Sampling

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Duality

- A periodic signal in time domain has the effect of sampling its spectrum in frequency domain.
- Sampling a time domain signal has the effect of making its spectrum periodic in frequency domain.

Intantaneous Sampling

- a uniform rate T_s seconds
- an infinite sequence of samples $\{g(nT_s)\}$
- mulitply $g(t)$ by the impuse train $\sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

ideal sampled signal

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s)$$

- instantaneous sampling
- natural sampling
- flat top sampling

A Periodic Spectrum : DTFT

DTFT time domain : a sampled signal

$$g_\delta(t) = [g(t)]\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right]$$ Multiplying an impulse train

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s)$$

DTFT frequency domain : a periodic spectrum

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s)exp(-j2\pi nT_s f)$$ Fourier transform of $g_\delta(t)$

$$G_\delta(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s)$$ Replicating $G(f)$, but why?

$G(f)$: Fourier transform of $g(t)$ $\qquad g(t) \leftrightarrows G(f)$

$f_s = 1/T_s$: the sampling rate

a periodic spectrum with a repetition frequency equal to the sampling rate

A Periodic Signal : CTFS

CTFS time domain : a periodic signal

$$g_{T_0}(t) = \sum_{m=-\infty}^{\infty} g(t - mT_s)$$ A periodic signal

$$g_{T_0}(t) = f_s \sum_{n=-\infty}^{\infty} G(nf_s) exp(+j2\pi n f_s t)$$ Fourier series exapansion

CTFS frequency domain : a sampled spectrum

$$c_n = f_s G(nf_s)$$

$G(f)$: Fourier transform of $g(t)$ $\quad g(t) \leftrightarrows G(f)$

$f_s = 1/T_s$: the sampling rate

Note the duality between DTFS and CTFS

A periodic signal $g_{T_o}(t)$ and a periodic spectrum $G_\delta(f)$

CTFS time domain : a periodic signal

$$g_{T_0}(t) = \sum_{m=-\infty}^{\infty} g(t - mT_s) = f_s \sum_{n=-\infty}^{\infty} G(nf_s)exp(+j2\pi nf_s t)$$

A Fourier series expansion of a periodic signal

DTFT frequency domain : a periodic spectrum

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s)exp(-j2\pi nT_s f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s))$$

Can prove through the duality property

DTFT and CTFT

The Fourier Transform of a Continuous Signal

$$G(f) = \int_{-\infty}^{+\infty} g(t) exp(-j2\pi f t) dt$$
$$g(t) = \int_{-\infty}^{+\infty} G(f) exp(+j2\pi f t) df$$

The Fourier Transform of a Sampled Signal

$$G_\delta(f) = \int_{-\infty}^{+\infty} g_\delta(t) exp(-j2\pi f t) dt$$
$$g_\delta(t) = \int_{-\infty}^{+\infty} G_\delta(f) exp(+j2\pi f t) df$$

$$G_\delta(f) = \int_{-\infty}^{+\infty} \left[\sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s) \right] exp(-j2\pi f t) dt$$
$$= \sum_{n=-\infty}^{\infty} g(nT_s) \int_{-\infty}^{+\infty} [\delta(t - nT_s) exp(-j2\pi f t)] dt$$
$$= \boxed{\sum_{n=-\infty}^{\infty} g(nT_s) exp(-j2\pi f nT_s)} \qquad \text{(Discrete Time Fourier Transform)}$$

Other Definitions of DTFT

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s) exp(-j2\pi f nT_s)$$

$$G_\delta(j\omega) = \sum_{n=-\infty}^{\infty} g(nT_s) exp(-j\omega nT_s) = \sum_{n=-\infty}^{\infty} g[n] exp(-j\omega nT_s)$$

$\hat{\omega} = \omega T_s$ (normalized radian frequency)

$$G(j\hat{\omega}) = \sum_{n=-\infty}^{\infty} g[n] exp(-j\hat{\omega} n)$$

Duality Between CTFS and DTFT

Continuous Time Fourier Series (CTFS)

