Characteristics of Multiple Random Variables

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June 22, 2020

Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Linear Transform
N Gaussian random variables

Definition

linear transform of Gaussian random variables $X_1, X_2, \cdots, X_N$

$$\begin{array}{lllll} Y_1 & = a_{11}X_1 & + a_{12}X_2 & + \cdots & + a_{1N}X_N \\ Y_2 & = a_{21}X_1 & + a_{22}X_2 & + \cdots & + a_{2N}X_N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_N & = a_{N1}X_1 & + a_{N2}X_2 & + \cdots & + a_{NN}X_N \end{array}$$

$$[Y] = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix}, \; [T] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, \; [X] = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$$

Linear Transform
N Gaussian random variables

Corollary

linear transform of Gaussian random variables $X_1, X_2, \cdots, X_N$

$$[Y] = [T][X], \qquad [Y - \overline{Y}] = [T][X - \overline{X}]$$

$$[X] = [T]^{-1}[Y], \qquad [X - \overline{X}] = [T]^{-1}[Y - \overline{Y}]$$

$$[Y] = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix}, \; [\overline{Y}] = \begin{bmatrix} \overline{Y_1} \\ \overline{Y_2} \\ \vdots \\ \overline{Y_N} \end{bmatrix}, \; [X] = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \; [\overline{X}] = \begin{bmatrix} \overline{X_1} \\ \overline{X_2} \\ \vdots \\ \overline{X_N} \end{bmatrix}$$

$[T]$ *is non-singular*

Linear Transform
N Gaussian random variables

Corollary

linear transform of Gaussian random variables $X_1, X_2, \cdots, X_N$

$$X_i = T_i^{-1}(Y_1, \cdots, Y_N) = a^{i1}Y_1 + \cdots + a^{iN}Y_N$$

$$\frac{\partial X_i}{\partial Y_j} = \frac{\partial T_i^{-1}}{\partial Y_j} = a^{ij}$$

$$X_i - \overline{X_i} = a^{i1}(Y_1 - \overline{Y_1}) + \cdots + a^{iN}(Y_N - \overline{Y_N})$$

a^{ij} *denotes the* $ij-th$ *element of* $[T]^{-1}$

Using Jacobian (1)
N Gaussian random variables

Corollary

linear transform of Gaussian random variables $X_1, X_2, \cdots, X_N$

$$|J| = \left\|[T]^{-1}\right\| = \frac{1}{\|[T]\|}$$

$$C_{X_i X_j} = E[(X_i - \overline{X_i})(X_j - \overline{X_j})]$$

$$= \sum_{k=1}^{N} a^{ik} \sum_{m=1}^{N} a^{im} E[(Y_k - \overline{Y_k})(Y_m - \overline{Y_m})]$$

$$= \sum_{k=1}^{N} a^{ik} \sum_{m=1}^{N} a^{im} C_{Y_k Y_m}$$

Using Jacobian (2)

N Gaussian random variables

Corollary

$C_{X_i X_j}$ is the ij-th element of the covariance matrix $[C_X]$ and $C_{Y_k Y_m}$ is the km-th element of the covariance matrix $[C_Y]$

$$[C_X] = [T]^{-1}[C_Y]([T]^t)^{-1}$$

$$[C_X]^{-1} = [T]^t[C_Y]^{-1}[T]$$

$$\left\|[C_X]^{-1}\right\| = \left\|[C_Y]^{-1}\right\| [T]^2$$

Using Jacobian (3)

N Gaussian random variables

Corollary

$$f_{X_1 \cdots X_N}(x_1 = T_1^{-1}, \cdots, x_N = T_N^{-1})$$

$$= \frac{\|[T]\| \, \left\|[C_Y]^{-1}\right\|^{1/2}}{(2\pi)^{N/2}} exp\left\{ -\frac{[x - \overline{X}][T]^t[C_Y]^{-1}[T][x - \overline{X}]}{2} \right\}$$

Using Jacobian (4)
N Gaussian random variables

Corollary

$$f_{Y_1 \cdots Y_N}(x_1 = T_1^{-1}, \cdots, x_N = T_N^{-1})$$

$$= \frac{\left\| [C_Y]^{-1} \right\|^{1/2}}{(2\pi)^{N/2}} exp\left\{ -\frac{[y - \overline{Y}]^t [C_Y]^{-1} [y - \overline{Y}]}{2} \right\}$$

Using Jacobian (5)

N Gaussian random variables

Corollary

a linear transformation of guassian random variables produces gaussian random variables
the new variables have the mean values

$$\overline{Y_j} = \sum_{k=1}^{N} a_{jk} \overline{X_k}$$

and covariance given by the elements of the covariance matrix

$$[C_Y] = [T][C_X][T]^t$$