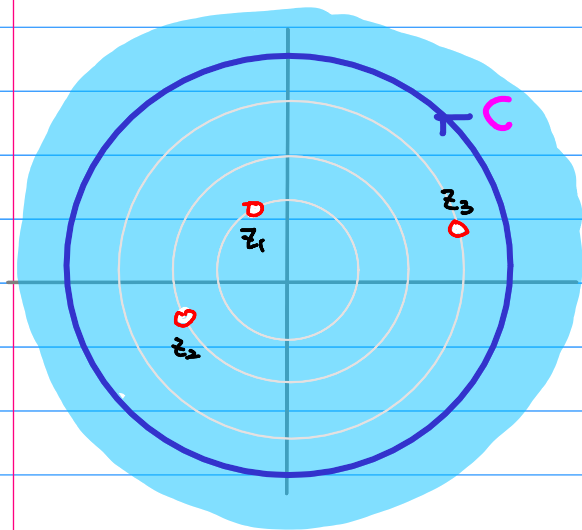
Laurent Series and Geometric Series

20170626

Series Expansion at $z=0$



$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} z^n}$$

$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right) \end{aligned}}$$

Poles z_k

$n \geq 0$	$z_1, z_2, z_3, 0$
$n < 0$	z_1, z_2, z_3

* General Series Expansion at $z=0$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n z^n}$$

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz = \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

* Z-transform

$$\boxed{X(z) = \sum_{k=0}^{\infty} x_k z^{-k}}$$

$$x_n = \frac{1}{2\pi i}\oint_C X(z) z^{n-1}\,dz = \sum_k \mathrm{Res}\left(X(z) z^{n-1}, z_k\right)$$

Z - Transform	$X(z)$
Laurent Series	$f(z)$

z-Transform $X(z) \longleftrightarrow x_n$

Laurent Series $f(z) \longleftrightarrow a_n$

$$\boxed{X(z) = f(z^{-1}) \longleftrightarrow x_n = a_n}$$

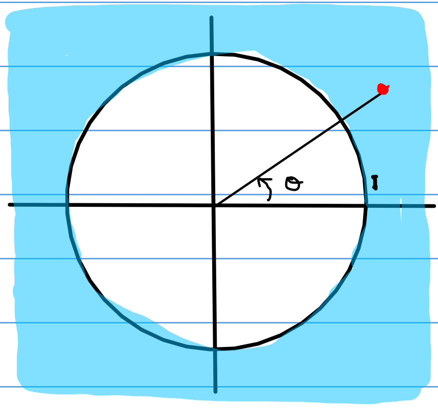
z-Transform $X(z) \longleftrightarrow x_n$

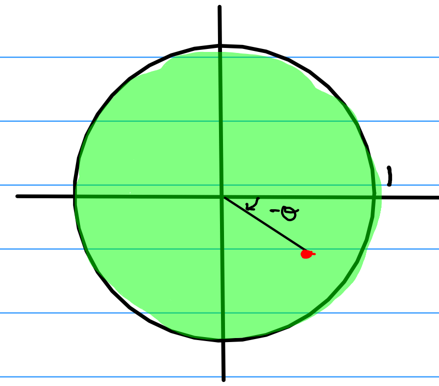
Laurent Series $f(z) \longleftrightarrow a_n$

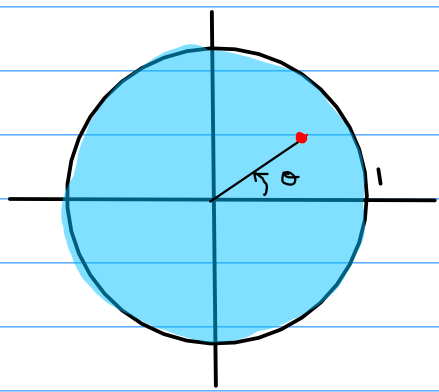
$$\boxed{X(z) = f(z) \longleftrightarrow x_n = a_{-n}}$$

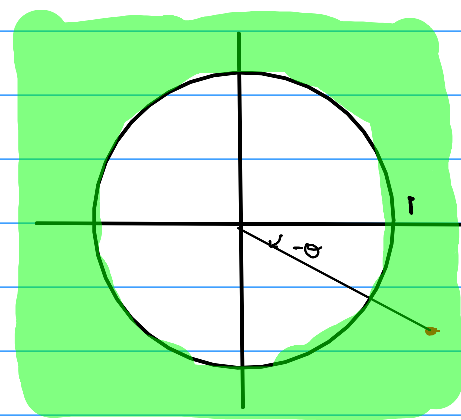
Mapping $w = \frac{1}{z}$

$z = \rho e^{j\theta}$ $\qquad$ $z^{-1} = \frac{1}{\rho} e^{-j\theta}$









- inverse magnitude
- negative phase

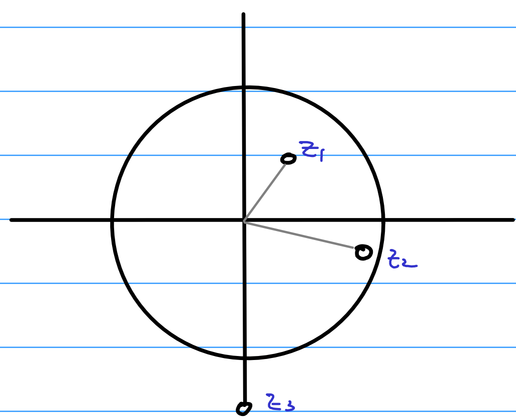
$$f(z) = \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)(z - p_3)}$$

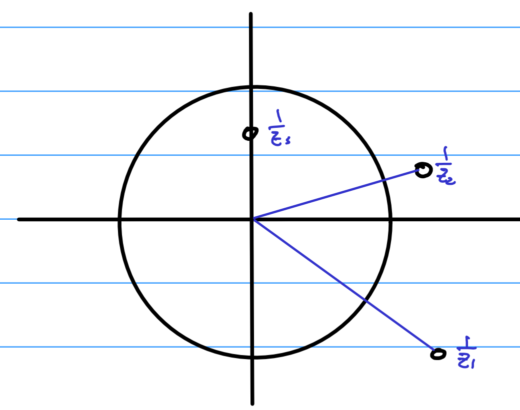
$$f(z^{-1}) = \frac{(\frac{1}{z} - z_1)(\frac{1}{z} - z_2)}{(\frac{1}{z} - p_1)(\frac{1}{z} - p_2)(\frac{1}{z} - p_3)}$$

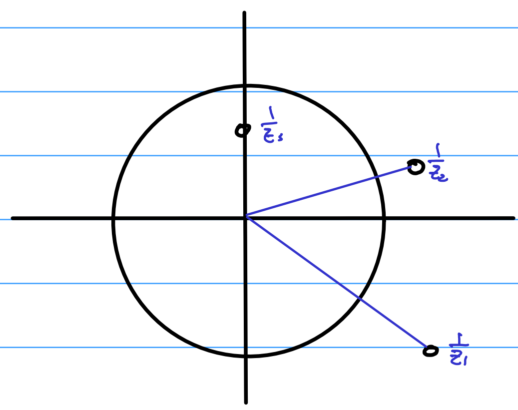
$$= \frac{(1 - z_1 z)(1 - z_2 z)}{(1 - p_1 z)(1 - p_3 z)(1 - p_3 z)}$$

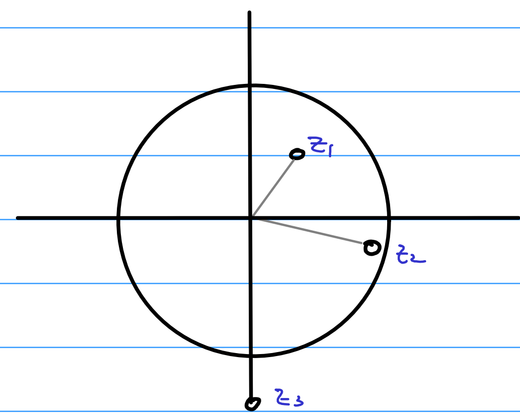
$\frac{1}{z_1}, \frac{1}{z_2}$

$\frac{1}{p_1}, \frac{1}{p_3}, \frac{1}{p_3}$



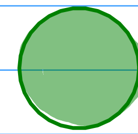


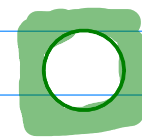




$g(z)$ with a simple pole

$b > 0$ assumed

$$g(z) = \frac{1}{1 - bz} = \boxed{\frac{b^{-1}}{b^{-1} - z}}$$ $|bz| < 1$  $|z| < \frac{1}{b}$

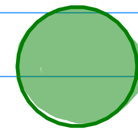
$$h(z) = \frac{1}{1 - \frac{b}{z}} = \boxed{\frac{z}{z - b}}$$ $\left|\frac{b}{z}\right| < 1$  $|z| > b$

$$g(z^{-1}) = \frac{b^{-1}}{b^{-1} - z^{-1}} = \frac{z}{z - b} = h(z)$$

$$h(z^{-1}) = \frac{z^{-1}}{z^{-1} - b} = \frac{b^{-1}}{b^{-1} - z} = g(z)$$

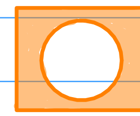
$$\boxed{\begin{aligned} g(z^{-1}) &= h(z) \\ h(z^{-1}) &= g(z) \end{aligned}}$$

$$g(z) = \boxed{\frac{b^{-1}}{b^{-1} - z}} = \frac{\bigcirc}{\bigcirc - z}$$



$|z| < \frac{1}{b}$

$$h(z) = \boxed{\frac{z}{z - b}} = \frac{z}{z - \square}$$



$|z| > b$

$$g(z^{-1}) = \frac{b^{-1}}{b^{-1} - z^{-1}} = \frac{z}{z - b} = h(z) \qquad \frac{\bigcirc}{\bigcirc - z^{-1}} = \frac{z}{z - \square}$$

$$h(z^{-1}) = \frac{z^{-1}}{z^{-1} - b} = \frac{b^{-1}}{b^{-1} - z} = g(z) \qquad \frac{z^{-1}}{z^{-1} - \square} = \frac{\bigcirc}{\bigcirc - z}$$

$$\bigcirc = b^{-1}$$

$$\square = b$$

$$\bigcirc^{-1} = \square$$

$$\square^{-1} = \bigcirc$$

Infinite Sum of G. P.