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} c_n exp(+j2\pi n f_0 t)$$

$g_{T_0}(t)$: a periodic function in the time domain
c_n: a sampled function in the frequency domain

Discrete Time Fourier Transform (DTFT)

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s) exp(-j2\pi n T_s f)$$

$G_\delta(f)$: a periodic function in the frequuency domain
$g(nT_s)$: a sampled function in the time domain

Periodic functions: $g_{T_0}(t) \longleftrightarrow G_\delta(f)$
Sampled functions: $c_{nf_0} \longleftrightarrow g(nT_s)$

CTFS and CTFT : Fourier Coefficients

Fourier Series

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} c_n exp(+j2\pi n f_0 t)$$

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} g_{T_0}(t) exp(-j2\pi n f_0 t) dt$$

$g(t)$ the one period portion of $g_{T_0}(t)$, at the origin
then there exists Fourier transform $g(t) \leftrightarrows G(f)$

$$g(t) = 0, (t < -\frac{T_0}{2}, t > +\frac{T_0}{2})$$

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} g(t) exp(-j2\pi n f_0 t) dt = f_0 G(n f_0)$$

$$\boxed{c_n = f_0 G(n f_0)}$$

Fourier Transform

$$G(f) = \int_{-\infty}^{+\infty} g(t) exp(-j2\pi f t) dt$$

$$g(t) = \int_{-\infty}^{+\infty} G(f) exp(+j2\pi f t) df$$

Fourier Series Expansion of $g_{T_0}(t)$

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} g(t) exp(-j2\pi n f_0 t) dt = f_0 G(nf_0)$$

The periodic signal $g_{T_0}(t)$ from the Fourier coeffiecients c_n

$c_n = f_0 G(nf_0)$: sampling in the frequency domain

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} c_n exp(+j2\pi n f_0 t) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0) exp(+j2\pi n f_0 t)$$

The periodic signal $g_{T_0}(t)$ obtained by replicating $g(t)$

$g_{T_0}(t) = \sum_{m=-\infty}^{\infty} g(t - mT_0)$: replication in the time domain

$$\sum_{m=-\infty}^{\infty} g(t - mT_0) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0) exp(+j2\pi n f_0 t)$$

CTFT of a periodic signal $g_{T_0}(t)$

Fourier Series

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} c_n exp(+j2\pi n f_0 t)$$

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} g_{T_0}(t) exp(-j2\pi n f_0 t) dt$$

$g(t)$ the one period portion of $g_{T_0}(t)$, at the origin
then there exists Fourier transform $g(t) \leftrightarrows G(f)$

$$\sum_{m=-\infty}^{\infty} g(t - mT_0) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0) exp(+j2\pi n f_0 t)$$

Also, note $exp(j2\pi f_c t) \leftrightarrows \delta(f - f_c)$

Fourier Transform of a Periodic Signal

$$\sum_{m=-\infty}^{\infty} g(t - mT_0) \leftrightarrows f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f - nf_0)$$

A Periodic Signal Summary: CTFS & CTFT

CTFS time domain : a periodic signal

$$g_{T_0}(t) = \sum_{m=-\infty}^{\infty} g(t - mT_s)$$ A periodic signal

$$g_{T_0}(t) = f_s \sum_{n=-\infty}^{\infty} G(nf_s) exp(+j2\pi n f_s t)$$ Fourier series exapansion

CTFS frequency domain : a sampled spectrum

$$c_n = f_s G(nf_s)$$

CFFT of a periodic signal

$$T_0 \sum_{m=-\infty}^{\infty} g(t - mT_0) = \sum_{n=-\infty}^{\infty} G(nf_0) exp(+j2\pi n f_0 t)$$

$$\leftrightarrows \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f - nf_0)$$: sampling in the frequency domain

