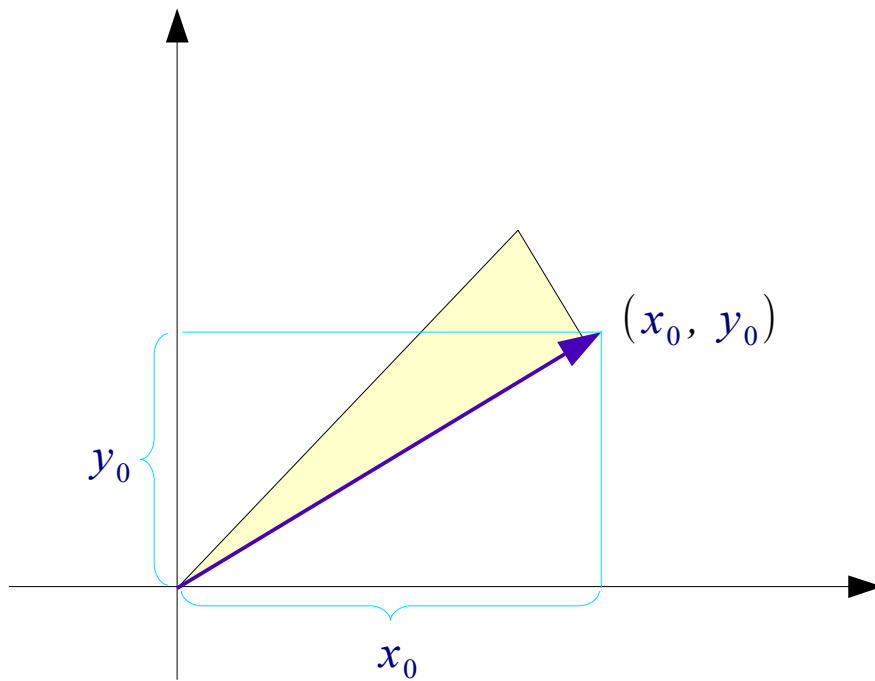


$$y_1 = x_0 \sin \alpha + y_0 \cos \alpha$$

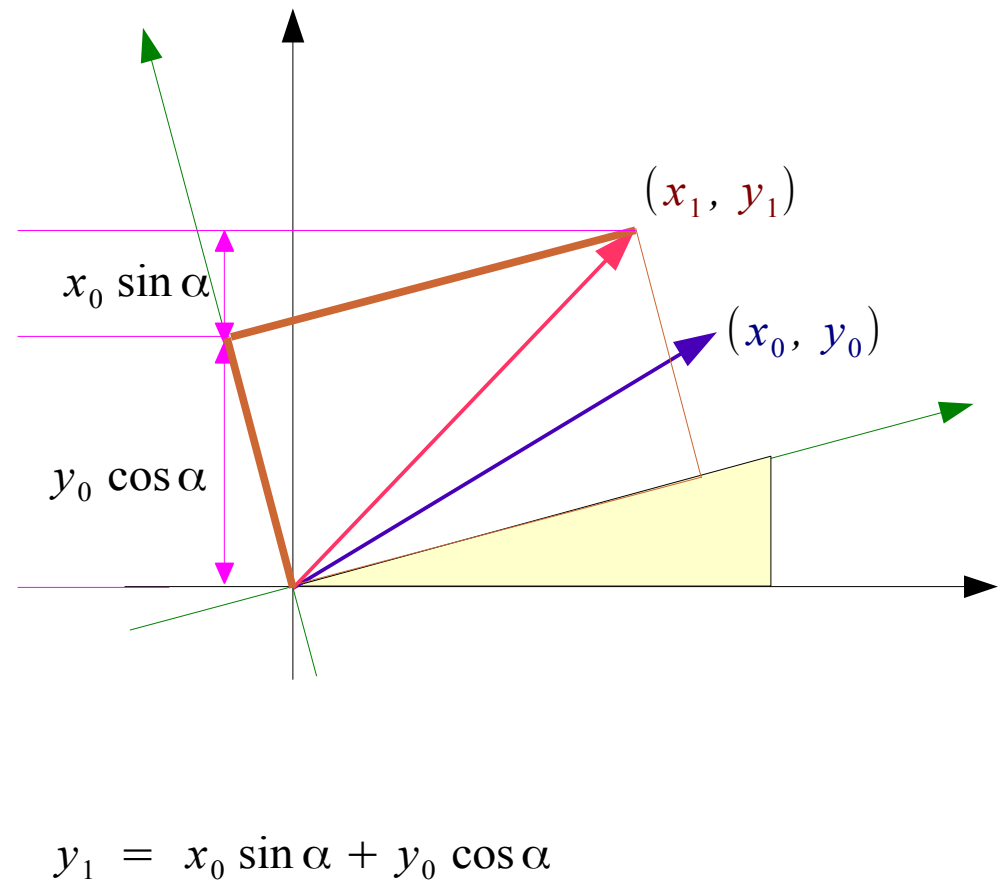
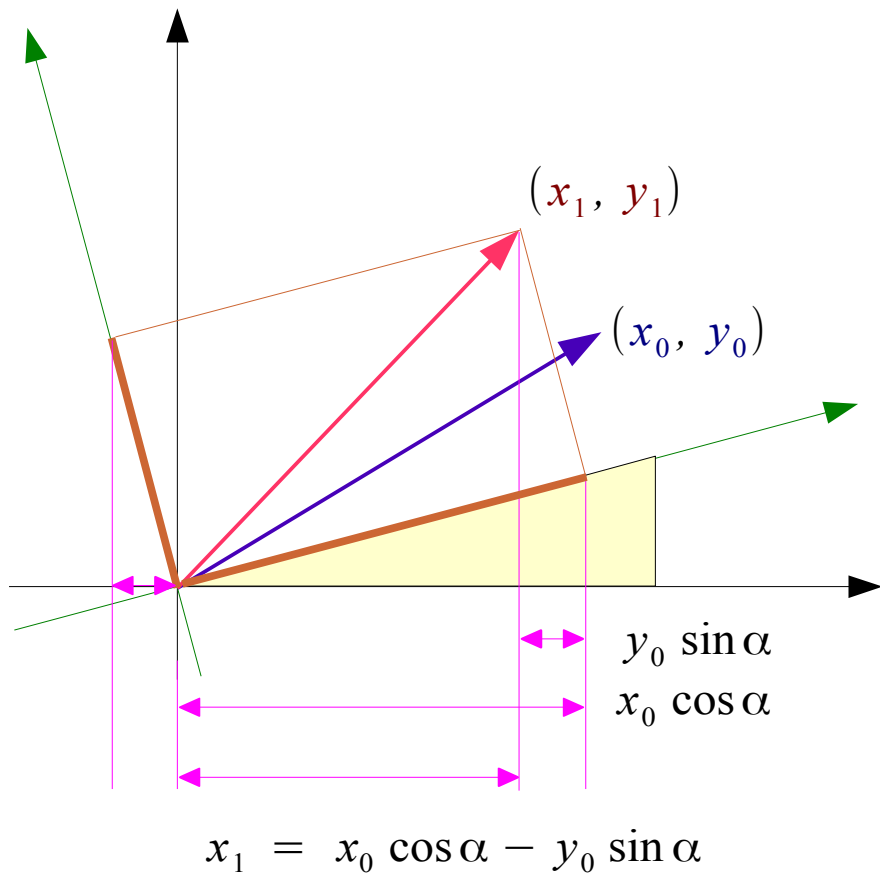