Simple pole $\Rightarrow$

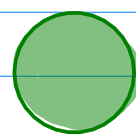
$$\frac{\;}{z-\square} \Rightarrow \frac{z}{z-\square} \Rightarrow \frac{1}{1-\frac{\square}{z}}$$ infinite sum of G.P

$$\frac{\;}{\triangle - z} \Rightarrow \frac{\triangle}{\triangle - z} \Rightarrow \frac{1}{1-\frac{z}{\triangle}}$$ infinite sum of G.P

Convergence Condition

$\frac{b^{-1}}{b^{-1}-z} \Rightarrow$ think this way $b^{-1}-z > 0$ $\boxed{b^{-1} > |z|}$ pole b^{-1}

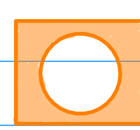
$$\boxed{\frac{b^{-1}}{b^{-1}-z}} = \frac{\bigcirc}{\bigcirc - z}$$



$|z| < \frac{1}{b}$

$\frac{z}{z-b} \Rightarrow$ think this way $z-b > 0$ $\boxed{|z| > b}$ pole b

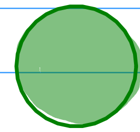
$$\boxed{\frac{z}{z-b}} = \frac{z}{z-\square}$$



$|z| > b$

Two Sequences are involved (causal, anti-causal)

$$\boxed{\frac{b^{-1}}{b^{-1} - z}} = \frac{\bigcirc}{\bigcirc - z}$$

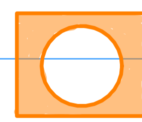


$|z| < \frac{1}{b}$

$$\boxed{① \quad (n \geq 0) \quad (bz)^0 + (bz)^1 + (bz)^2 + \cdots = \sum_{n=0}^{\infty} b^n z^n \quad \text{L.S.}}$$

$$\boxed{② \quad (n \leq 0) \quad (b^{-1}z^{-1})^0 + (b^{-1}z^{-1})^{-1} + (b^{-1}z^{-1})^{-2} + \cdots = \sum_{n=0}^{-\infty} b^{-n} z^{-n} \quad \text{Z.T.}}$$

$$\boxed{\frac{z}{z - b}} = \frac{z}{z - \square}$$



$|z| > b$

$$\boxed{① \quad (n \geq 0) \quad (bz^{-1})^0 + (bz^{-1})^1 + (bz^{-1})^2 + \cdots = \sum_{n=0}^{\infty} b^n z^{-n} \quad \text{Z.T.}}$$

$$\boxed{② \quad (n \leq 0) \quad (b^{-1}z)^0 + (b^{-1}z)^{-1} + (b^{-1}z)^{-2} + \cdots = \sum_{n=0}^{-\infty} b^{-n} z^{n} \quad \text{L.S.}}$$

$\boxed{n \geq 0}$ $\boxed{n \leq 0}$ $\boxed{L.S.}$ $\boxed{Z.T.}$

$(\quad)^0 + (\quad)^1 + (\quad)^2 + \cdots \rightarrow (n \geq 0)$

$(\quad)^0 + (\quad)^{-1} + (\quad)^{-2} + \cdots \rightarrow (n \leq 0)$

$\sum \;\; \boxed{z^n} \longrightarrow L.S.$

$\sum \;\; \boxed{z^{-n}} \longrightarrow Z.T.$

	$\frac{\bigcirc}{\bigcirc - z}$	$\frac{z}{z - \square}$
	pole $p = \bigcirc$	pole $p = \square$
	c.r $\frac{z}{\bigcirc}$	c.r $\frac{\square}{z}$
	r.o.c $\lvert z \rvert < \bigcirc$	r.o.c $\lvert z \rvert > \square$
$(n \geq 0)$	$\sum_{n=0}^{\infty} \left(\frac{z}{\bigcirc}\right)^{n} = \sum_{n=0}^{\infty} \left(\frac{1}{\bigcirc}\right)^{n} z^{n}$	$\sum_{n=0}^{\infty} \left(\frac{\square}{z}\right)^{n} = \sum_{n=0}^{\infty} \square^{n} z^{-n}$
$(n \leq 0)$	$\sum_{n=0}^{-\infty} \left(\frac{z}{\bigcirc}\right)^{-n} = \sum_{n=0}^{-\infty} \bigcirc^{n} z^{-n}$	$\sum_{n=0}^{-\infty} \left(\frac{\square}{z}\right)^{-n} = \sum_{n=0}^{-\infty} \left(\frac{1}{\square}\right)^{n} z^{n}$
	L.S: $b^{n} z^{n}$ $(n \geq 0)$	Z.T: $b^{n} z^{-n}$ $(n \geq 0)$
	Z.T: $b^{-n} z^{-n}$ $(n \leq 0)$	L.S: $b^{-n} z^{n}$ $(n \leq 0)$
	$p = \bigcirc = b^{-1}$	$p = \square = b$

$$\sum_{n=0}^{\infty} (\quad)^n = \sum_{n=0}^{-\infty} (\quad)^{-n}$$

L.S.

$$\sum_{n=0}^{\infty} \left(\frac{z}{\bigcirc}\right)^n = \boxed{\sum_{n=0}^{\infty}} \left(\frac{1}{\bigcirc}\right)^n \boxed{z^n}$$

$(n \geq 0)$ $\boxed{b^n}\boxed{z^n}$

Z.T

$$\sum_{n=0}^{\infty} \left(\frac{\square}{z}\right)^n = \boxed{\sum_{n=0}^{\infty}} \square^n \boxed{z^{-n}}$$

$(n \geq 0)$ $\boxed{b^n}\boxed{z^{-n}}$

Z.T

$$\sum_{n=0}^{-\infty} \left(\frac{z}{\bigcirc}\right)^{-n} = \boxed{\sum_{n=0}^{-\infty}} \bigcirc^n \boxed{z^{-n}}$$

$(n \leq 0)$ $\boxed{b^{-n}}\boxed{z^{-n}}$

L.S.

$$\sum_{n=0}^{-\infty} \left(\frac{\square}{z}\right)^{-n} = \boxed{\sum_{n=0}^{-\infty}} \left(\frac{1}{\square}\right)^n \boxed{z^n}$$

$(n \leq 0)$ $\boxed{b^{-n}}\boxed{z^n}$

$$p = \bigcirc = b^{-1}$$

$$p = \square = b$$

$n \geq 0$ & $n \leq 0$

	$\dfrac{\bigcirc}{\bigcirc - z}$	$\dfrac{z}{z - \square}$
$(n \geq 0)$	L.S. $\left(\dfrac{z}{\bigcirc}\right)^{n} \Longleftrightarrow b^{n} z^{n}$ $(n \geq 0)$	Z.T $\Longleftrightarrow b^{n} z^{-n}$ $(n \geq 0)$
$(n \leq 0)$	Z.T $\left(\dfrac{z}{\bigcirc}\right)^{-n} \Longleftrightarrow b^{-n} z^{-n}$ $(n < 0)$	L.S. $\Longleftrightarrow b^{-n} z^{n}$ $(n < 0)$
	$p = \bigcirc = b^{-1}$	$p = \square = b$

$(n \geq 0)$ $\quad n$

$(n \leq 0)$ $\quad -n$

$\left(\frac{z}{\bigcirc}\right)^{n}, \left(\frac{z}{\bigcirc}\right)^{-n}, \left(\frac{\square}{z}\right)^{n}, \left(\frac{\square}{z}\right)^{-n}$ $\boxed{p}$

$\frac{z \quad +}{\bigcirc \quad -}$ $\frac{z}{p}$ $(n \geq 0)$ $p^{-n} z^{n}$ L.S $(n \geq 0)$ $b^{n} z^{n}$ L.S. $p = \bigcirc = b^{-1}$	$\frac{\square \quad +}{z \quad -}$ $\frac{p}{z}$ $(n \geq 0)$ $p^{n} z^{-n}$ Z.T. $(n \geq 0)$ $b^{n} z^{-n}$ Z.T. $p = \square = b$
$\frac{\bigcirc \quad +}{z \quad -}$ $\frac{p}{z}$ $(n \leq 0)$ $p^{n} z^{-n}$ Z.T. $(n \leq 0)$ $b^{-n} z^{-n}$ Z.T. $p = \bigcirc = b^{-1}$	$\frac{z \quad +}{\square \quad -}$ $\frac{z}{p}$ $(n \leq 0)$ $p^{-n} z^{n}$ L.S. $(n \leq 0)$ $b^{-n} z^{n}$ L.S. $= b$

$\frac{z \quad +}{p \quad -}$	$\frac{p \quad +}{z \quad -}$
L.S.	Z.T.

