Z Transform (H.1)
Definition

20170107

Based on
Complex Analysis for Mathematics and Engineering
J. Mathews

Z - Transform

$$X(z) = \sum_{k=-\infty}^{+\infty} x[k]\, z^{-k}$$

$$z = |r| e^{j2\pi F} = |r| e^{j\Omega}$$

$$x[n] \longleftrightarrow X(z)$$

One Sided z-transform

$$X(z) = \sum_{k=0}^{+\infty} x[k]\, z^{-k}$$

Inverse z- Transform

$$X(z) = \mathcal{Z}\left[\{x_n\}_{n=0}^{\infty}\right] = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$x_n = x[n] = \mathcal{Z}^{-1}[X(z)] = \frac{1}{2\pi i}\int_C X(z) z^{n-1} \, dz$$

Admissible Form of z-transform

$$X(z) = \sum_{k=0}^{\infty} x[n]\, z^{-n}$$

admissible z-transform

if $X(z)$ is a rational function

$$X(z) = \frac{P(z)}{Q(z)} = \frac{b_0 + b_1 z^1 + b_2 z^2 + \cdots + b_{p-1} z^{p-1} + b_p z^p}{a_0 + a_1 z^1 + a_2 z^2 + \cdots + a_{q-1} z^{q-1} + a_q z^q}$$

$P(z)$: a polynomial of degree p

$Q(z)$: a polynomial of degree q

D: simply connected domain

C: simple closed contour (CCW) in D

if $f(z)$ is analytic inside C and on C

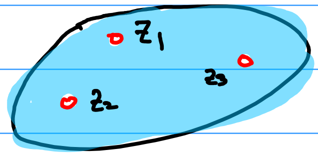
except at the points $z_1, z_2, \cdots, z_k$ in C

then

$$\frac{1}{2\pi i} \int_C f(z)\, dz = \sum_{j=1}^{k} \operatorname{Res}(f(z), z_j)$$

$$\oint_c f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

finite number k of singular points z_k



$$\oint_c f(z)\,dz = 0$$

if $f(z)$ is continuous in D and

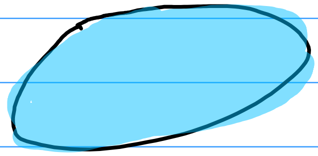
$f(z) = F'(z)$: $F(z)$ is an antiderivative of $f(z)$

fundamental theorem of calculus

$$\oint_c f(z)\,dz = 0$$

if $f(z)$ is analytic within and on C

no singularity



Thomas J. Cavicchi
Digital Signal Processing, Wiley, 2000

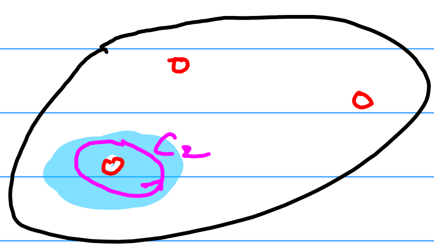
$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

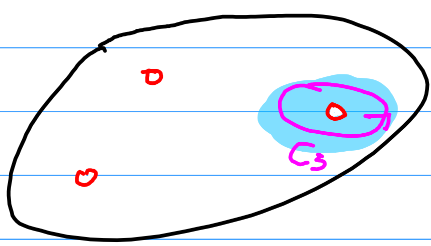
$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k \right) \qquad z_k \text{ within } C$$