$$x_1 = x_0 \cos \alpha - y_0 \sin \alpha$$

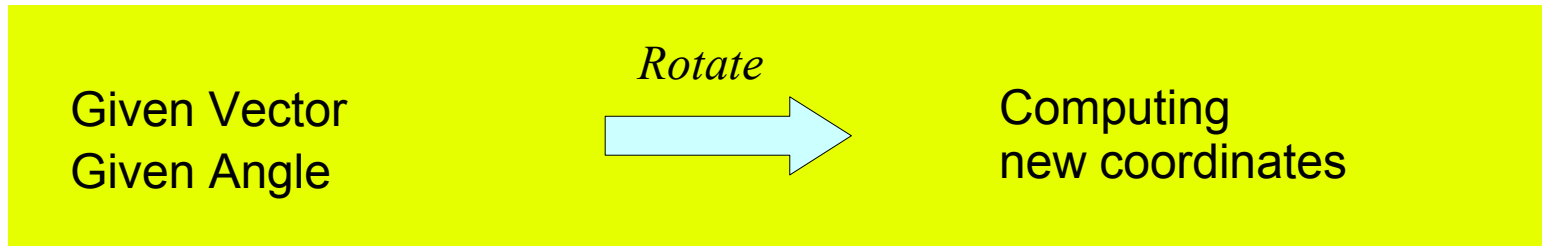
Vector Rotation (2)



Vector Rotation (3)

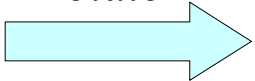


Iterative Rotation



Given Unit Vector
Given Angle α

Rotate




$$x = \cos \alpha$$

$$y = \sin \alpha$$

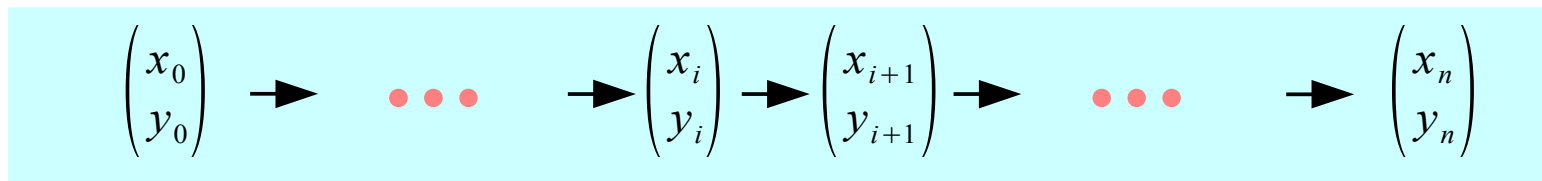
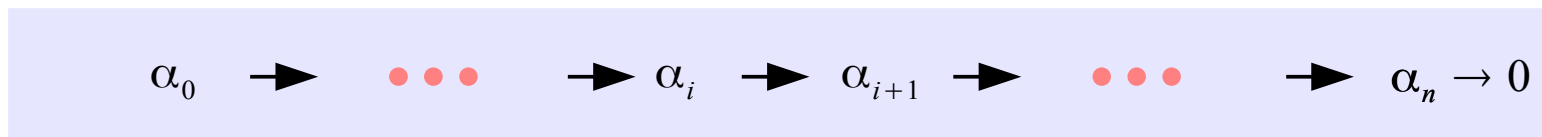
Given Vector (x_0, y_0)
Given Angle α

Rotate

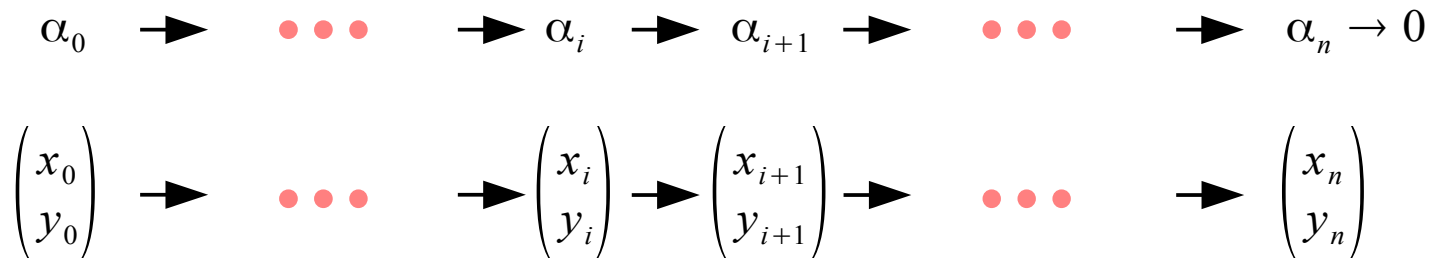


$$x_n = x_0 \cos \alpha - y_0 \sin \alpha$$

$$y_n = x_0 \sin \alpha + y_0 \cos \alpha$$



CORDIC Rotation



$$\begin{cases} x_{i+1} = x_i \cos \alpha_i - y_i \sin \alpha_i \\ y_{i+1} = x_i \sin \alpha_i + y_i \cos \alpha_i \end{cases}$$

$$\begin{cases} x_{i+1} = \cos \alpha_i (x_i - y_i \tan \alpha_i) \\ y_{i+1} = \cos \alpha_i (x_i \tan \alpha_i + y_i) \end{cases}$$

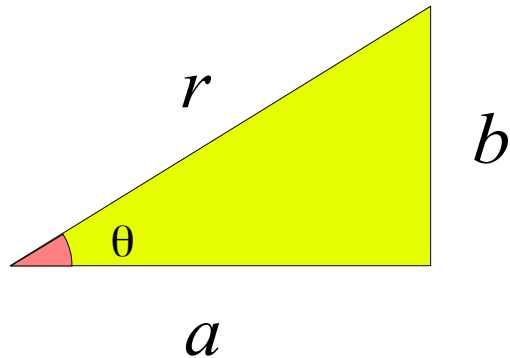
$$\begin{cases} x_{i+1} = \frac{1}{\sqrt{1 + \tan^2 \alpha_i}} (x_i - y_i \tan \alpha_i) \\ y_{i+1} = \frac{1}{\sqrt{1 + \tan^2 \alpha_i}} (x_i \tan \alpha_i + y_i) \end{cases}$$



Pseudo-rotation

$$\begin{cases} x'_{i+1} = (x_i - y_i \tan \alpha_i) \\ y'_{i+1} = (x_i \tan \alpha_i + y_i) \end{cases}$$