$p = \bigcirc = \square^{-1}$

$p = \square = \bigcirc^{-1}$

	L.S.	Z.T.
$(n \geq 0)$		
$(n \leq 0)$		

	L.S.	Z.T.
$(n \geq 0)$	$\frac{z^{\;+}}{\bigcirc^{\;-}}$ $\frac{z}{p}$ $(n \geq 0)$ $p^{-n} z^{n}$ L.S $(n \geq 0)$ $b^{n} z^{n}$ L.S. $p = \bigcirc = b^{-1}$	$\frac{\square^{\;+}}{z^{\;-}}$ $\frac{p}{z}$ $(n \geq 0)$ $p^{n} z^{-n}$ Z.T. $(n \geq 0)$ $b^{n} z^{-n}$ Z.T. $p = \square = b$
$(n \leq 0)$	$\frac{z^{\;+}}{\square^{\;-}}$ $\frac{z}{p}$ $(n \leq 0)$ $p^{-n} z^{n}$ L.S. $(n \leq 0)$ $b^{-n} z^{n}$ L.S. $p = \square = b$	$\frac{\bigcirc^{\;+}}{z^{\;-}}$ $\frac{p}{z}$ $(n \leq 0)$ $p^{n} z^{-n}$ Z.T. $(n \leq 0)$ $b^{-n} z^{-n}$ Z.T. $p = \bigcirc = b^{-1}$

	L.S. $p^{-n}z^{n}$	Z.T. $p^{n}z^{-n}$
$(n \geq 0)$	$\left(\frac{z}{\bigcirc}\right)^{n}$ $\quad \left(\frac{1}{\bigcirc}\right)^{n} \boxed{z^{n}}$ $\boxed{\frac{\bigcirc}{\bigcirc - z}}$ $\quad p^{-n}z^{n}$	$\left(\frac{\square}{z}\right)^{n}$ $\quad \square^{n} \boxed{z^{-n}}$ $\boxed{\frac{z}{z - \square}}$ $\quad p^{n}z^{-n}$
$(n \leq 0)$	$\left(\frac{\square}{z}\right)^{-n}$ $\quad \left(\frac{1}{\square}\right)^{n} \boxed{z^{n}}$ $\boxed{\frac{z}{z - \square}}$ $\quad p^{-n}z^{n}$	$\left(\frac{z}{\bigcirc}\right)^{-n}$ $\quad \bigcirc^{n} \boxed{z^{-n}}$ $\boxed{\frac{\bigcirc}{\bigcirc - z}}$ $\quad p^{n}z^{-n}$
	$p = \bigcirc = b^{-1}$ $p = \square = b$	$p = \square = b$ $p = \bigcirc = b^{-1}$

	L.S. $\bigcirc = \square^{-1}$ $\quad p^{-n} z^{n}$	Z.T. $\bigcirc = \square^{-1}$ $\quad p^{n} z^{-n}$
$(n \geq 0)$	$\left(\frac{z}{\square^{-1}}\right)^{n}$ $\quad \left(\frac{1}{\square^{-1}}\right)^{n} \boxed{z^{n}}$	$\left(\frac{\square}{z}\right)^{n}$ $\quad \square^{n} \boxed{z^{-n}}$
	$\boxed{\frac{\square^{-1}}{\square^{-1} - z}}$ $\quad p^{-n} z^{n}$	$\boxed{\frac{z}{z - \square}}$ $\quad p^{n} z^{-n}$
$(n \leq 0)$	$\left(\frac{\square}{z}\right)^{-n}$ $\quad \left(\frac{1}{\square}\right)^{n} \boxed{z^{n}}$	$\left(\frac{z}{\square^{-1}}\right)^{-n}$ $\quad \square^{-n} \boxed{z^{-n}}$
	$\boxed{\frac{z}{z - \square}}$ $\quad p^{-n} z^{n}$	$\boxed{\frac{\square^{-1}}{\square^{-1} - z}}$ $\quad p^{n} z^{-n}$
	$p = \bigcirc = b^{-1}$ $p = \square = b$	$p = \square = b$ $p = \bigcirc = b^{-1}$

b & p with Σ notations

	L.S.	Z.T.
$(n \geq 0)$	$\sum_{n=0}^{\infty}\left(\frac{z}{\bigcirc}\right)^{n} = \sum_{n=0}^{\infty} p^{-n} z^{n}$ $= \sum_{n=0}^{\infty} b^{n} z^{n}$	$\sum_{n=0}^{\infty}\left(\frac{\square}{z}\right)^{n} = \sum_{n=0}^{\infty} p^{n} z^{-n}$ $= \sum_{n=0}^{\infty} b^{n} z^{-n}$
$(n \leq 0)$	$\sum_{n=0}^{-\infty}\left(\frac{\square}{z}\right)^{-n} = \sum_{n=0}^{-\infty} p^{-n} z^{n}$ $= \sum_{n=0}^{-\infty} b^{-n} z^{n}$	$\sum_{n=0}^{-\infty}\left(\frac{z}{\bigcirc}\right)^{-n} = \sum_{n=0}^{-\infty} p^{n} z^{-n}$ $= \sum_{n=0}^{-\infty} b^{-n} z^{-n}$
	$p = \bigcirc = b^{-1}$ $p = \square = b$	$p = \square = b$ $p = \bigcirc = b^{-1}$

L.S.	Z.T.
L.S.	Z.T.

$(n \geq 0)$	$(n \geq 0)$
$(n \leq 0)$	$(n \leq 0)$

p^{-n}	p^{n}
p^{-n}	p^{n}

b^{n}	b^{n}
b^{-n}	b^{-n}

$p = \bigcirc$	$p = \square$
$p = \square$	$p = \bigcirc$

$\bigcirc = b^{-1}$	$\square = b$
$\square = b$	$\bigcirc = b^{-1}$

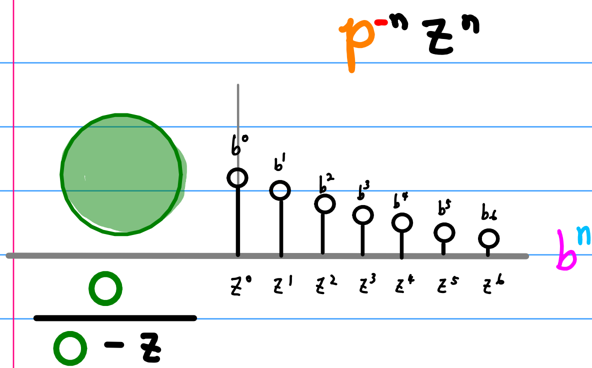
$\square^{n}$	$\square^{n}$
$\square^{-n}$	$\square^{-n}$

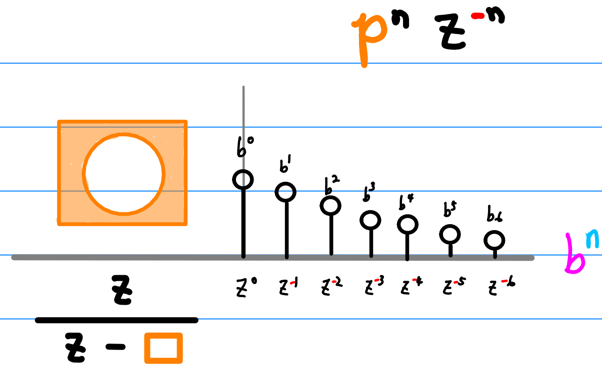
$\square^{n}$	$\square^{n}$
$\square^{-n}$	$\square^{-n}$

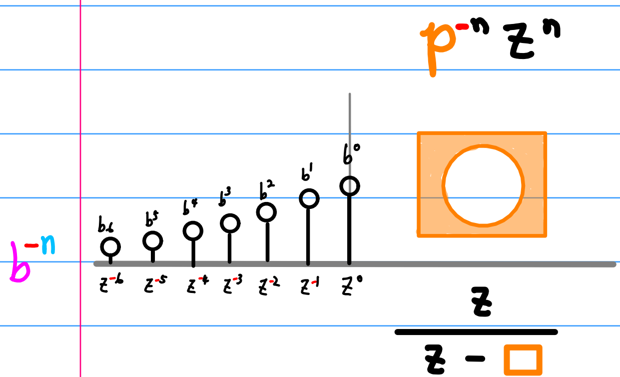
$\dfrac{\bigcirc}{\bigcirc - z}$	$\dfrac{z}{z - \square}$
$\dfrac{z}{z - \square}$	$\dfrac{\bigcirc}{\bigcirc - z}$

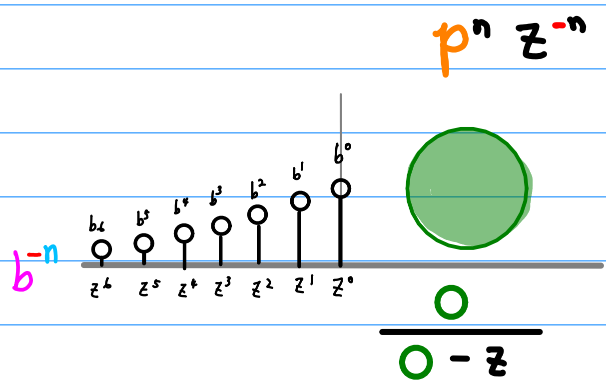
$\dfrac{\square^{-1}}{\square^{-1} - z}$	$\dfrac{z}{z - \square}$
$\dfrac{z}{z - \square}$	$\dfrac{\square^{-1}}{\square^{-1} - z}$