$$= \frac{1}{n!} f^{(n)}(z_m) \qquad n \geq 0$$

$a_n^{\{0\}}$ expansion at z_0



$a_n^{\{1\}}$ expansion at z_1



$a_n^{\{2\}}$ expansion at z_2

$$a_n^{\{m\}} = \mathrm{Res}(f(z), z_m)$$

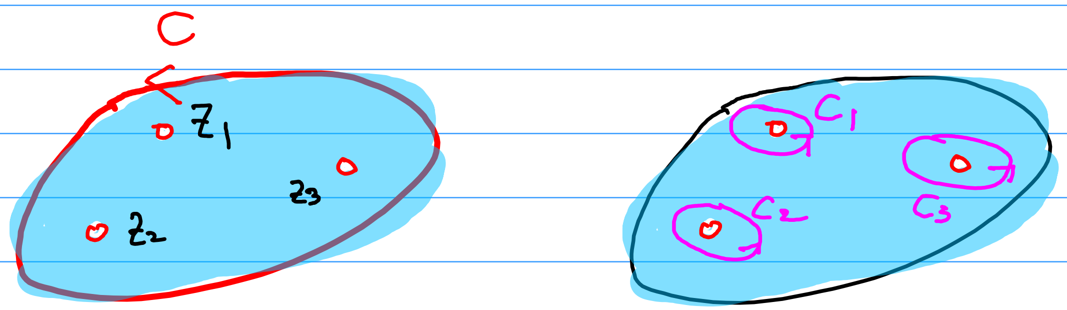
the residue of $f(z)$ at z_m using C_m

assumed that

there are several (m) singularities (poles) of $f(z)$ in a region

but that

C is taken to enclose only the pole z_m : C_m



Laurent's Theorem

f : analytic within the annular domain D

$$r < |z - z_0| < R$$

then

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k \quad , \text{ valid for } \quad r < |z - z_0| < R$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} \, ds , \qquad k = 0, \pm 1, \pm 2, \cdots$$

C : a simple closed curve
that lies entirely within D
that encloses z_0

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) \, ds \Rightarrow \oint_C f(s) \, ds = 2\pi i \cdot a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) \, ds = Res(f(z), z_0)$$

$$= \begin{cases} \lim_{z \to z_0} (z - z_0) f(z) & (\text{simple}) \\ \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) & (\text{order } n) \end{cases}$$

Cauchy's Residue Theorem

$f(z)$: analytic on and within C

except a finite number of singular points

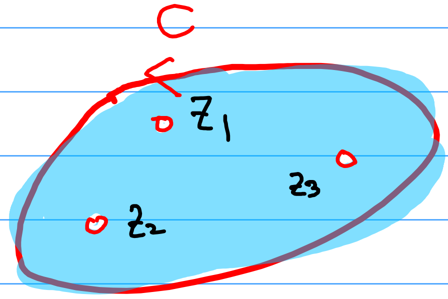
$z_1, z_2, \cdots, z_n$ within C

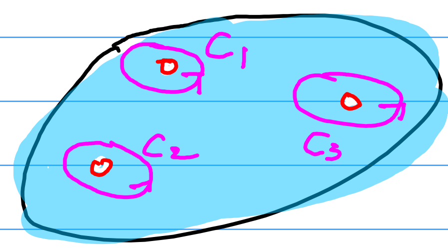
then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k)$$

D : a simply connected domain

C : a simple closed contour in D





$\boxed{z_1}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_1)^k \qquad a_{-1}^{\{1\}} = \frac{1}{2\pi i}\oint_{C_1} f(s)\,ds = \operatorname{Res}(f(z), z_1)$

$\boxed{z_2}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_2)^k \qquad a_{-1}^{\{2\}} = \frac{1}{2\pi i}\oint_{C_2} f(s)\,ds = \operatorname{Res}(f(z), z_2)$

$\boxed{z_3}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_3)^k \qquad a_{-1}^{\{3\}} = \frac{1}{2\pi i}\oint_{C_3} f(s)\,ds = \operatorname{Res}(f(z), z_3)$

$\boxed{z_1}$

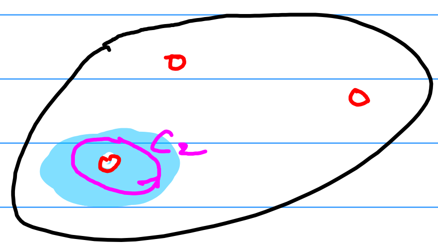
Laurent series expansion at z_1

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_1)^k$$

$$a_{-1}^{\{1\}} = \frac{1}{2\pi i} \oint_{C_1} f(s)\, ds = \mathrm{Res}(f(z), z_1)$$