COS θ



$$\cos \theta = \frac{a}{r}$$

$$\sin \theta = \frac{b}{r}$$

$$\tan \theta = \frac{b}{a}$$

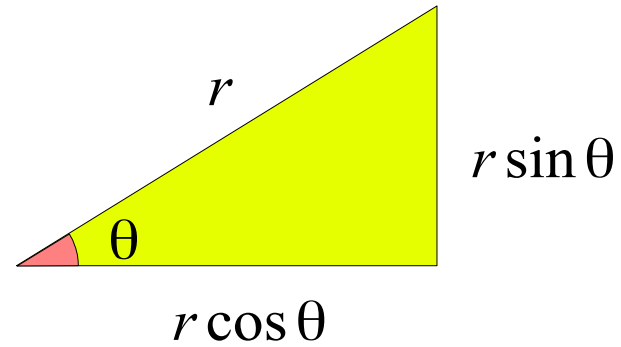
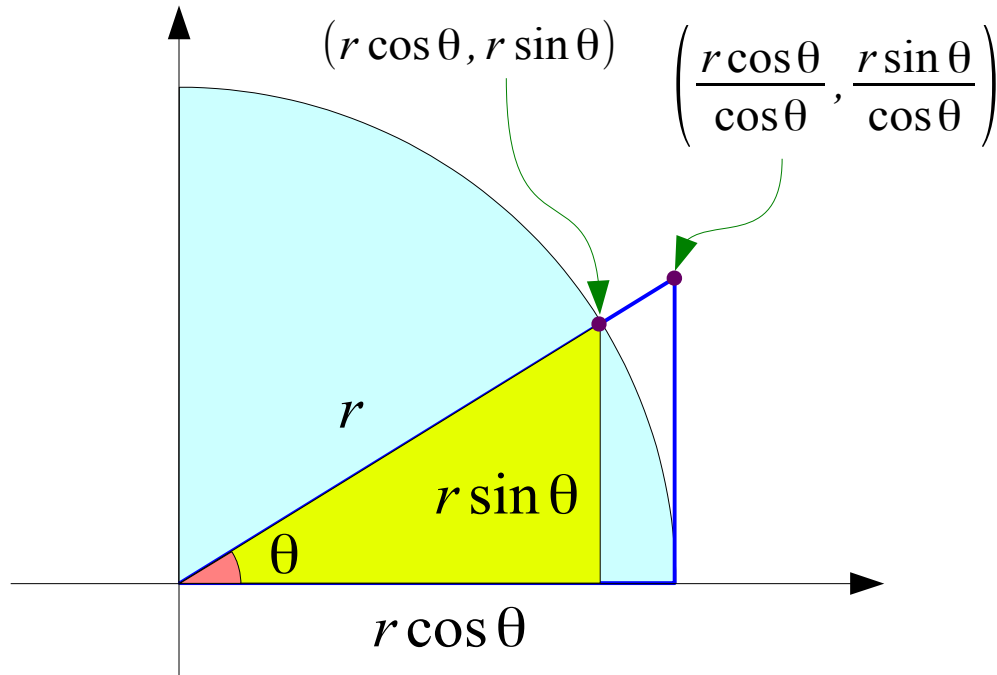
$$r = \sqrt{a^2 + b^2}$$

$$\cos \theta = \frac{a}{r} = \frac{a}{\sqrt{a^2 + b^2}}$$

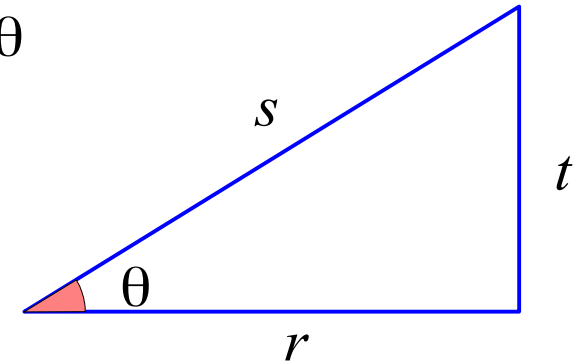
$$= \frac{1}{\sqrt{1 + (b/a)^2}}$$

$$= \frac{1}{\sqrt{1 + \tan^2 \theta}}$$

Pseudo-rotation – factor of $1/\cos \theta$



$r \cos \theta$



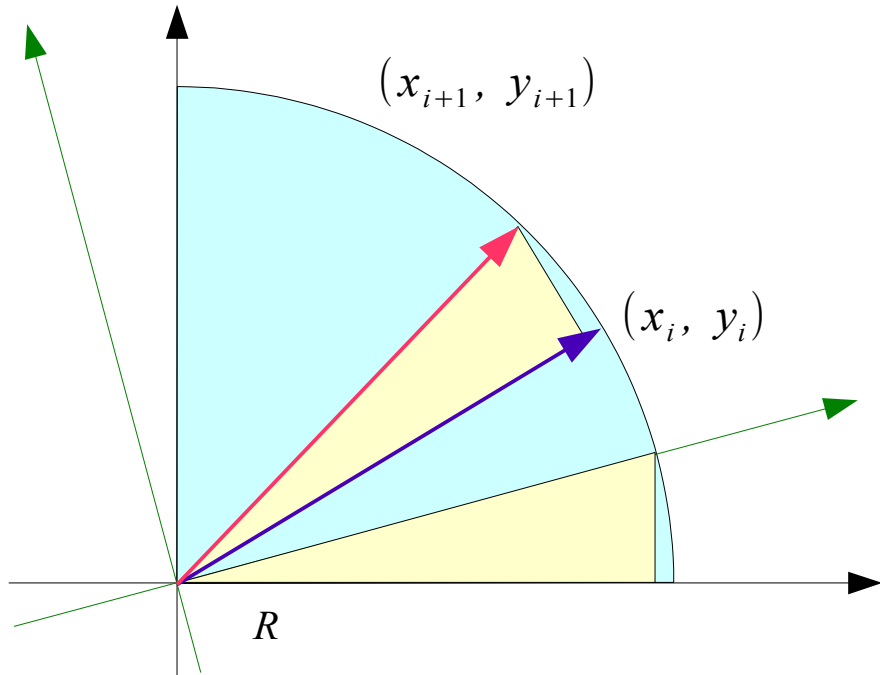
$$r : r \cos \theta = s : r$$

$$r \cos \theta : r \sin \theta = r : t$$

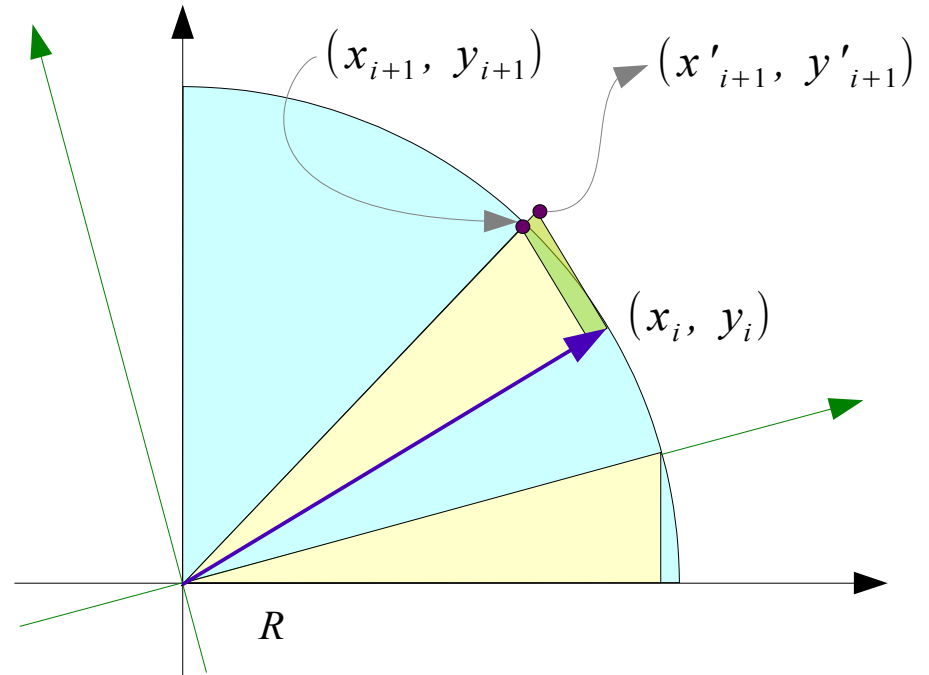
$$s = \frac{r}{\cos \theta}$$

$$s = \frac{r \sin \theta}{\cos \theta}$$

Pseudo-rotation (1)