Duality (Time Shift $\Rightarrow$ Frequency Shift)

$$\boxed{g(t-t_0) \leftrightarrows G(f)exp(-j2\pi f t_0)}$$

$[h(t) = g(t-t_0)]$, $[H(f) = G(f)exp(-j2\pi f t_0)]$ $\qquad h(t) \rightleftarrows H(f)$

change the order (the left and right hand sides)

$[H(f) = G(f)exp(-j2\pi f t_0)]_{f \leftarrow -t}$, $[h(t) = g(t-t_0)]_{t \leftarrow f}$ $\qquad H(-t) \rightleftarrows h(f)$

change t_0 and $f_0$with each other

$[G(-t)exp(+j2\pi f_0 t)]$, $[g(f-f_0)]$ $\qquad G(-t) \rightleftarrows g(f)$

change $g()$ and $G()$:

$G(-t) \leftarrow g(t)$, $g(f-f_0) \leftarrow G(f-f_0)$ $\qquad g(t) \rightleftarrows G(f)$

$$\boxed{g(t)exp(+j2\pi f_0 t) \leftrightarrows G(f-f_0)}$$

Duality (Frequency Shift $\Rightarrow$ Time Shift)

$$\boxed{g(t)exp(+j2\pi f_0 t) \leftrightarrows G(f-f_0)}$$

$[h(t) = g(t)exp(+j2\pi f_0 t)]$, $[H(f) = G(f-f_0)]$ $\qquad h(t) \rightleftarrows H(f)$

change the order (the left and right hand sides)

$[H(f) = G(f-f_0)]_{f \leftarrow t}$, $[h(t) = g(t)exp(+j2\pi f_0 t)]_{t \leftarrow -f}$ $\qquad H(t) \rightleftarrows h(-f)$

change t_0 and $f_0$with each other

$[G(t-t_0)]$, $[g(-f)exp(-j2\pi t_0 f)]$ $\qquad G(t) \rightleftarrows g(-f)$

change $g()$ and $G()$:

$G(t-t_0) \leftarrow g(t-t_0)$, $g(-f) \leftarrow G(f)$ $\qquad g(t) \rightleftarrows G(f)$

$$\boxed{g(t-t_0) \leftrightarrows G(f)exp(-j2\pi f t_0)}$$

Duality (Frequecy Sampling⇒ Time Sampling)

$$\sum_{m=-\infty}^{\infty} g(t-mT_0) \leftrightarrows f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f-nf_0)$$ Frequency Sampling

$$\left[h(t) = \sum_{m=-\infty}^{\infty} g(t-mT_0)\right], \left[H(f) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f-nf_0)\right]$$

change the order (the left and right hand sides) $h(t) \rightleftarrows H(f)$

$$\left[f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f-nf_0)\right]_{f\leftarrow t}, \left[\sum_{m=-\infty}^{\infty} g(t-mT_0)\right]_{t\leftarrow -f} \quad H(t) \rightleftarrows h(-f)$$

change T_0 and f_0 with each other

$$\left[\sum_{n=-\infty}^{\infty} G(nT_0)\delta(t-nT_0)\right], \left[f_0 \sum_{m=-\infty}^{\infty} g(-f-mf_0)\right] \quad G(t) \rightleftarrows g(-f)$$

$$G(nT_0) \leftarrow g(nT_0),\ g(-f-mf_0) \leftarrow G(f-mf_0) \quad g(t) \rightleftarrows G(f)$$

$$\sum_{n=-\infty}^{\infty} g(nT_s)\delta(t-nT_s) \leftrightarrows f_s \sum_{m=-\infty}^{\infty} G(f-mf_s)$$ Time Sampling