$\boxed{z_2}$

Laurent series expansion at z_2

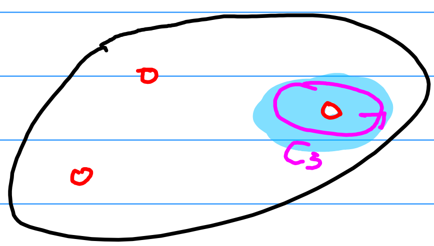


$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_2)^k$$

$$a_{-1}^{\{2\}} = \frac{1}{2\pi i} \oint_{C_2} f(s)\, ds = \mathrm{Res}(f(z), z_2)$$

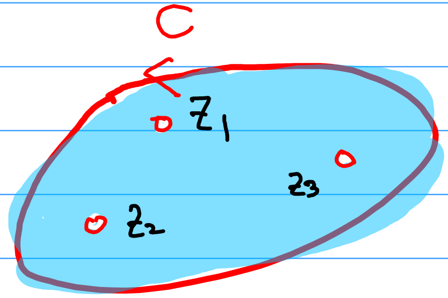
$\boxed{z_3}$

Laurent series expansion at z_3

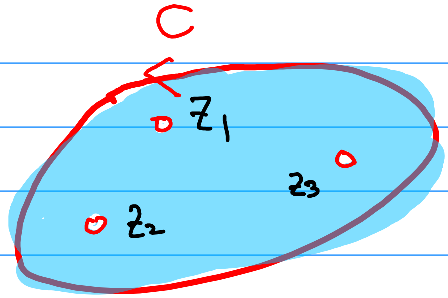


$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_3)^k$$

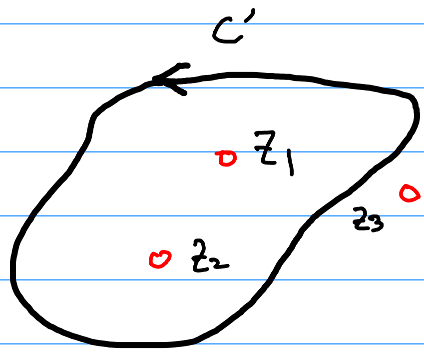
$$a_{-1}^{\{3\}} = \frac{1}{2\pi i} \oint_{C_3} f(s)\, ds = \mathrm{Res}(f(z), z_3)$$



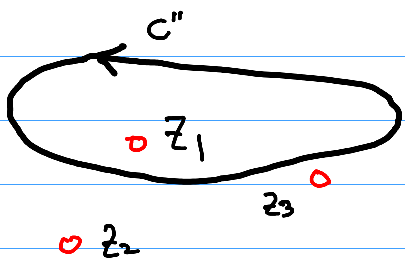
$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$



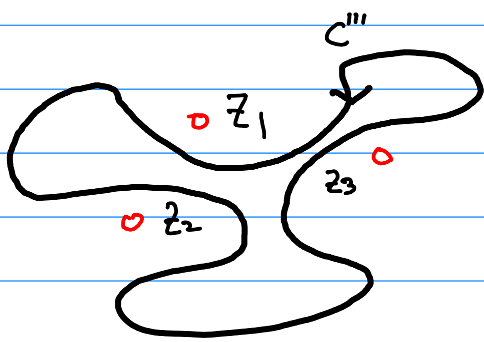
$$\int_C f(z)\,dz = 2\pi i \operatorname{Res}(f(z), z_1) + 2\pi i \operatorname{Res}(f(z), z_2) + 2\pi i \operatorname{Res}(f(z), z_3)$$



$$\int_{C'} f(z)\,dz = 2\pi i \operatorname{Res}(f(z), z_1) + 2\pi i \operatorname{Res}(f(z), z_2)$$