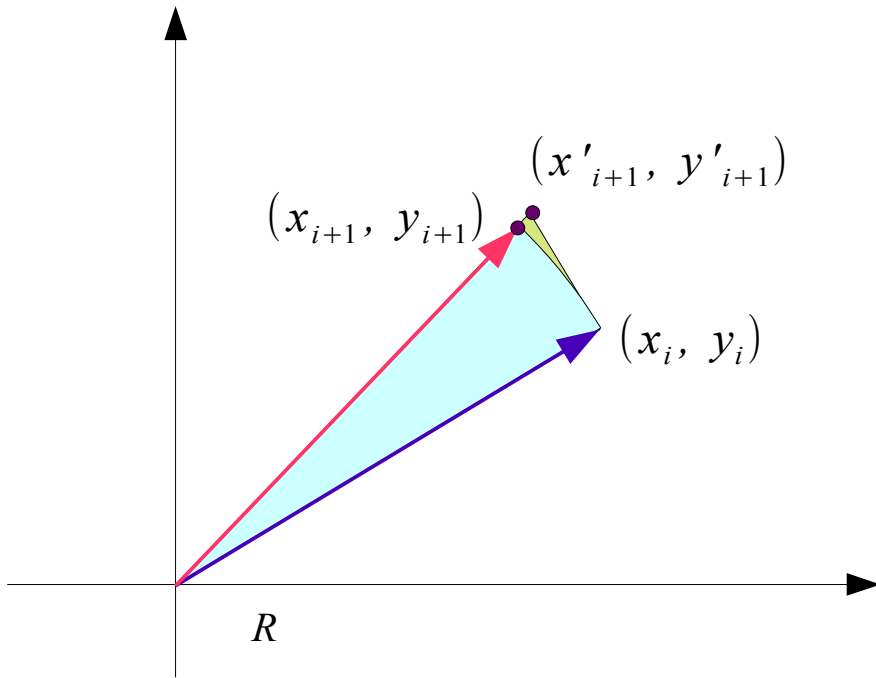


$$\begin{cases} x_{i+1} = \cos \alpha_i (x_i - y_i \tan \alpha_i) \\ y_{i+1} = \cos \alpha_i (x_i \tan \alpha_i + y_i) \end{cases}$$

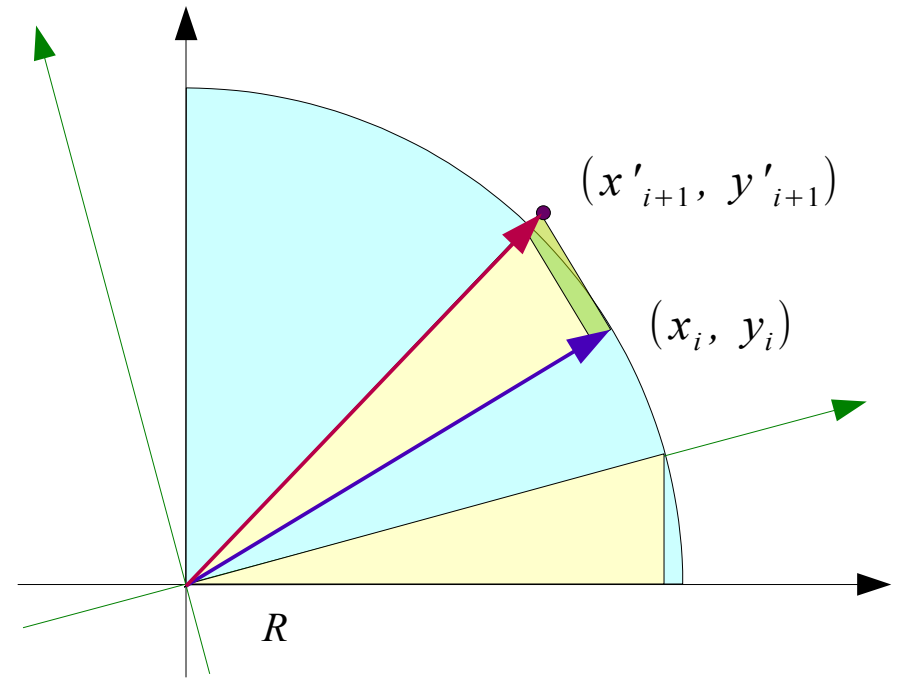


$$\begin{cases} x'_{i+1} = (x_i - y_i \tan \alpha_i) \\ y'_{i+1} = (x_i \tan \alpha_i + y_i) \end{cases}$$

Pseudo-rotation (2)

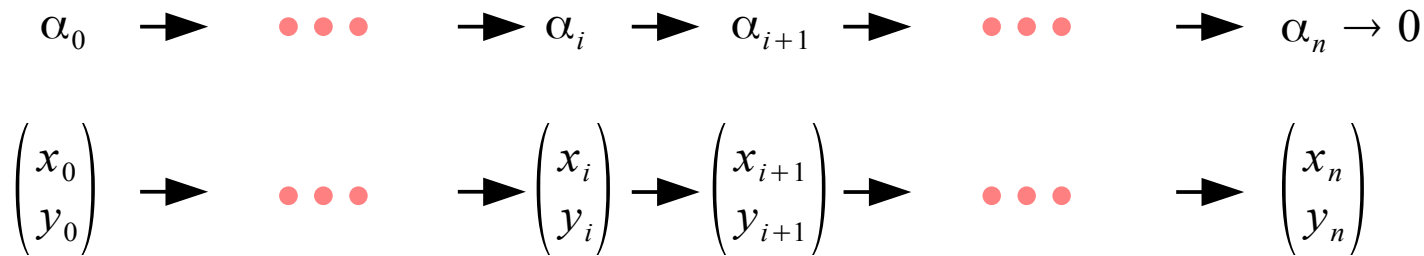


$$\begin{aligned}x'_{i+1} &= x_{i+1} / \cos \alpha_i \\y'_{i+1} &= y_{i+1} / \cos \alpha_i\end{aligned}$$



$$\begin{aligned}x'_{i+1} &> x_{i+1} \\y'_{i+1} &> y_{i+1}\end{aligned}$$

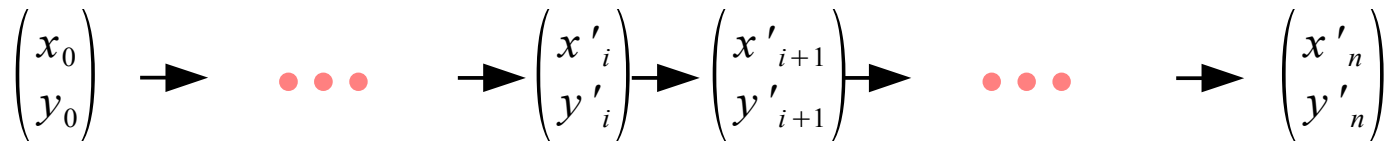
CORDIC Iteration Equations (1)



$$x_{i+1} = x_i \cos \alpha_i - y_i \sin \alpha_i = \cos \alpha_i (x_i - y_i \tan \alpha_i) = (1/\sqrt{1 + \tan^2 \alpha_i}) (x_i - y_i \tan \alpha_i)$$

$$y_{i+1} = x_i \sin \alpha_i + y_i \cos \alpha_i = \cos \alpha_i (x_i \tan \alpha_i + y_i) = (1/\sqrt{1 + \tan^2 \alpha_i}) (x_i \tan \alpha_i + y_i)$$

Pseudo-rotation



$$x'_{i+1} = (x'_i - y'_i \tan \alpha_i) \sqrt{1 + \tan^2 \alpha_i} = (x'_i \cos \alpha_i - y'_i \sin \alpha_i) \sqrt{1 + \tan^2 \alpha_i}$$

$$y'_{i+1} = (x'_i \tan \alpha_i + y'_i) \sqrt{1 + \tan^2 \alpha_i} = (x'_i \sin \alpha_i + y'_i \cos \alpha_i) \sqrt{1 + \tan^2 \alpha_i}$$