L.S.: $a_n z^n$ Z.T.: $x_n z^{-n}$

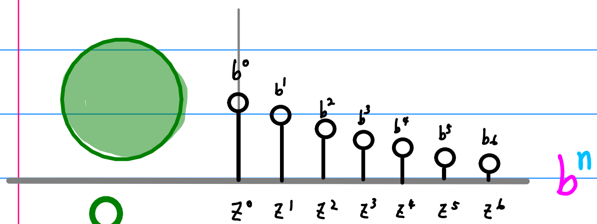






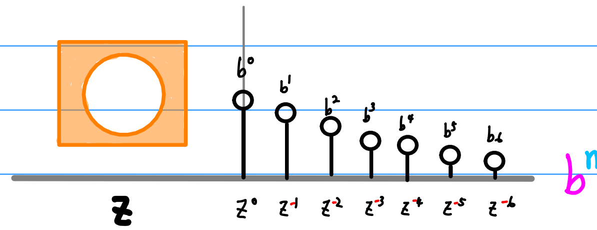


$b^n z^n$



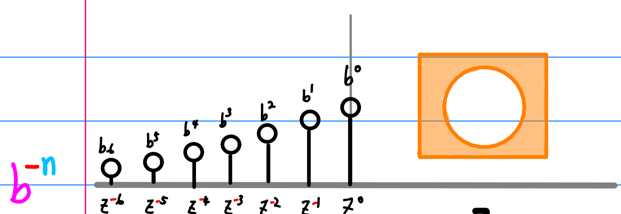
$$\frac{\bigcirc}{\bigcirc - z}$$

$b^n z^{-n}$



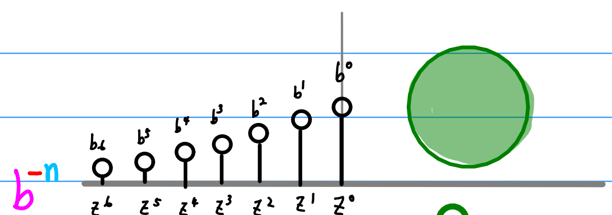
$$\frac{z}{z - \square}$$

$b^{-n} z^n$



$$\frac{z}{z - \square}$$

$b^{-n} z^{-n}$

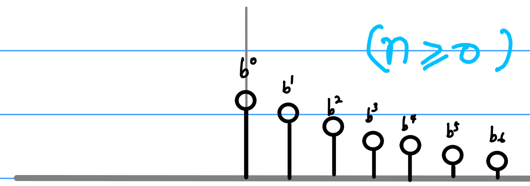


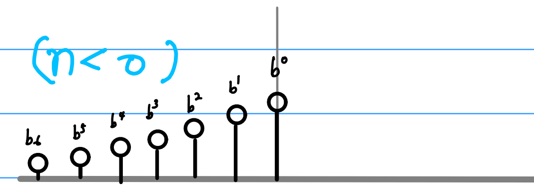
$$\frac{\bigcirc}{\bigcirc - z}$$

b^{n} & b^{-n}

$0 < b < 1$ assumed

$\boxed{\begin{array}{ll} (n \geq 0) & a_n = x_n = b^{n} \\ (n < 0) & a_n = x_n = b^{-n} \\ & \\ x_n = p^{n} & (n \geq 0,\ n < 0) \\ a_n = p^{-n} & (n \geq 0,\ n < 0) \end{array}}$





a_n Laurent Series Coefficient

x_n input to z-Transform

causal signal $\begin{cases} x_n = 0 & n < 0 \\ x_n \neq 0 & n \geq 0 \end{cases}$

anti-causal signal $\begin{cases} x_n \neq 0 & n < 0 \\ x_n = 0 & n \geq 0 \end{cases}$

p the simple pole of $f(z)$ or $X(z)$

$\boxed{\dfrac{b^{-1}}{b^{-1} - z}}$ $p = b^{-1}$

$\boxed{\dfrac{z}{z - b}}$ $p = b$

$Z.T.: x_n z^{-n}$ $L.S.: a_n z^n$

Z. T	z^{-n} $(n \geq 0)$	$b^n z^{-n}$ $(n \geq 0)$	$+\ -$
	z^n $(n < 0)$	$b^{-n} z^n$ $(n < 0)$	$-\ +$
L. S	z^n $(n \geq 0)$	$b^n z^n$ $(n \geq 0)$	$+\ +$
	z^{-n} $(n < 0)$	$b^{-n} z^{-n}$ $(n < 0)$	$-\ -$

$(n \geq 0)$	Z. T. z^{-n}	Z. T. $b^n z^{-n}$	$+\ -$
	L. S. z^n	L. S. $b^n z^n$	$+\ +$
$(n < 0)$	Z. T. z^n	Z. T. $b^{-n} z^n$	$-\ +$
	L. S. z^{-n}	L. S. $b^{-n} z^{-n}$	$-\ -$

Laurent Series a_n	Z-Transform x_n
$a_n = b^n \quad (n \geq 0)$	$x_n = b^n \quad (n \geq 0)$
$bz = \frac{z}{p} \qquad p = b^{-1}$	$bz^{-1} = \frac{p}{z} \qquad p = b$
$\left\lvert \frac{z}{p} \right\rvert < 1 \qquad \lvert z \rvert < p$	$\left\lvert \frac{p}{z} \right\rvert < 1 \qquad \lvert z \rvert > p$
$a_n = p^{-n}$	$x_n = p^n$

$a_n = b^{-n} \quad (n < 0)$	$x_n = b^{-n} \quad (n < 0)$
$bz^{-1} = \frac{p}{z} \qquad p = b$	$bz = \frac{z}{p} \qquad p = b^{-1}$
$\left\lvert \frac{p}{z} \right\rvert < 1 \qquad \lvert z \rvert > p$	$\left\lvert \frac{z}{p} \right\rvert < 1 \qquad \lvert z \rvert < p$
$a_n = p^{-n}$	$x_n = p^n$
$(n < 0) \rightarrow (k > 0)$	$(n < 0) \rightarrow (k > 0)$