Duality (Time Sampling$\Rightarrow$ Frequency Sampling)

$$\sum_{n=-\infty}^{\infty} g(nT_s)\delta(t-nT_s) \leftrightarrows f_s \sum_{m=-\infty}^{\infty} G(f-mf_s) \qquad \text{Time Sampling}$$

$$\left[h(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t-nT_s)\right], \left[H(f) = f_s \sum_{m=-\infty}^{\infty} G(f-mf_s)\right]$$

$h(t) \rightleftarrows H(f)$

change the order (the left and right hand sides)

$$\left[f_s \sum_{m=-\infty}^{\infty} G(f-mf_s)\right]_{f\leftarrow -t}, \left[\sum_{n=-\infty}^{\infty} g(nT_s)\delta(t-nT_s)\right]_{t\leftarrow f} \qquad H(-t) \rightleftarrows h(f)$$

change T_0 and $f_0$with each other

$$\left[\sum_{m=-\infty}^{\infty} G(-t-mT_s)\right], \left[f_s \sum_{n=-\infty}^{\infty} g(nf_s)\delta(f-nf_s)\right] \qquad G(-t) \rightleftarrows g(f)$$

$G(-t-mT_0) \leftarrow g(t-mT_0)$, $g(nf_s) \leftarrow G(nf_s)$ $\qquad g(t) \rightleftarrows G(f)$

$$\sum_{m=-\infty}^{\infty} g(t-mT_0) \leftrightarrows f_s \sum_{n=-\infty}^{\infty} G(nf_s)\delta(f-nf_s) \qquad \text{Frequency Sampling}$$

Sampling Duality

periodic in the time domain: fundamental period $T_0 = 1/f_0$

$$T_0 \sum_{m=-\infty}^{\infty} g(t - mT_0) = \sum_{n=-\infty}^{\infty} G(nf_0) exp(+j2\pi nf_0 t)$$

$$\leftrightarrows \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f - nf_0) \quad : \text{ sampling in the frequency domain}$$

periodic in the frequency domain : sampling rate $f_s = 1/T_s$

$$\sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s) \quad : \text{ sampling in the time domain}$$

$$\leftrightarrows f_s \sum_{m=-\infty}^{\infty} G(f - mf_s) = \sum_{n=-\infty}^{\infty} g(nT_s) exp(-j2\pi nT_s f)$$

Analysis (Time Shift $\Rightarrow$ Frequency Shift) (1)

$$g(t - t_0) \leftrightarrows G(f) exp(-j2\pi f t_0)$$

$$G(-t) exp(+j2\pi t f_0) \rightleftarrows g(f - f_0)$$

$$G(-t) exp(+j2\pi t f_0)$$

$$= \left[\int_{-\infty}^{+\infty} [g(f)] \, exp(+j2\pi f t) df \right] exp(+j2\pi f_0 t) \qquad (G(-t) \rightleftarrows g(f))$$

$$= \int_{-\infty}^{+\infty} g(f) exp(+j2\pi (f + f_0) t) df \qquad (\nu = f + f_0)$$

$$= \int_{-\infty}^{+\infty} g(\nu - f_0) exp(+j2\pi \nu t) d\nu$$

$$[G(-t) exp(+j2\pi t f_0)] = \int_{-\infty}^{+\infty} [g(f - f_0)] \, exp(+j2\pi f t) df$$

$$g(t) exp(+j2\pi f_0 t) \leftrightarrows G(f - f_0)$$

Analysis (Time Shift $\Rightarrow$ Frequency Shift) (2)

$$g(t-t_0) \leftrightarrows G(f)exp(-j2\pi f t_0)$$

$$\int_{-\infty}^{+\infty} [G(-t)exp(+j2\pi t f_0)]\, exp(-j2\pi f t) dt = [g(f-f_0)]$$

$$\int_{-\infty}^{+\infty} G(-t)exp(-j2\pi t(f-f_0)) dt = [g(f-f_0)] \qquad (\nu = f - f_0)$$