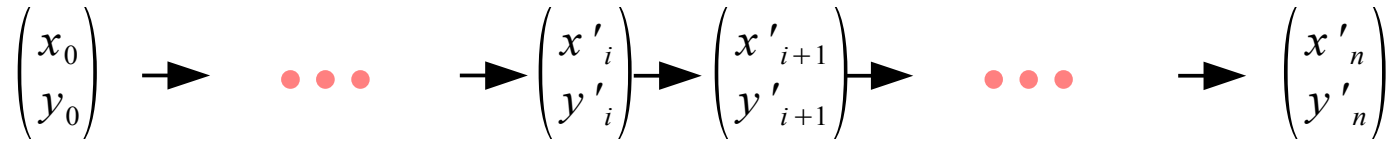
$$x'_n = \{x_0 \cos(\sum \alpha_i) - y_0 \sin(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\}$$

$$y'_n = \{x_0 \sin(\sum \alpha_i) + y_0 \cos(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\}$$

$$\alpha_n = \alpha - \sum \alpha_i$$

CORDIC Iteration Equations (2)

Pseudo-rotation



$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i \tan \alpha_i) = (x'_i \cos \alpha_i - y'_i \sin \alpha_i) \sqrt{1 + \tan^2 \alpha_i} \\ y'_{i+1} &= (x'_i \tan \alpha_i + y'_i) = (x'_i \sin \alpha_i + y'_i \cos \alpha_i) \sqrt{1 + \tan^2 \alpha_i} \end{aligned}$$

$$x'_n = \{x_0 \cos(\sum \alpha_i) - y_0 \sin(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\}$$

$$y'_n = \{x_0 \sin(\sum \alpha_i) + y_0 \cos(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\}$$

$$\alpha_n = \alpha - \sum \alpha_i$$

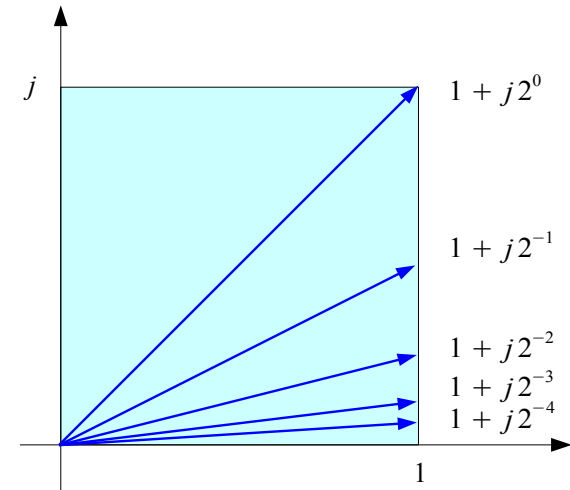
Choose α_i such that $\tan \alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$$\tan \alpha_i = \sigma_i 2^{-i} \quad \sigma_i \in \{+1, -1\}$$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

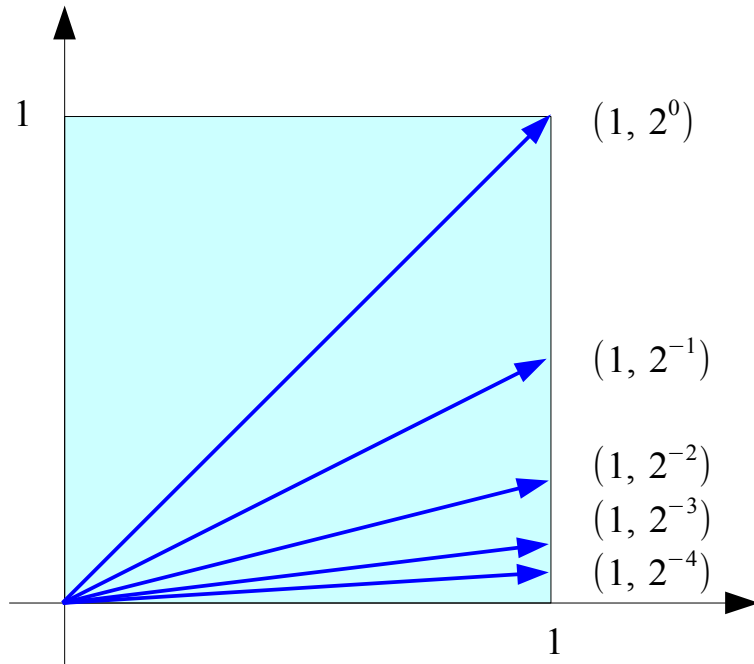
$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan(\sigma_i 2^{-i})$$



CORDIC Iteration Equations (3)

Choose α_i such that $\tan \alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$
 $\tan \alpha_i = \sigma_i 2^{-i} \quad \sigma_i \in \{+1, -1\}$



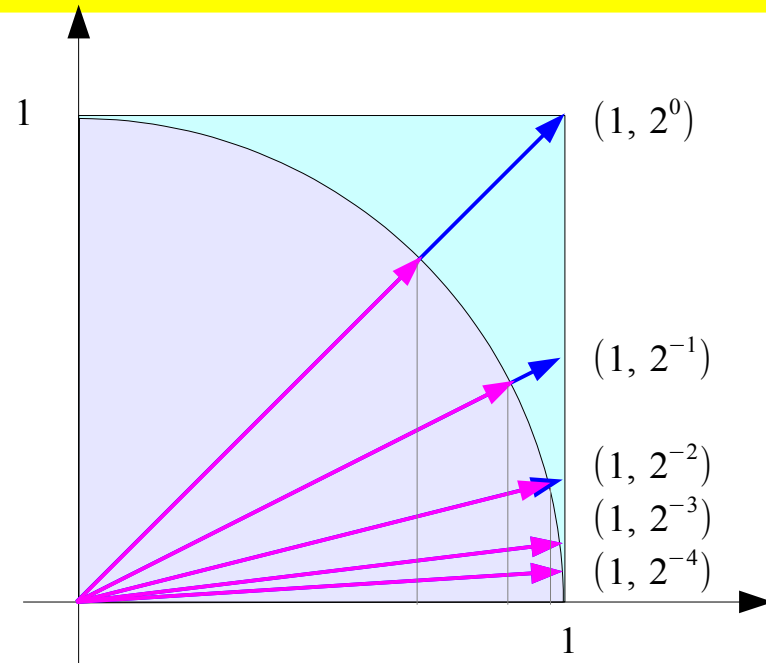
$$\tan \alpha_i = \pm 2^{-i} \quad \cos \alpha_i = \frac{+1}{\sqrt{1 + 2^{-2i}}}$$

$$\sin \alpha_i = \frac{\pm 2^{-i}}{\sqrt{1 + 2^{-2i}}}$$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

