

# Multiple Random Variables

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles, Jr. and B. Shi

# Outline

- 1 Central Limit Theorem
  - Unequal Distributions
  - Equal Distributions

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# Central Limit Theorem

## Definition

the central limit theorem says that the probability distribution function of the sum of large number of random variables approaches a Gaussian distribution.

This theorem is known to apply some cases of statistically independent random variables.

# Central Limit Theorem

## Unequal Distribution Case

### Definition

the sum  $Y$  of  $N$  independent random variables  $X_1, X_2, \dots, X_N$   
Let  $Y = X_1 + X_2 + \dots + X_N$ , then

$$\bar{Y}_N = \bar{X}_1 + \bar{X}_2 + \dots + \bar{X}_N$$

$$\sigma_{Y_N}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \dots + \sigma_{X_N}^2$$

the probability distribution of  $Y$  asymptotically approaches to Gaussian distribution function as  $N \rightarrow \infty$

# Sufficient Conditions

## Unequal Distribution Case

### Definition

$$\sigma_{X_i}^2 > B_1 > 0 \quad i = 1, 2, \dots, N$$

$$E[|X_i - \bar{X}_i|^3] < B_2 \quad i = 1, 2, \dots, N$$

where  $B_1$  and  $B_2$  are positive numbers

these conditions guarantee that no one random variable in the sum dominates

# Distribution vs density functions

## Unequal Distribution Case

the central limit theorem guarantees

- only that the distribution of the sum of random variables become Gaussian
- the density of the sum of random variables is not always Gaussian
- the sum of continuous random variables :
  - under certain conditions on individual random variables the density of the sum is always Gaussian
- the sum of discrete random variables :
  - the density function may contain impulses and thus is not Gaussian.



# Discrete Random Variable Examples

distribution may contain impulses

the sum  $Y$  of  $N$  independent discrete random variables  $X_1, X_2, \dots, X_N$

- discrete random variable
- density function may contain impulses
- therefore the density function is not Gaussian
- although the distribution approaches Gaussian
  
- when the possible discrete values of each random variable are  $kb, k = 0, \pm 1, \pm 2, \dots$ , where  $b$  is a constant
  - the envelope of the impulses in the density of the sum will be Gaussian
  - with the mean  $Y_N$  and variance  $\sigma_{Y_N}^2$

# Central Limit Theorem

## Equal Distribution Case

### Definition

the sum  $Y$  of  $N$  independent random variables  
 assume that  $X_1, X_2, \dots, X_N$  have the same distribution function.  
 Let  $Y_N = X_1 + X_2 + \dots + X_N$ ,  
 then  $W_N = (Y_N - \bar{Y}_N) / \sigma_{Y_N}$  is  
 the zero-mean, unit-variance random variable

$$W_N = (Y_N - \bar{Y}_N) / \sigma_{Y_N} = \sum_{i=1}^N (X_i - \bar{X}_i) / \left[ \sum_{i=1}^N \sigma_{\bar{X}_i}^2 \right]^{1/2}$$

$$W_N = \frac{1}{\sqrt{N} \sigma_X} \sum_{i=1}^N (X_i - \bar{X}_i)$$

where  $X_i = \bar{X}$  and  $\sigma_{\bar{X}_i}^2 = \sigma_{\bar{X}}$

# Characteristic Function(1)

## Equal Distribution Case

the characteristic function of  $W_N$   
a zero mean, unit variance Gaussian random variable

$$\Phi_{W_N}(\omega) = \exp(-\omega^2/2)$$

$W_N$  is the density of the Gaussian random variable  
Fourier transforms are unique

$$\begin{aligned}\Phi_{W_N}(\omega) &= E[e^{j\omega W_N}] \\ &= E \left[ \exp \left( \frac{j\omega}{\sqrt{N}\sigma_X} \sum_{i=1}^N (X_i - \bar{X}) \right) \right] \\ &= \left\{ E \left[ \exp \left( \frac{j\omega}{\sqrt{N}\sigma_X} (X_i - \bar{X}) \right) \right] \right\}^N\end{aligned}$$

## Characteristic Function (2)

Equal Distribution Case

$$\begin{aligned} & E \left[ \exp \left( \frac{j\omega}{\sqrt{N}\sigma_X} (X_i - \bar{X}) \right) \right] \\ &= E \left[ 1 + \frac{j\omega}{\sqrt{N}\sigma_X} (X_i - \bar{X}) + \left( \frac{j\omega}{\sqrt{N}\sigma_X} \right)^2 (X_i - \bar{X})^2 + \frac{R_N}{N} \right] \\ &= 1 - \frac{\omega^2}{2N} + \frac{E[R_N]}{N} \end{aligned}$$

where  $E[R_N]$  approaches zero as  $N \rightarrow \infty$

## Characteristic Function (3)

## Equal Distribution Case

$$\ln[\Phi_{W_N}(\omega)] = N \ln \left[ 1 - \frac{\omega^2}{2N} + \frac{E[R_N]}{N} \right]$$

$$\ln[1 - z] = - \left[ z + \frac{z^2}{2} + \frac{z^3}{3} + \dots \right], |z| < 1$$

$$\ln[\Phi_{W_N}(\omega)] = -\frac{\omega^2}{2} + E[R_N] - \frac{N}{2} \left[ \frac{\omega^2}{2N} + \frac{E[R_N]}{N} \right]^2 + \dots$$

$$\ln[\Phi_{W_N}(\omega)] = -\frac{\omega^2}{2} + E[R_N] - \frac{N}{2} \left[ \frac{\omega^2}{2N} + \frac{E[R_N]}{N} \right]^2 + \dots$$

$$\lim_{N \rightarrow \infty} \ln[\Phi_{W_N}(\omega)] = \ln \left[ \lim_{N \rightarrow \infty} \Phi_{W_N}(\omega) \right] = -\frac{\omega^2}{2}$$

$$\lim_{N \rightarrow \infty} \Phi_{W_N}(\omega) = e^{-\frac{\omega^2}{2}}$$