$$\int_{-\infty}^{+\infty} G(-t)exp(-j2\pi t \nu)) dt = g(\nu) \qquad (-t = \tau, \nu = t)$$

$$\int_{-\infty}^{+\infty} G(\tau)exp(+j2\pi \tau t) d\tau = g(t) \qquad (\tau = f - f_0)$$

$$\int_{-\infty}^{+\infty} G(f-f_0)exp(+j2\pi(f-f_0)t) df = g(t)$$

$$\left[\int_{-\infty}^{+\infty} G(f-f_0)exp(+j2\pi f t) df\right] exp(-j2\pi f_0 t) = g(t)$$

$$\int_{-\infty}^{+\infty} [G(f-f_0)]\, exp(+j2\pi f t) df = [g(t)exp(+j2\pi f_0 t)]$$

$$g(t)exp(+j2\pi f_0 t) \leftrightarrows G(f-f_0)$$

CTFS, CTFT, and DTFT

CTFS $\quad c_n = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} g_{T_0}(t) exp(-j2\pi n f_0 t) dt$

CTFT $\quad G(f) = \int_{-\infty}^{+\infty} g(t) exp(-j2\pi f t) dt$

DTFT $\quad G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s) exp(-j2\pi n T_s f)$

Continuous Time Fourier Series (CTFS)

- $g_{T_0}(t)$: a continuous time periodic function
- c_n: a discrete frequency function

Continuous Time Fourier Transform (CTFT)

- $g(t)$: a continuous time aperiodic function
- $G(f)$: a continuous frequency aperiodic function

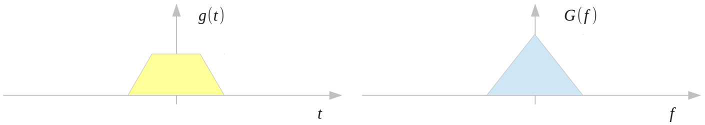
Discrete Time Fourier Transform (DTFT)

- $g(nT_s)$: a discrete time (sampled) aperiodic function
- $G_\delta(f)$: a continuous frequency periodic function

CTFT: Continuous Time Fourier Transform

CTFT

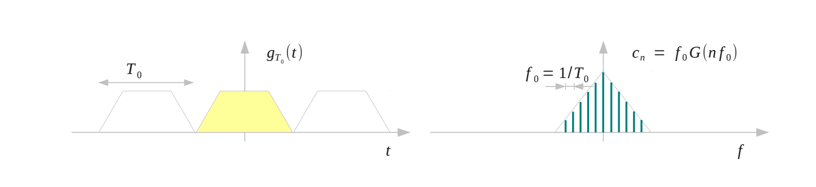
$$G(f) = \int_{-\infty}^{+\infty} g(t) exp(-j2\pi f t) dt$$



CTFS: Continuous Time Fourier Series

CTFS

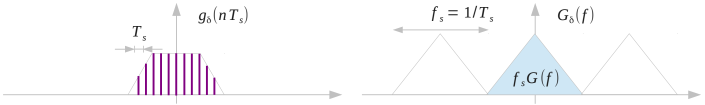
$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} g_{T_0}(t) exp(-j2\pi n f_0 t) dt$$



DTFT: Discrete Time Fourier Transform

DTFT

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s) exp(-j2\pi nT_s f)$$



Fourier Series of an Impulse Train

Fourier Series

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} c_n exp(+j2\pi n f_0 t)$$

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} g_{T_0}(t) exp(-j2\pi n f_0 t) dt$$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} \delta(t) exp(-j2\pi n f_0 t) dt = f_0$$