Converging Geometric Series

$$\boxed{\left|\frac{z}{p}\right| < 1 \qquad |z| < p}$$

$$\left|\frac{p}{z}\right| > 1$$

$$\frac{1}{1-\frac{z}{p}} = \frac{p}{p-z}$$

think $p - z > 0$

$p > |z|$

$$\boxed{\left|\frac{p}{z}\right| < 1 \qquad p < |z|}$$

$$\left|\frac{z}{p}\right| > 1$$

$$\frac{1}{1-\frac{p}{z}} = \frac{z}{z-p}$$

think $z - p > 0$

$|z| > p$

Z- Transform

$\boxed{\frac{z}{p}} = bz \qquad p = b^{-1}$

anticausal

$(n<0)$

$\boxed{\frac{p}{z}} = \boxed{bz^{-1}} \qquad p = b$

causal

$(n \geq 0)$

Laurent Series

$\boxed{\frac{z}{p}} = \boxed{bz} \qquad p = b^{-1}$

$(n \geq 0)$

$\boxed{\frac{p}{z}} = bz^{-1} \qquad p = b$

$(n<0)$

Simple pole p & common ratio b

$$\left|\frac{z}{p}\right| < 1$$

$$|z| < p$$

$$\frac{1}{1-\frac{z}{p}} = \frac{p}{p-z}$$

$$\boxed{\frac{z}{p}} = b\,z \qquad p = b^{-1}$$

$$a_n = b^n \quad (n \geq 0)$$

$$\left|\frac{p}{z}\right| < 1$$

$$|z| > p$$

$$\frac{1}{1-\frac{p}{z}} = \frac{z}{z-p}$$

$$\boxed{\frac{p}{z}} = b\,z^{-1} \qquad p = b$$

$$x_n = b^n \quad (n \geq 0)$$

$$\left|\frac{p}{z}\right| < 1$$

$$p < |z|$$

$$\frac{1}{1-\frac{p}{z}} = \frac{z}{z-p}$$

$$\boxed{\frac{p}{z}} = b\,z^{-1} \qquad p = b$$

$$a_n = b^{-n} \quad (n < 0)$$

$$\left|\frac{z}{p}\right| < 1$$

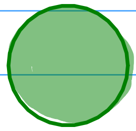
$$|z| < p$$

$$\frac{1}{1-\frac{z}{p}} = \frac{p}{p-z}$$

$$\boxed{\frac{z}{p}} = b\,z \qquad p = b^{-1}$$

$$x_n = b^{-n} \quad (n < 0)$$

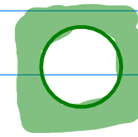
$n \geq 0$



$\left|\frac{z}{p}\right| < 1$

$\boxed{\frac{p}{p-z}}$

$a_n = p^{-n} = b^n$

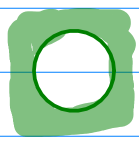


$\left|\frac{p}{z}\right| < 1$

$\boxed{\frac{z}{z-p}}$

$x_n = p^n = b^n$

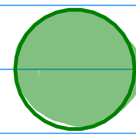
$n < 0$



$\left|\frac{p}{z}\right| < 1$

$\boxed{\frac{z}{z-p}}$

$a_n = p^{-n} = b^{-n}$



$\left|\frac{z}{p}\right| < 1$

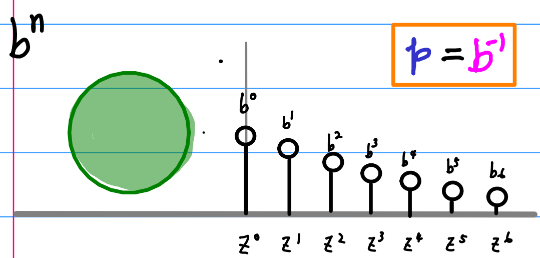
$\boxed{\frac{p}{p-z}}$

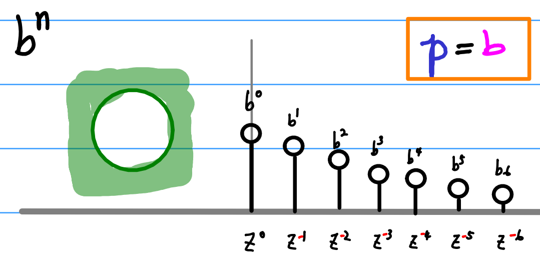
$x_n = p^n = b^{-n}$

$(n \geq 0)$	$a_n = x_n = b^n$
$(n < 0)$	$a_n = x_n = b^{-n}$

Laurent Series $\quad x_n = p^n \quad (n \geq 0,\ n < 0)$

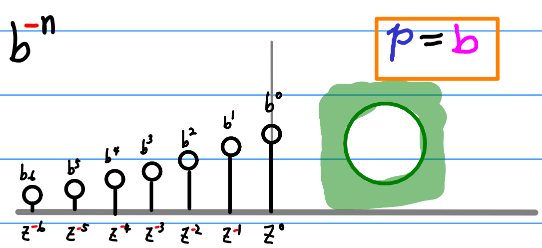
z- Transform $\quad a_n = p^{-n} \quad (n \geq 0,\ n < 0)$

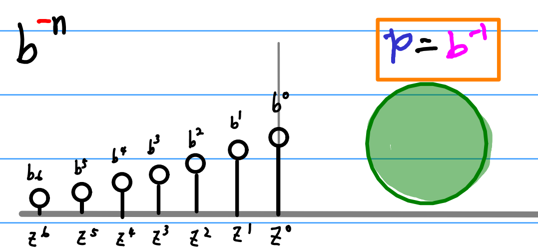




$$\boxed{a_n = p^{-n} \quad (n \geq 0)}$$

$$\boxed{x_n = p^{n} \quad (n \geq 0)}$$





$$\boxed{a_n = p^{-n} \quad (n \leq 0)}$$

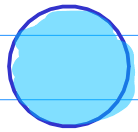
$$\boxed{x_n = p^{n} \quad (n \leq 0)}$$

$(n \geq 0)$ $\quad a_n = x_n = b^n$

$(n < 0)$ $\quad a_n = x_n = b^{-n}$

Laurent Series	$x_n = p^n$	$(n \geq 0,\ n < 0)$
z- Transform	$a_n = p^{-n}$	$(n \geq 0,\ n < 0)$

Laurent Series



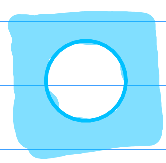
$|z| < p$

$\left|\frac{z}{p}\right| < 1$

① $z^1, z^2, z^3, \dots$

② $\frac{1}{1-\frac{z}{p}} = \frac{p}{p-z}$

③ $a_n = p^{-n} = b^n \quad p = b^{-1}$



$|z| > p$

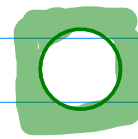
$\left|\frac{p}{z}\right| < 1$

① $z^{-1}, z^{-2}, z^{-3}, \dots$

② $\frac{1}{1-\frac{p}{z}} = \frac{z}{z-p}$

③ $a_n = p^{-n} = b^{-n} \quad p = b$

Z - Transform



$|z| > p$

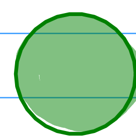
$\left|\frac{p}{z}\right| < 1$

① $z^{-1}, z^{-2}, z^{-3}, \dots$

causal signal

② $\frac{1}{1-\frac{p}{z}} = \frac{z}{z-p}$

③ $x_n = p^n = b^n \quad (p = b)$



$|z| < p$

$\left|\frac{z}{p}\right| < 1$

① $z^1, z^2, z^3, \dots$

anti-causal

② $\frac{1}{1-\frac{z}{p}} = \frac{p}{p-z}$

③ $x_n = p^n = b^{-n} \quad p = b^{-1}$

$$a_n = \left(\tfrac{1}{2}\right)^n \quad (n \geq 0)$$
$$= p^{-n} \quad (n \geq 0) \quad p = 2$$
$$f(z) = \frac{2}{2-z}$$

$$x_n = \left(\tfrac{1}{2}\right)^n \quad (n \geq 0)$$
$$= p^n \quad (n \geq 0) \quad p = \tfrac{1}{2}$$
$$X(z) = \frac{z}{z-0.5}$$

$$a_n = \left(\tfrac{1}{2}\right)^{-n} \quad (n \leq 0)$$
$$= p^{-n} \quad (n \leq 0) \quad p = \tfrac{1}{2}$$
$$f(z) = \frac{z}{z-0.5}$$

$$x_n = \left(\tfrac{1}{2}\right)^{-n} \quad (n \leq 0)$$
$$= p^n \quad (n \leq 0) \quad p = 2$$
$$X(z) = \frac{2}{2-z}$$

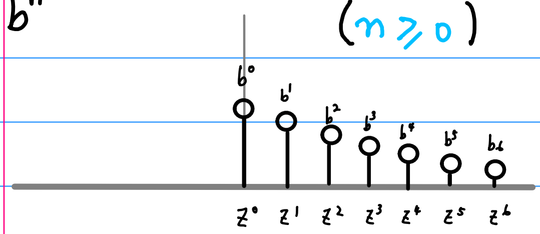
$$a_n = b^n \quad (n \geq 0)$$
$$= p^{-n} \quad (n \geq 0) \quad p = b^{-1}$$
$$f(z) = \frac{b^{-1}}{b^{-1}-z}$$

$$x_n = b^n \quad (n \geq 0)$$
$$= p^n \quad (n \geq 0) \quad p = b$$
$$X(z) = \frac{z}{z-b}$$

$$a_n = b^{-n} \quad (n \leq 0)$$
$$= p^{-n} \quad (n \leq 0) \quad p = b$$
$$f(z) = \frac{z}{z-b}$$

$$x_n = b^{-n} \quad (n \leq 0)$$
$$= p^n \quad (n \leq 0) \quad p = b^{-1}$$
$$X(z) = \frac{b^{-1}}{b^{-1}-z}$$

b^n $(n \geq 0)$

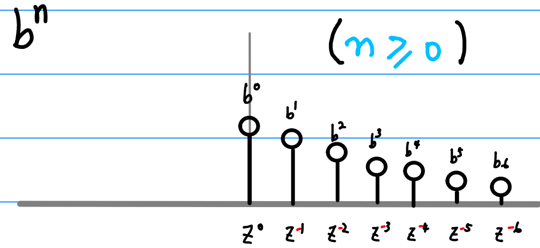


$$X(z^{-1}) = \frac{z^{-1}}{z^{-1} - 0.5} \qquad |z| < 2$$

$$f(z) = \frac{2}{2-z} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n$$

$$a_n = \left(\frac{1}{2}\right)^n$$
$$= p^{-n} \qquad \boxed{p = 2}$$

b^n $(n \geq 0)$

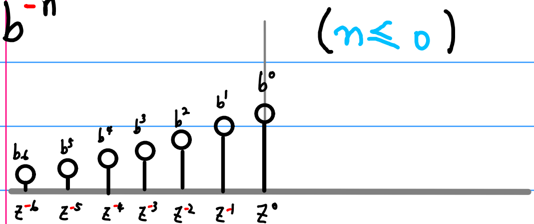


$$|z| > \frac{1}{2}$$

$$X(z) = \frac{z}{z - 0.5} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$x_n = \left(\frac{1}{2}\right)^n$$
$$= p^n \qquad \boxed{p = \frac{1}{2}}$$

b^{-n} $(n \leq 0)$

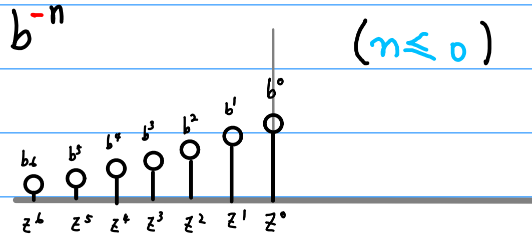


$$X(z^{-1}) = \frac{2}{2 - z^{-1}} \qquad |z| > \frac{1}{2}$$

$$f(z) = \frac{z}{z - 0.5} = \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} z^n$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$a_n = \left(\frac{1}{2}\right)^{-n}$$
$$= p^n \qquad \boxed{p = \frac{1}{2}}$$

b^{-n} $(n \leq 0)$

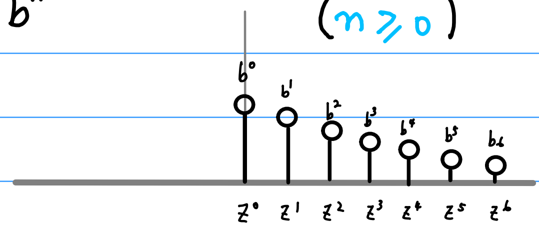


$$|z| < 2$$

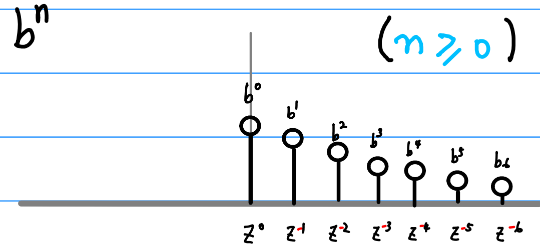
$$X(z) = \frac{2}{2 - z} = \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} z^{-n}$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n$$