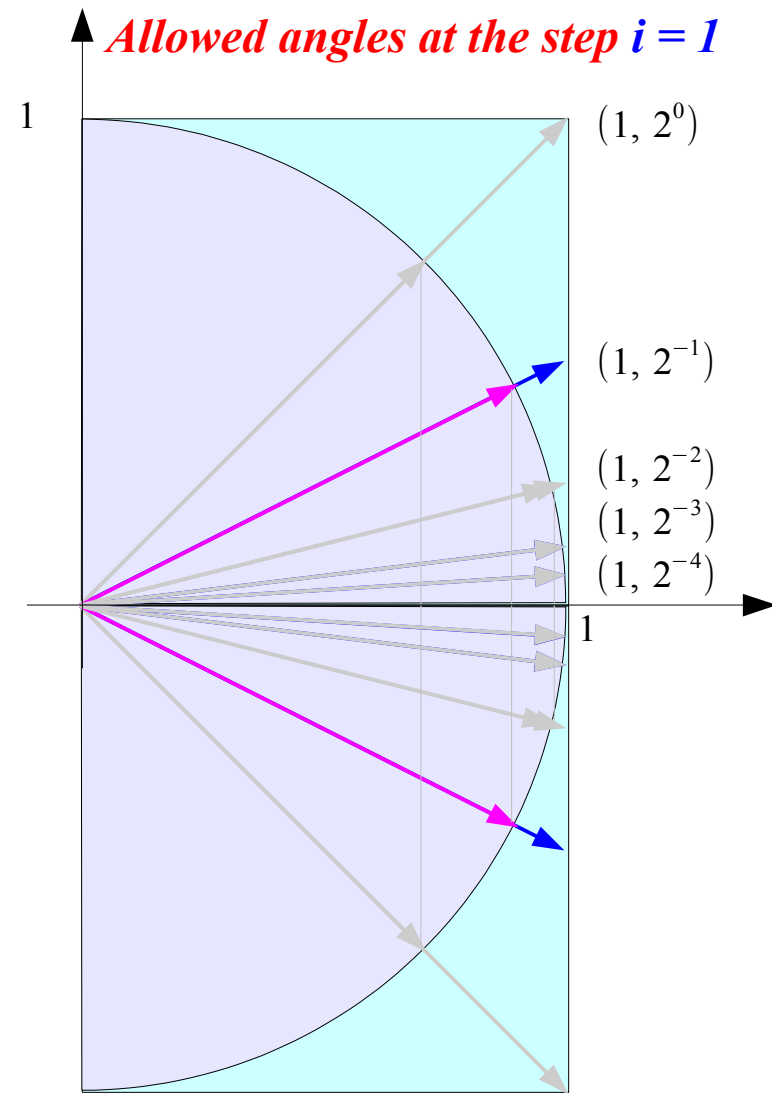
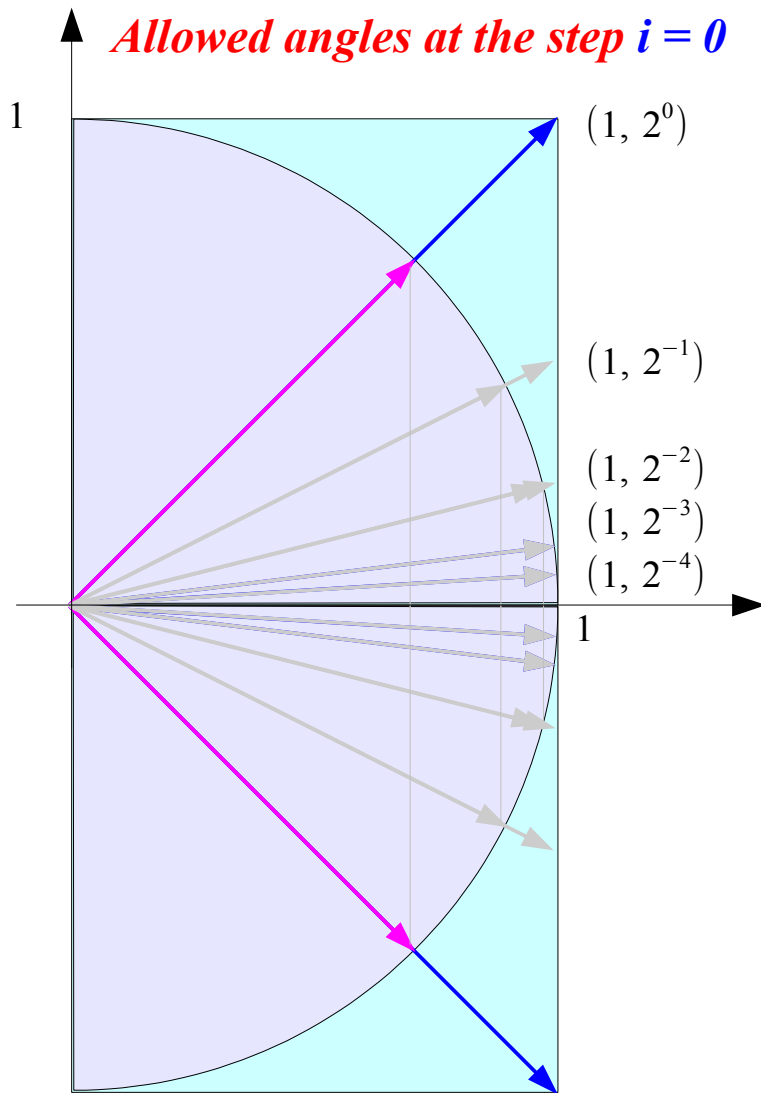
$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan(\sigma_i 2^{-i})$$

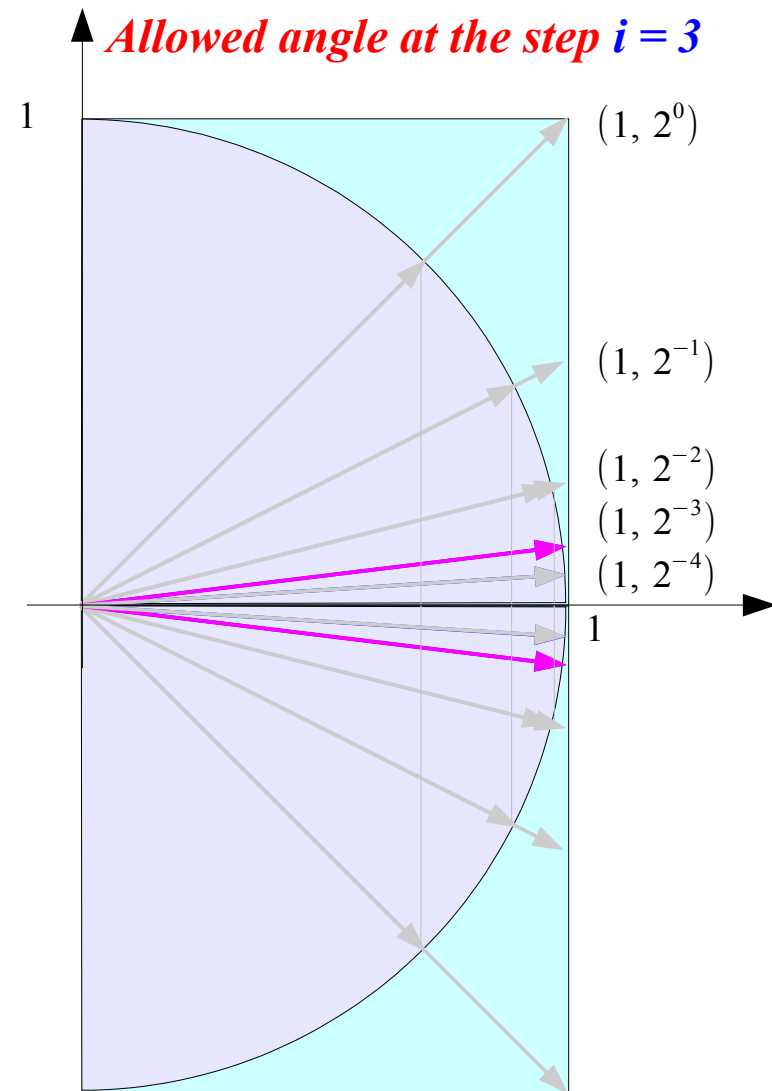
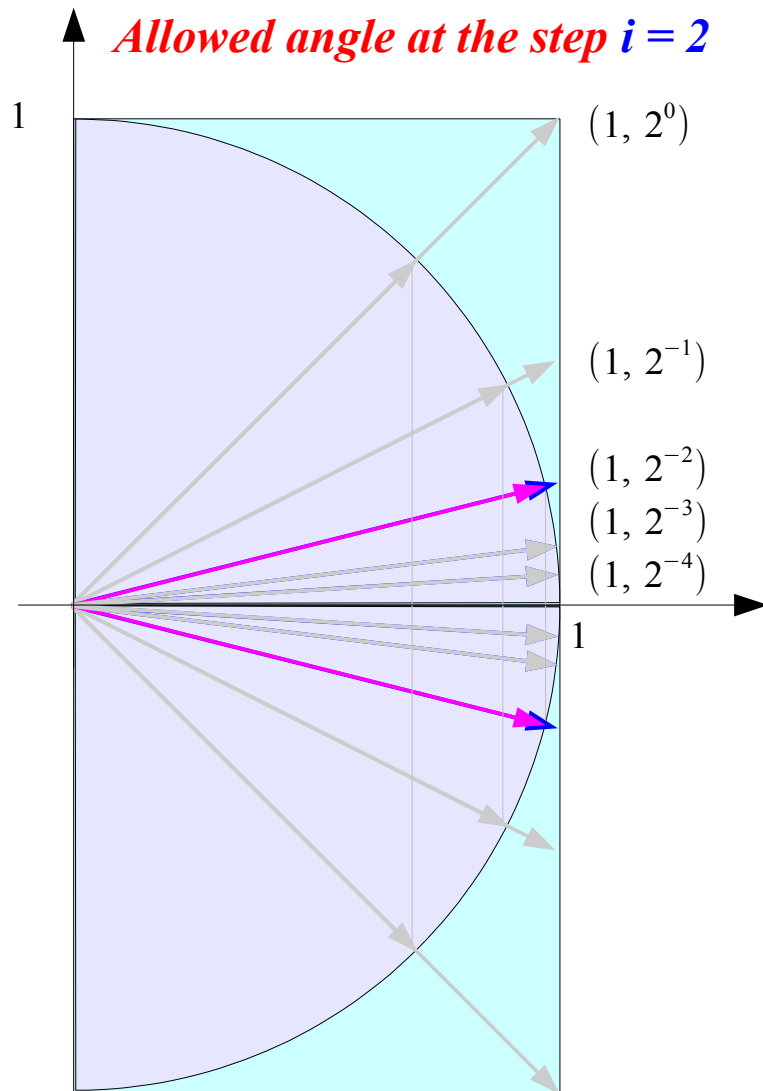