Two Representations of an Impulse Train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

$$p(t) = f_0 \sum_{n=-\infty}^{\infty} exp(+j2\pi n f_0 t)$$

Fourier Transform of an Impulse Train

Two Representations of an Impulse Train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \rightleftarrows P(f) = \sum_{n=-\infty}^{\infty} exp(+j2\pi nT_0 f)$$

$$p(t) = f_0 \sum_{n=-\infty}^{\infty} exp(+j2\pi nf_0 t) \rightleftarrows P(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

$$p(t) \leftrightarrows P(f)$$

$$exp(+j2\pi nf_0 t) \rightleftarrows \delta(f - nf_0)$$

$$P(f) = \int_{-\infty}^{+\infty} p(t) exp(-j2\pi f t) dt$$

$$P(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

Fourier Transform

$$G(f) = \int_{-\infty}^{+\infty} g(t) exp(-j2\pi f t) dt$$

$$g(t) = \int_{-\infty}^{+\infty} G(f) exp(+j2\pi f t) df$$

Other Derivation of DTFT

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \qquad \text{(Method I)}$$

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_0)\delta(t - nT_0)$$

$$\delta(t - nT_0) \rightleftarrows exp(-j2\pi f nT_0)$$

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s) exp(-j2\pi f nT_s)$$

$$p(t) = f_0 \sum_{n=-\infty}^{\infty} exp(+j2\pi n f_0 t) \qquad \text{(Method II)}$$

$$g_\delta(t) = g(t) \sum_{n=-\infty}^{\infty} \frac{1}{T_0} exp(+j2\pi n f_0 t)$$

$$g(t)p(t) \rightleftarrows G(f) \star P(f)$$

$$G_\delta(f) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} G(f - nf_0)$$

Sampling Theorem

Strictly band-limited signal $g(t)$: $G(f) = 0$ for $|f| \geq W$

- Sampling Period $\quad T_s \leq 1/2W \quad$ $max\ T_s = 1/2W$
- Sampling Frequency $\quad f_s \geq 2W \quad$ $min\ f_s = 2W$

$g_\delta(t) \rightleftarrows G_\delta(f)$

$$\boxed{\sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s)} \rightleftarrows \boxed{\sum_{n=-\infty}^{\infty} g(nT_s)exp(-j2\pi nT_s f)} \qquad \text{(DTFT)}$$

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right)exp\left(-j\frac{\pi nf}{W}\right)$$

: *Fourier Transform of the sequence* $g(nT_s) = g(n/2W)$

Discrete Fourier Transform

Discrete Time Fourier Transform (DTFT)

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) exp\left(-j\frac{\pi n f}{W}\right)$$

$$f = k \times 2W/N$$

- $g_n = g\left(\frac{n}{2W}\right) = g(nT_s)$
- $G_k = G_\delta\left(\frac{2kW}{N}\right) = G_\delta\left(\frac{k}{NT_s}\right)$

Discrete Fourier Transform (DFT)

$$G_k = \sum_{n=0}^{N-1} g_n exp\left(-j\frac{2\pi}{N} nk\right)$$

Recovering $G(f)$ from $G_\delta(f)$

$$G_\delta(f) = f_s \sum_{m=-\infty}^{\infty} G(f - mf_s)$$

- $f_s = 2W$
- $G(f) = 0$ for $|f| \geq W$

- $$G_\delta(f) = 2W \sum_{m=-\infty}^{\infty} G(f - mf_s)$$
  $$= 2W\left[G(f) + G(f \pm f_s) + G(f \pm 2f_s) + \cdots\right]$$

- $G(f) = \frac{1}{2W} G_\delta(f) \qquad -W < f < +W$

Ideal Low Pass Filter

The knowledge of the all the sample values of $g(n/2W)$
can determine the Fourier Transform $G(f)$ with the scaling factor $1/2W$
by the DFT $G_\delta(f)$ with a focus to the interval $-W < f < +W$
: the sampled sequence $g(n/2W)$ contains all the information of $g(t)$

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(\frac{n}{2W}) exp(-j\frac{\pi n f}{W})$$

$$G(f) = \frac{1}{2W} G_\delta(f) \qquad -W < f < +W$$

Ideal Low Pass Filter

$$G(f) = H(f) G_\delta(f) = \frac{1}{2W} G_\delta(f) \qquad -W < f < +W$$

$$G(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} g(\frac{n}{2W}) exp(-j\frac{\pi n f}{W}) \qquad -W < f < +W$$