$$x_n = \left(\frac{1}{2}\right)^{-n}$$
$$= p^n \qquad \boxed{p = 2}$$

b^n $(n \geq 0)$



b^n $(n \geq 0)$



$$X(z^{-1}) = \frac{z^{-1}}{z^{-1} - b} \qquad |z| < b^{-1}$$

$$f(z) = \frac{b^{-1}}{b^{-1} - z} = \sum_{n=0}^{\infty} b^n z^n$$

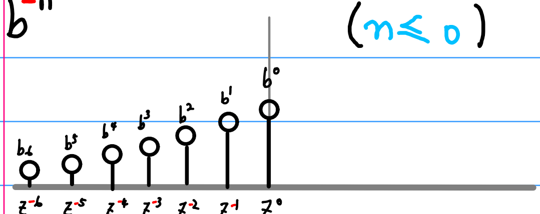
$$a_n = b^n$$
$$= p^{-n} \qquad \boxed{p = b^{-1}}$$

$$|z| > b$$

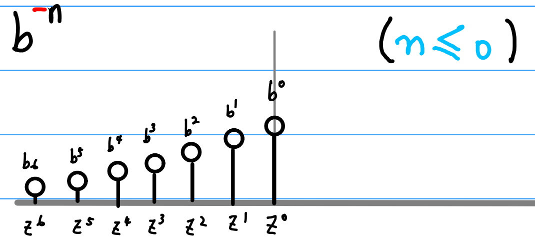
$$X(z) = \frac{z}{z - b} = \sum_{n=0}^{\infty} b^n z^{-n}$$

$$x_n = b^n$$
$$= p^n \qquad \boxed{p = b}$$

b^{-n} $(n \leq 0)$



b^{-n} $(n \leq 0)$



$$X(z^{-1}) = \frac{b^{-1}}{b^{-1} - z^{-1}} \qquad |z| > b$$

$$f(z) = \frac{z}{z - b} = \sum_{n=-\infty}^{0} b^{-n} z^n$$
$$= \sum_{n=0}^{\infty} b^n z^{-n}$$

$$a_n = b^{-n}$$
$$= p^{-n} \qquad \boxed{p = b}$$

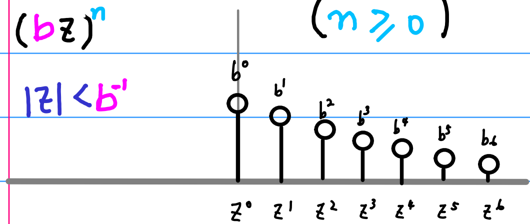
$$|z| < b^{-1}$$

$$X(z) = \frac{b^{-1}}{b^{-1} - z} = \sum_{n=-\infty}^{0} b^{-n} z^{-n}$$
$$= \sum_{n=0}^{\infty} b^n z^n$$

$$x_n = b^{-n}$$
$$= p^n \qquad \boxed{p = b^{-1}}$$

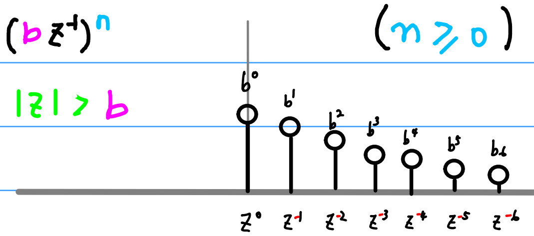
$(bz)^n$ $\quad (n \geq 0)$

$|z| < b^{-1}$



$(bz^{-1})^n$ $\quad (n \geq 0)$

$|z| > b$



$$f(z) = \frac{1}{1 - bz} = \frac{b^{-1}}{b^{-1} - z}$$

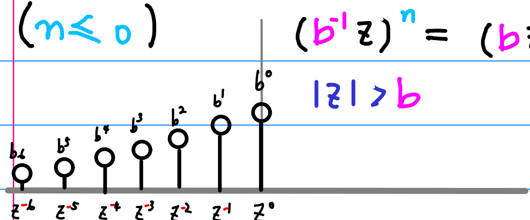
$$X(z) = \frac{1}{1 - b/z} = \frac{z}{z - b}$$

$$a_n = b^n$$
$$= p^{-n} \qquad \boxed{p = b^{-1}}$$

$$x_n = b^n$$
$$= p^n \qquad \boxed{p = b}$$

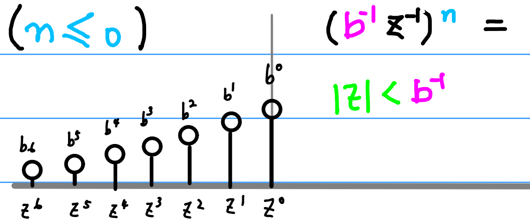
$(n \leq 0)$ $\quad (b^{-1}z)^n = (bz^{-1})^{-n}$

$|z| > b$



$(n \leq 0)$ $\quad (b^{-1}z^{-1})^n = (bz)^{-n}$

$|z| < b^{-1}$

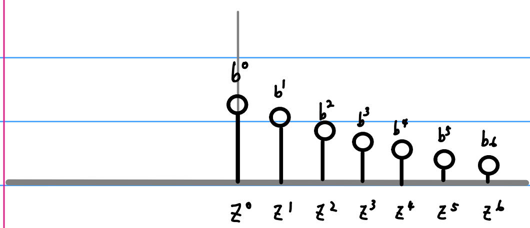


$$f(z) = \frac{1}{1 - (bz^{-1})} = \frac{z}{z - b}$$

$$X(z) = \frac{1}{1 - (bz)} = \frac{b^{-1}}{b^{-1} - z}$$

$$a_n = b^{-n}$$
$$= p^{-n} \qquad \boxed{p = b}$$

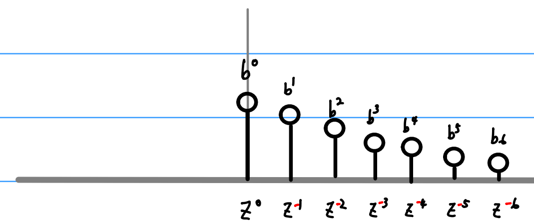
$$x_n = b^{-n}$$
$$= p^n \qquad \boxed{p = b^{-1}}$$



$$f(z) = \sum_{n=0}^{\infty} (bz)^n \qquad |bz| < 1$$

$$a_n = b^n$$

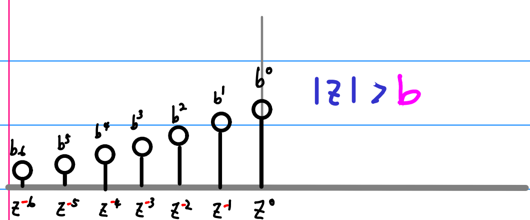
$$= p^{-n} \qquad \boxed{p = b^{-1}}$$



$$X(z) = \sum_{n=0}^{\infty} (bz^{-1})^n \qquad |bz^{-1}| < 1$$

$$x_n = b^n$$

$$= p^n \qquad \boxed{p = b}$$



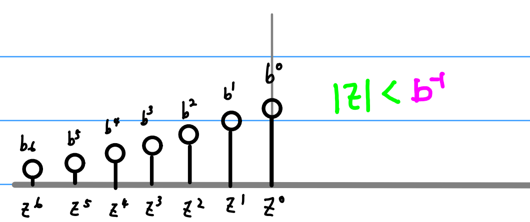
$|z| > b$

$$f(z) = \sum_{n=-\infty}^{0} (bz^{-1})^{-n} \qquad |bz^{-1}| < 1$$

$$= \sum_{n=0}^{\infty} (bz^{-1})^n$$

$$a_n = b^{-n}$$

$$= p^{-n} \qquad \boxed{p = b}$$



$|z| < b^{-1}$

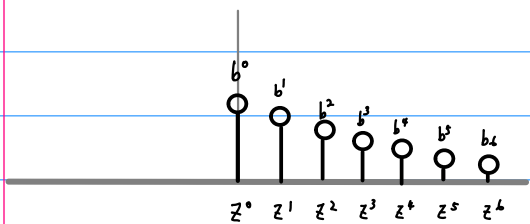
$$X(z) = \sum_{n=-\infty}^{0} (bz)^{-n} \qquad |bz| < 1$$