$$\begin{pmatrix} +\cos \alpha_i & -\sin \alpha_i \\ +\sin \alpha_i & +\cos \alpha_i \end{pmatrix} = \frac{1}{\sqrt{1 + 2^{-2i}}} \begin{pmatrix} +1 & \mp 2^{-i} \\ \pm 2^{-i} & +1 \end{pmatrix}$$

CORDIC Iteration Equations (4)



CORDIC Iteration Equations (5)



CORDIC Iteration Equations (6)

Choose α_i such that $\tan \alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$$\tan \alpha_i = \sigma_i 2^{-i} \quad \sigma_i \in \{+1, -1\}$$

$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i \sigma_i 2^{-i}) \\ y'_{i+1} &= (x'_i \sigma_i 2^{-i} + y'_i) \\ \alpha_{i+1} &= \alpha_i - \tan(\sigma_i 2^{-i}) \end{aligned}$$

$$\begin{aligned} &\begin{pmatrix} +\cos \alpha & -\sin \alpha \\ +\sin \alpha & +\cos \alpha \end{pmatrix} \\ &= \frac{1}{\sqrt{1 + 2^{-2 \cdot n}}} \begin{pmatrix} +1 & \mp 2^{-n} \\ \pm 2^{-n} & +1 \end{pmatrix} \cdots \frac{1}{\sqrt{1 + 2^{-2 \cdot 1}}} \begin{pmatrix} +1 & \mp 2^{-1} \\ \pm 2^{-1} & +1 \end{pmatrix} \cdot \frac{1}{\sqrt{1 + 2^0}} \begin{pmatrix} +1 & \mp 2^0 \\ \pm 2^0 & +1 \end{pmatrix} \\ &= \frac{1}{\sqrt{1 + 2^{-2 \cdot n}}} \cdots \frac{1}{\sqrt{1 + 2^{-2 \cdot 1}}} \cdot \frac{1}{\sqrt{1 + 2^{-2 \cdot 0}}} \begin{pmatrix} +1 & \mp 2^{-n} \\ \pm 2^{-n} & +1 \end{pmatrix} \cdots \begin{pmatrix} +1 & \mp 2^{-1} \\ \pm 2^{-1} & +1 \end{pmatrix} \begin{pmatrix} +1 & \mp 2^{-0} \\ \pm 2^{-0} & +1 \end{pmatrix} \end{aligned}$$

➡ $1 / K = 0.607$

➡ $\begin{pmatrix} +\cos(\sum \alpha_i) & -\sin(\sum \alpha_i) \\ +\sin(\sum \alpha_i) & +\cos(\sum \alpha_i) \end{pmatrix}$

$$\begin{pmatrix} +\cos(\sum \alpha_i) & -\sin(\sum \alpha_i) \\ +\sin(\sum \alpha_i) & +\cos(\sum \alpha_i) \end{pmatrix} = K \cdot \begin{pmatrix} +\cos \alpha & -\sin \alpha \\ +\sin \alpha & +\cos \alpha \end{pmatrix}$$

$$K = \prod \{\sqrt{1 + \tan^2 \alpha_i}\} = 1.647$$

CORDIC Iteration Equations (7)

Choose α_i such that $\tan \alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$$\tan \alpha_i = \sigma_i 2^{-i} \quad \sigma_i \in \{+1, -1\}$$

$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i \sigma_i 2^{-i}) \\ y'_{i+1} &= (x'_i \sigma_i 2^{-i} + y'_i) \\ \alpha_{i+1} &= \alpha_i - \tan(\sigma_i 2^{-i}) \end{aligned}$$

$$\begin{pmatrix} +\cos \alpha & -\sin \alpha \\ +\sin \alpha & +\cos \alpha \end{pmatrix} = \frac{1}{\sqrt{1 + 2^{-2 \cdot n}}} \cdots \frac{1}{\sqrt{1 + 2^{-2 \cdot 1}}} \cdot \frac{1}{\sqrt{1 + 2^{-2 \cdot 0}}} \begin{pmatrix} +1 & \mp 2^{-n} \\ \pm 2^{-n} & +1 \end{pmatrix} \cdots \begin{pmatrix} +1 & \mp 2^{-1} \\ \pm 2^{-1} & +1 \end{pmatrix} \begin{pmatrix} +1 & \mp 2^{-0} \\ \pm 2^{-0} & +1 \end{pmatrix}$$

➡ Use pre-computed value

➡ n iterations
each iteration involves only **adding and shifting**

Similar to shift-and-add multiplication

$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i \sigma_i 2^{-i}) \\ y'_{i+1} &= (x'_i \sigma_i 2^{-i} + y'_i) \end{aligned}$$

CORDIC Iteration Equations (8)

Pseudo-rotation

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \dots \rightarrow \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} \rightarrow \begin{pmatrix} x'_{i+1} \\ y'_{i+1} \end{pmatrix} \rightarrow \dots \rightarrow \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$$