Reconstruction $g(t)$ from $\{g(n/2W)\}$

$$g(t) = \int_{-\infty}^{+\infty} [G(f)]\, exp(j2\pi f t) df$$

$$= \int_{-W}^{+W} \left[\frac{1}{2W} \sum_{n=-\infty}^{\infty} g(\tfrac{n}{2W}) exp(-j\tfrac{\pi n f}{W}) \right] exp(j2\pi f t) df$$

$$= \sum_{n=-\infty}^{\infty} g(\tfrac{n}{2W}) \left\{ \frac{1}{2W} \int_{-W}^{+W} exp[j2\pi f(t - \tfrac{n}{2W})] df \right\}$$

$$= \sum_{n=-\infty}^{\infty} g(\tfrac{n}{2W}) \left[\frac{1}{2W} \frac{exp[j2\pi f(t - \frac{n}{2W})]}{j2\pi(t - \frac{n}{2W})} \right]_{-W}^{+W}$$

$$= \sum_{n=-\infty}^{\infty} g(\tfrac{n}{2W}) \left[\frac{exp[+j2\pi W(t - \frac{n}{2W})] - exp[-j2\pi W(t - \frac{n}{2W})]}{2j \cdot 2\pi W(t - \frac{n}{2W})} \right]$$

$$= \sum_{n=-\infty}^{\infty} g(\tfrac{n}{2W}) \left[\frac{sin[2\pi W(t - \frac{n}{2W})]}{2\pi W(t - \frac{n}{2W})} \right] = \sum_{n=-\infty}^{\infty} g(\tfrac{n}{2W}) \left[\frac{sin[2\pi W t - n\pi)]}{2\pi W t - n\pi)} \right]$$

$$= \sum_{n=-\infty}^{\infty} g(\tfrac{n}{2W}) [sinc[2Wt - n]]$$

Interpolation Formula

$$g(t) = \sum_{n=-\infty}^{\infty} g(\tfrac{n}{2W}) sinc[2Wt - n]$$

$$= \boxed{\sum_{n=-\infty}^{\infty} g(\tfrac{n}{2W})\, sinc\left[2W(t - \tfrac{n}{2W})\right]} \qquad \text{(interpolation formula)}$$

$$= \left[\sum_{n=-\infty}^{\infty} g(\tfrac{n}{2W})\delta(t - \tfrac{n}{2W})\right] \star [sinc(2Wt)]$$

Ideal Low Pass Filter : Synthesis / Reconstruction Filter

- $h(t) = sinc(2Wt)$ for $-\infty < t < +\infty$
- $H(f) = \frac{1}{2W}$ for $-W < f < +W$
- $h(t) = sinc(2Wt)$

Analysis and Synthesis

Analysis : at the transmitter

A band-limited signal (only for $-W < f < +W$)
is completely described by sample values
whose time instants are separated by $1/2W$seconds.

$$G(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} g(\frac{n}{2W}) exp(-j\frac{\pi n f}{W}) \qquad -W < f < +W$$

Synthesis : at the receiver

A band-limited signal (only for $-W < f < +W$)
is completely recovered by sample values
whose time instants are separated by $1/2W$seconds.

$$g(t) = \sum_{n=-\infty}^{\infty} g(\frac{n}{2W}) sinc[2Wt - n] \qquad -\infty < t < +\infty$$

Nyquist Rate and Interval

Nyquist Rate

The sampling rate of $2W$ samples per second
for a signal bandwidth W hertz

Nyquist Interval

The reciprocal of the sampling rate : $1/2W$.

Reference

[1] S. Haykin, M Moher, “Introduction to Analog and Digital Communications”, 2ed