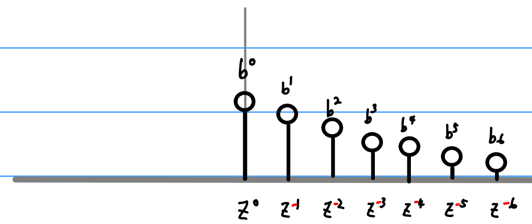
$$= \sum_{n=0}^{\infty} (bz)^n$$

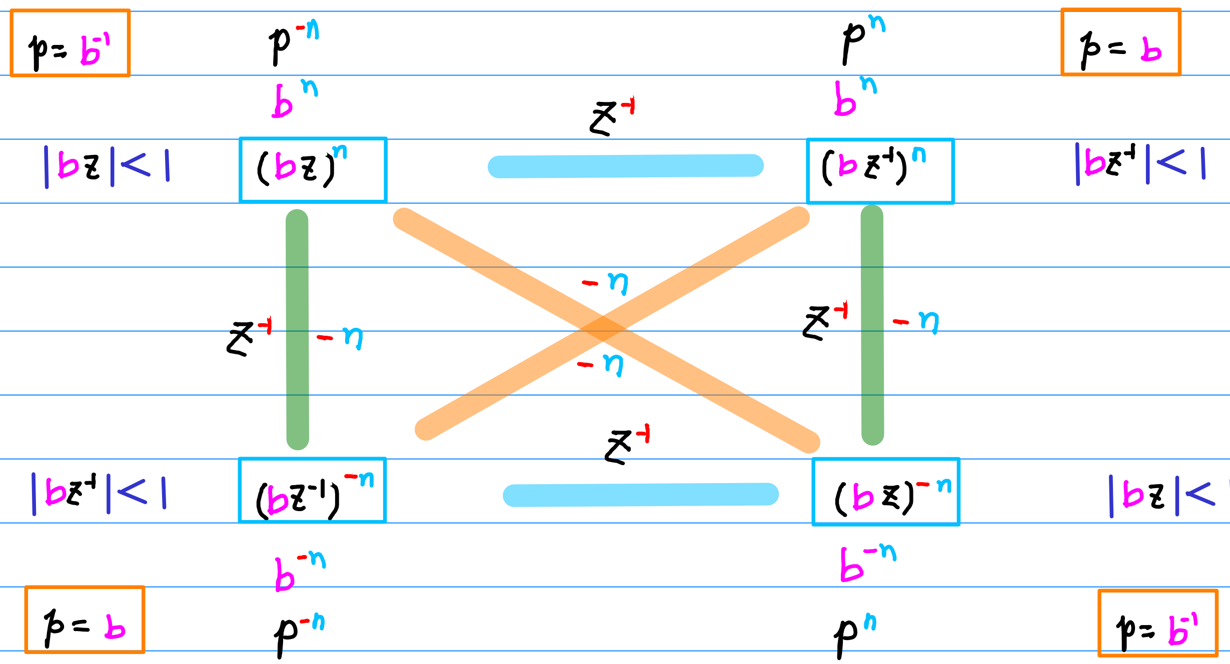
$$x_n = b^{-n}$$

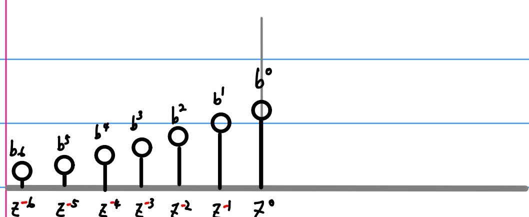
$$= p^n \qquad \boxed{p = b^{-1}}$$

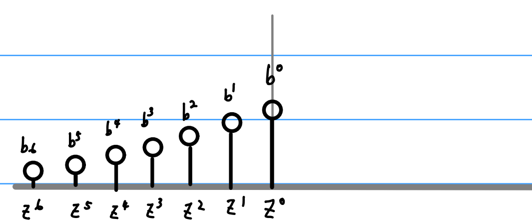
$$\boxed{\frac{1}{1-(bz)} = \frac{b^{-1}}{b^{-1}-z}} \qquad \boxed{\frac{1}{1-(bz^{-1})} = \frac{z}{z-b}}$$



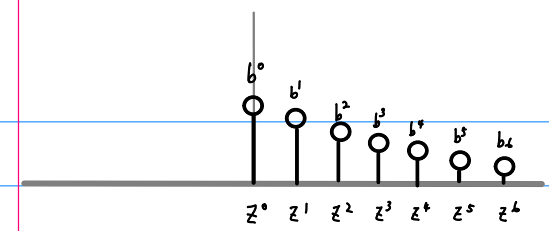


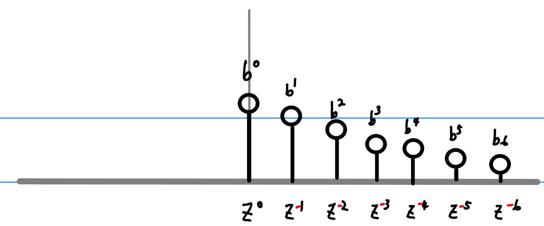






$$\boxed{\frac{1}{1-(bz^{-1})} = \frac{z}{z-b}} \qquad \boxed{\frac{1}{1-(bz)} = \frac{b^{-1}}{b^{-1}-z}}$$



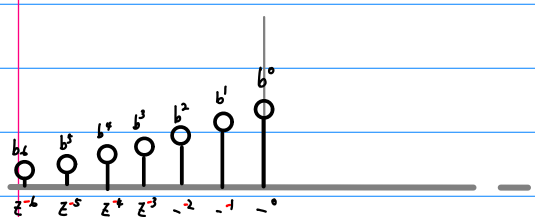


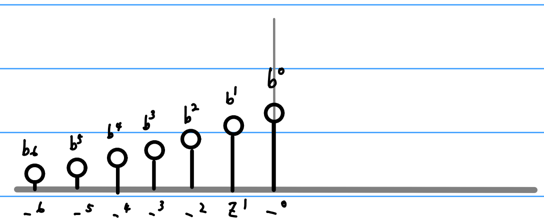
$$\boxed{\frac{1}{1-(bz)} = \frac{b^{-1}}{b^{-1}-z}}$$

$$\boxed{\frac{1}{1-(bz^{-1})} = \frac{z}{z-b}}$$

$$\boxed{\frac{1}{1-(bz^{-1})} = \frac{z}{z-b}}$$

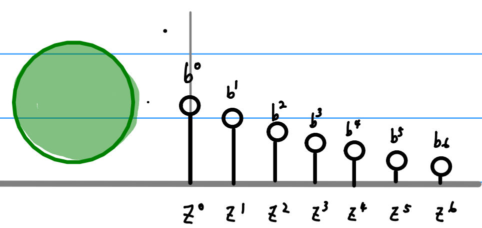
$$\boxed{\frac{1}{1-(bz)} = \frac{b^{-1}}{b^{-1}-z}}$$



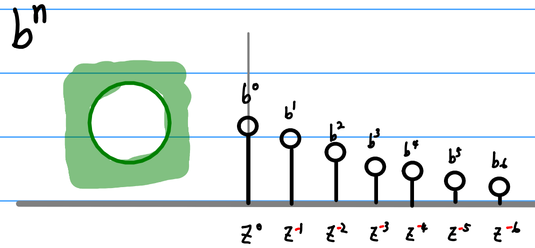


$$x_n = a_{-n} \qquad X(z) = f(z)$$

b^n



b^n



$$f(z) = \frac{1}{1-(bz)} \qquad |z| < b^{-1}$$

$$X(z) = \frac{1}{1-(b/z)} \qquad |z| > b$$

$$\boxed{a_n = b^n \quad (n \geqslant 0)}$$

$$= p^{-n} \qquad (p = b^{-1})$$

$$\boxed{x_n = b^n \quad (n \geqslant 0)}$$

$$= p^n \qquad (p = b)$$

$$f(z) = \frac{1}{1-(b/z)} \qquad |z| > b$$

$$X(z) = \frac{1}{1-(bz)} \qquad |z| < b^{-1}$$

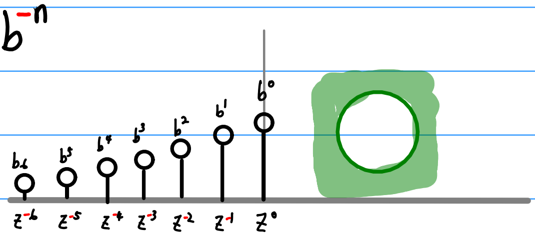
$$\boxed{a_n = b^{-n} \quad (n \leqslant 0)}$$

$$= p^{-n} \qquad (p = b)$$

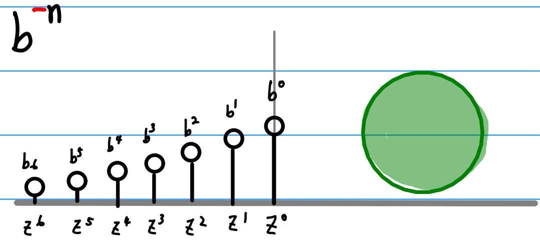
$$\boxed{x_n = b^{-n} \quad (n \leqslant 0)}$$

$$= p^n \qquad (p = b^{-1})$$

b^{-n}

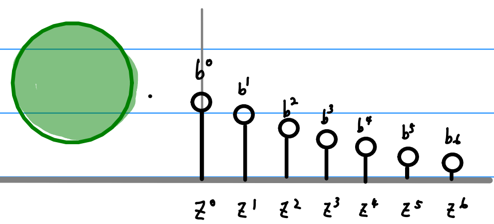


b^{-n}

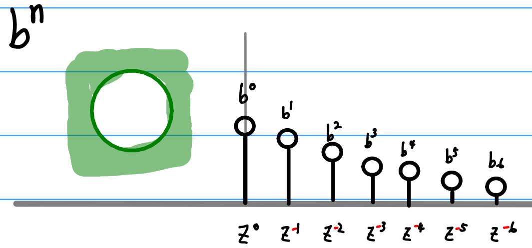


$x_n = a_n \qquad X(z) = f(z^{-1})$

b^n



b^n



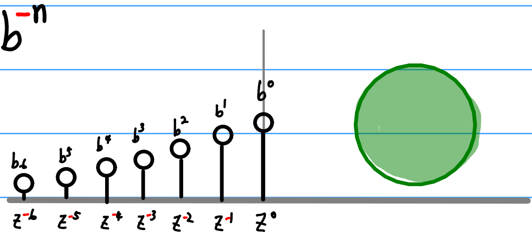
$$\boxed{f(z) = \frac{1}{1-(bz)} \quad |z| < b^{-1}} \Longleftrightarrow \boxed{X(z) = \frac{1}{1-(b/z)} \quad |z| > b}$$