$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i \tan \alpha_i) &= (x'_i \cos \alpha_i - y'_i \sin \alpha_i) \sqrt{1 + \tan^2 \alpha_i} \\ y'_{i+1} &= (x'_i \tan \alpha_i + y'_i) &= (x'_i \sin \alpha_i + y'_i \cos \alpha_i) \sqrt{1 + \tan^2 \alpha_i} \end{aligned}$$

$$x'_n = \{x_0 \cos(\sum \alpha_i) - y_0 \sin(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\}$$

$$y'_n = \{x_0 \sin(\sum \alpha_i) + y_0 \cos(\sum \alpha_i)\} \cdot \prod \{\sqrt{1 + \tan^2 \alpha_i}\}$$

$$\alpha_n = \alpha - \sum \alpha_i$$

Choose α_i such that $\tan \alpha_i = \sigma_i 2^{-i}$ $\sigma_i \in \{+1, -1\}$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_n = K(x_0 \cos \alpha - y_0 \sin \alpha)$$

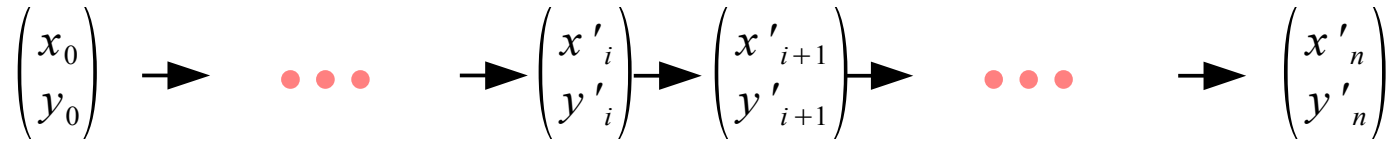
$$y'_n = K(x_0 \sin \alpha + y_0 \cos \alpha)$$

$$\alpha_n = \alpha - \sum \tan^{-1}(\sigma_i 2^{-i})$$

$$K = \prod \{\sqrt{1 + \tan^2 \alpha_i}\}$$

CORDIC Iteration Equations (9)

Pseudo-rotation



Choose α_i such that $\tan \alpha_i = \sigma_i 2^{-i}$ $\sigma_i \in \{+1, -1\}$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_n = K(x_0 \cos \alpha - y_0 \sin \alpha)$$

$$y'_n = K(x_0 \sin \alpha + y_0 \cos \alpha)$$

$$\alpha_n = \alpha - \sum \tan^{-1}(\sigma_i 2^{-i})$$

pre-compute $K = \prod \{\sqrt{1 + \tan^2 \alpha_i}\} = 1.647$

set $x_0 = 1 / K = 0.607$

$$y_0 = 0$$

then $x'_n = K(1/K \cos \alpha - 0 \cdot \sin \alpha) = \cos \alpha$

$$y'_n = K(1/K \sin \alpha + 0 \cdot \cos \alpha) = \sin \alpha$$

Example: $\cos 75^\circ$, $\sin 75^\circ$ - Step 0

$$i=0 \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0.607 \\ 0 \end{pmatrix} \quad \alpha_0 = 75^\circ$$

$$(\alpha_i > 0) \quad ? \quad \sigma_i = +1 \quad : \quad \sigma_i = -1$$

$$(\alpha_0 = 75)^\circ > 0^\circ \quad \Rightarrow \quad \sigma_0 = +1$$

$$2^{-i} = 2^0 = 1 \quad \Rightarrow \quad \tan^{-1} 2^{-i} = \tan^{-1} 1 = 45^\circ$$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$x'_1 = 0.607 - 0 \cdot (+1) \cdot 1 = 0.607$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$y'_1 = 0.607 \cdot (+1) \cdot 1 + 0 = 0.607$$

$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$\alpha_1 = \alpha_0 - 45^\circ = 30^\circ$$

Example: $\cos 75^\circ$, $\sin 75^\circ$ - Step 1

$$i=1 \quad \begin{pmatrix} x'_1 \\ y'_1 \end{pmatrix} = \begin{pmatrix} 0.607 \\ 0.607 \end{pmatrix} \quad \alpha_1 = 30^\circ$$

$$(\alpha_i > 0) \quad ? \quad \sigma_i = +1 \quad : \quad \sigma_i = -1$$

$$(\alpha_1 = 30)^\circ > 0^\circ \quad \Rightarrow \quad \sigma_1 = +1$$

$$2^{-i} = 2^{-1} = 0.5 \quad \Rightarrow \quad \tan^{-1} 2^{-i} = \tan^{-1} 0.5 = 26.565^\circ$$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_2 = 0.607 - 0.607 \cdot (+1) \cdot 0.5 = 0.3035$$

$$y'_2 = 0.607 \cdot (+1) \cdot 0.5 + 0.607 = 0.9105$$

$$\alpha_2 = \alpha_1 - 26.565^\circ = 3.435^\circ$$

Example: $\cos 75^\circ$, $\sin 75^\circ$ - Step 2

$$i=2 \quad \begin{pmatrix} x'_2 \\ y'_2 \end{pmatrix} = \begin{pmatrix} 0.3035 \\ 0.9105 \end{pmatrix} \quad \alpha_2 = 3.435^\circ$$

$$(\alpha_i > 0) \quad ? \quad \sigma_i = +1 \quad : \quad \sigma_i = -1$$

$$(\alpha_2 = 3.435)^\circ > 0^\circ \quad \Rightarrow \quad \sigma_2 = +1$$

$$2^{-i} = 2^{-2} = 0.25 \quad \Rightarrow \quad \tan^{-1} 2^{-2} = \tan^{-1} 0.25 = 14.036^\circ$$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_3 = 0.3035 - 0.9105 \cdot (+1) \cdot 0.25 = 0.0759$$

$$y'_3 = 0.3035 \cdot (+1) \cdot 0.25 + 0.9105 = 0.9864$$

$$\alpha_3 = \alpha_2 - 14.036^\circ = -10.601^\circ$$

Example: $\cos 75^\circ$, $\sin 75^\circ$ - Step 3

$$i=3 \quad \begin{pmatrix} x'_3 \\ y'_3 \end{pmatrix} = \begin{pmatrix} 0.0759 \\ 0.9864 \end{pmatrix} \quad \alpha_3 = -10.601^\circ$$

$$(\alpha_i > 0) \text{ ? } \sigma_i = +1 : \sigma_i = -1$$

$$(\alpha_3 = -10.601)^\circ < 0^\circ \Rightarrow \sigma_3 = -1$$

$$2^{-i} = 2^{-3} = 0.125 \quad \Rightarrow \tan^{-1} 2^{-3} = \tan^{-1} 0.125 = 7.125^\circ$$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_4 = 0.0759 - 0.9864 \cdot (-1) \cdot 0.125 = 0.1992$$

$$y'_4 = 0.0759 \cdot (-1) \cdot 0.125 + 0.9864 = 0.9373$$

$$\alpha_4 = \alpha_3 + 7.125^\circ = -3.476^\circ$$

Example: $\cos 75^\circ$, $\sin 75^\circ$ - Step 4

$$i=4 \quad \begin{pmatrix} x'_4 \\ y'_4 \end{pmatrix} = \begin{pmatrix} 0.1992 \\ 0.9373 \end{pmatrix} \quad \alpha_4 = -3.476^\circ$$

$$(\alpha_i > 0) \text{ ? } \sigma_i = +1 : \sigma_i = -1$$

$$(\alpha_4 = -3.476)^\circ < 0^\circ \Rightarrow \sigma_4 = -1$$

$$2^{-i} = 2^{-4} = 0.0625 \Rightarrow \tan^{-1} 2^{-4} = \tan^{-1} 0.0625 = 3.576^\circ$$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_5 = 0.1992 - 0.9373 \cdot (-1) \cdot 0.0625 = 0.2578$$

$$y'_5 = 0.1992 \cdot (-1) \cdot 0.0625 + 0.9373 = 0.9249$$

$$\alpha_5 = \alpha_4 + 3.576^\circ = 0.1^\circ$$

References

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