$a_n = b^n \quad (n \geq 0) \qquad x_n = b^n \quad (n \geq 0)$

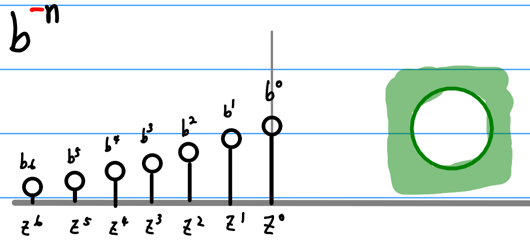
$$\boxed{f(z) = \frac{1}{1-(b/z)} \quad |z| > b} \Longleftrightarrow \boxed{X(z) = \frac{1}{1-(bz)} \quad |z| < b^{-1}}$$

$a_n = b^{-n} \quad (n \leq 0) \qquad x_n = b^{-n} \quad (n \leq 0)$

b^{-n}

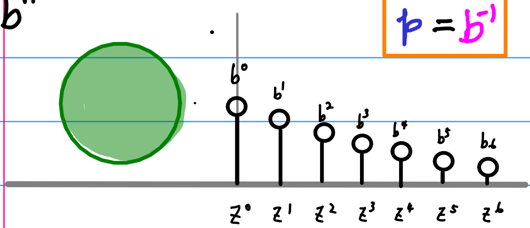


b^{-n}



$$a_n = p^n \qquad\qquad x_n = p^{-n}$$

b^n $\boxed{p = b^{-1}}$



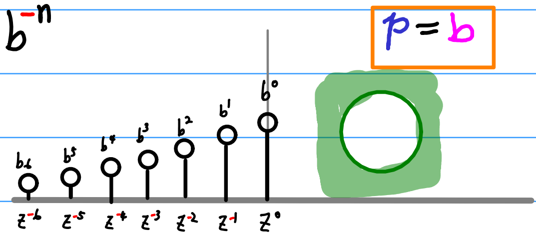
$$f(z) = \frac{1}{1-(bz)} \qquad |z| < b^{-1}$$

$$\boxed{a_n = p^{-n} \quad (n \geq 0)}$$

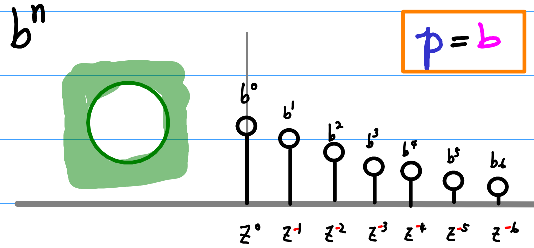
$$f(z) = \frac{1}{1-(b/z)} \qquad |z| > b$$

$$\boxed{a_n = p^{-n} \quad (n \leq 0)}$$

b^{-n} $\boxed{p = b}$



b^n $\boxed{p = b}$



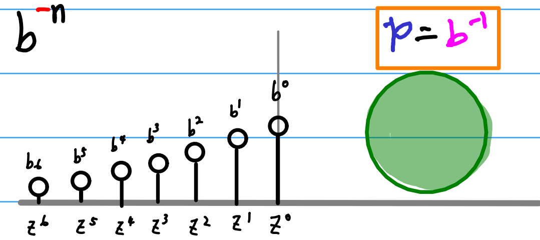
$$X(z) = \frac{1}{1-(b/z)} \qquad |z| > b$$

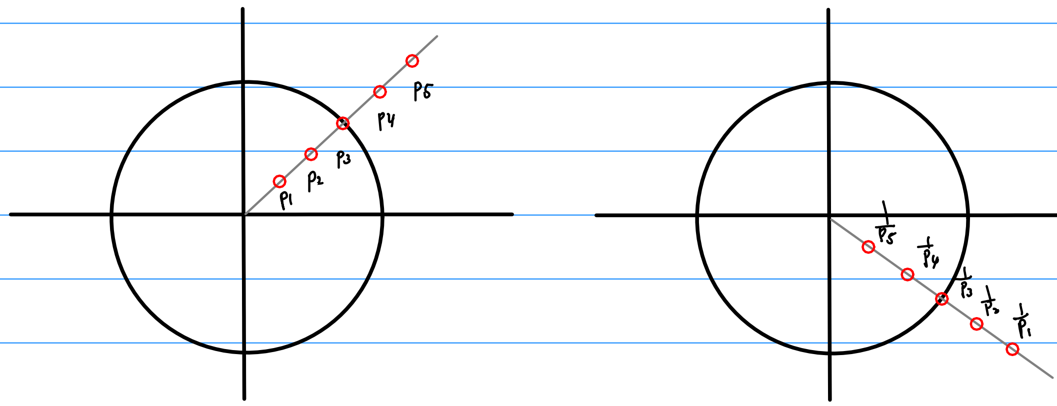
$$\boxed{x_n = p^n \quad (n \geq 0)}$$

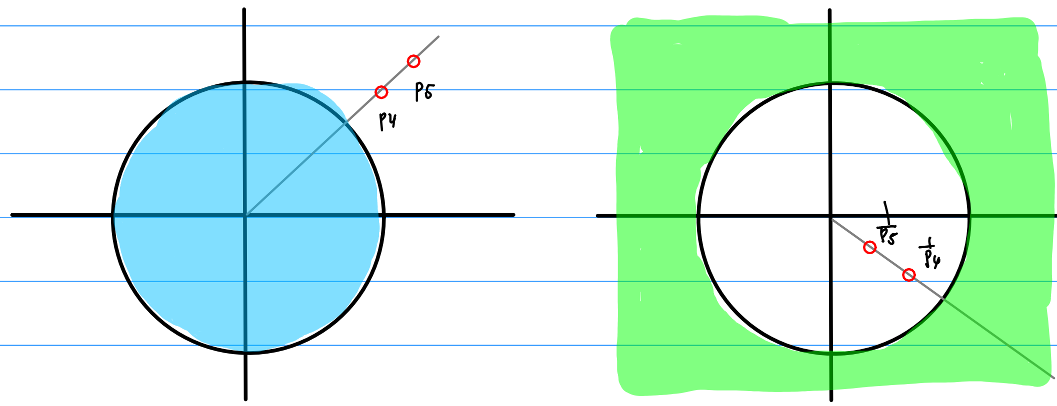
$$X(z) = \frac{1}{1-(bz)} \qquad |z| < b^{-1}$$

$$\boxed{x_n = p^n \quad (n \leq 0)}$$

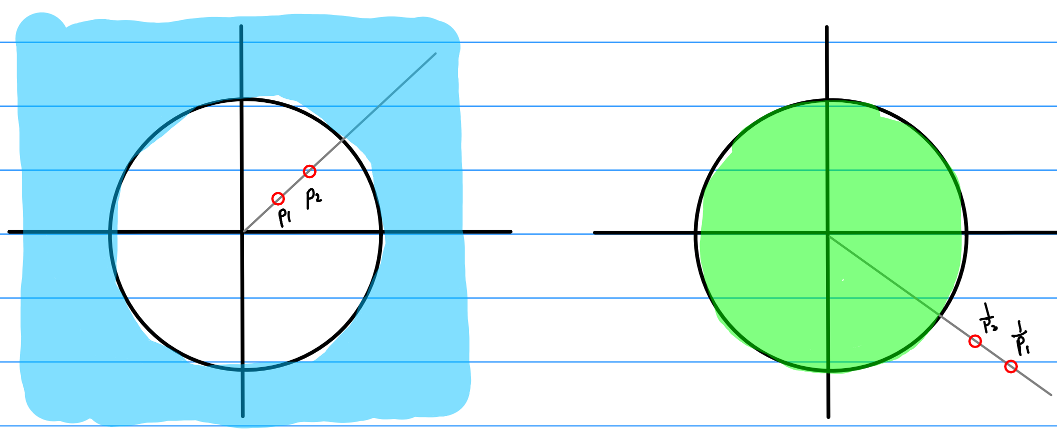
b^{-n} $\boxed{p = b^{-1}}$



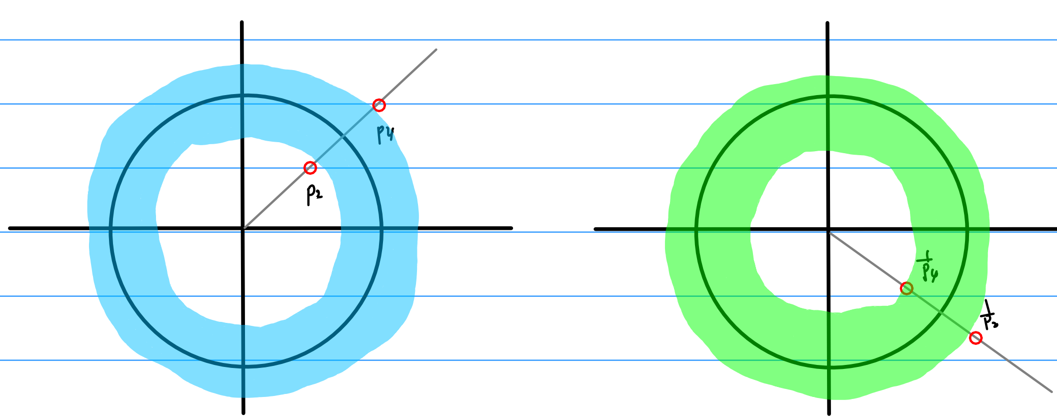




causal



anticausal



two-sided