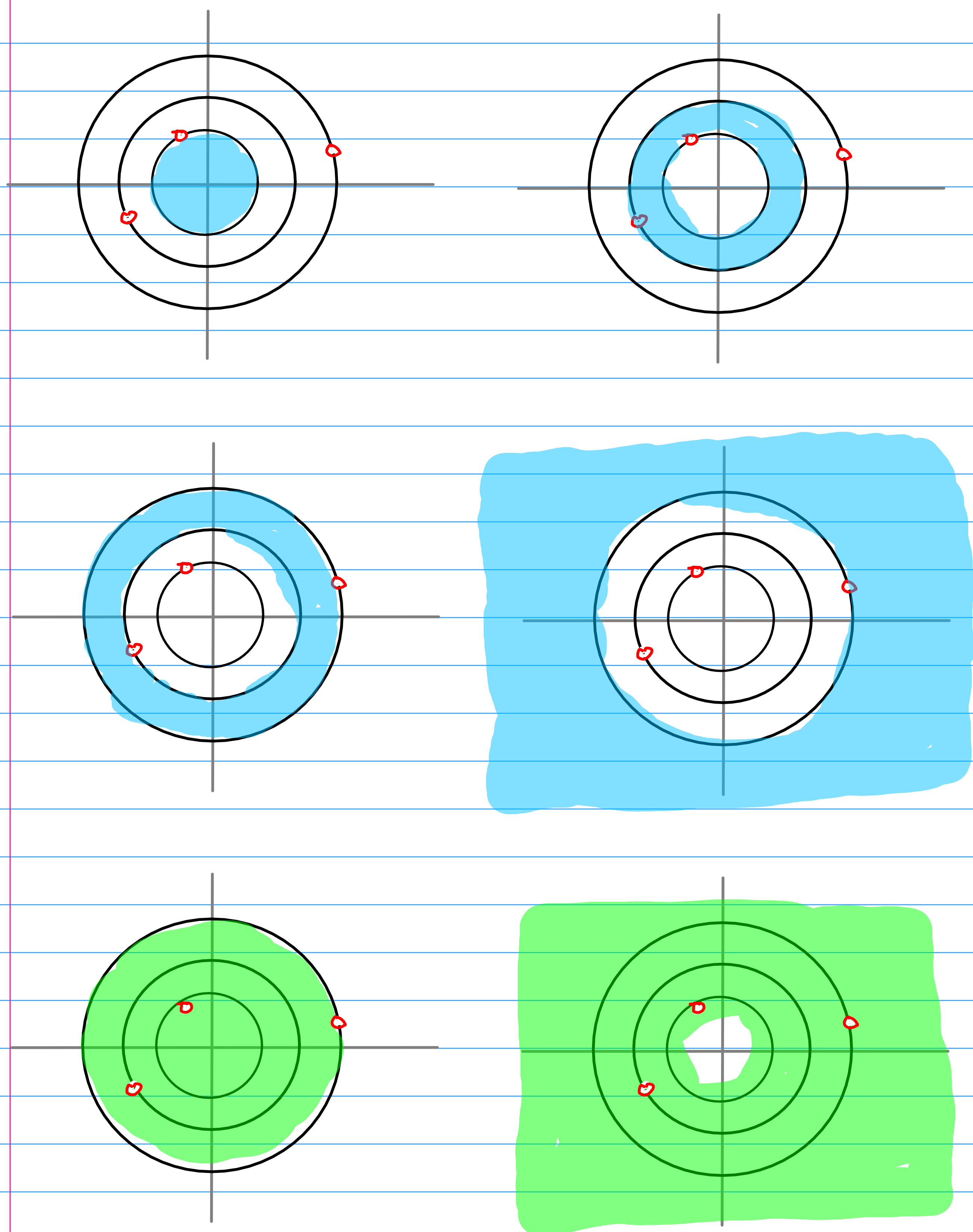


$$\int_{C''} f(z)\,dz = 2\pi i \operatorname{Res}(f(z), z_1)$$



$$\int_{C''} f(z)\,dz = 0$$

Different D, Different Laurent Series

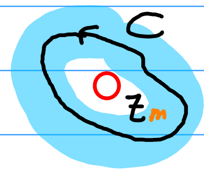


z-transform

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} \, dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$



C is in the same region of analyticity of $f(z)$

typically a circle centered on z_m

z_k within C : singularities of $\frac{f(z)}{(z - z_m)^{n+1}}$

$n_1 = n_{f,m}$ depends on $f(z)$, z_m

$a_n^{\{m\}}$ depends on $f(z)$, z_m, region of analyticity

Whether $f(z)$ is singular at $z = z_m$ or not

or at other points between z and z_m

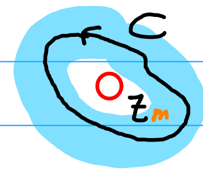
We can expand $f(z)$ about any point z_m

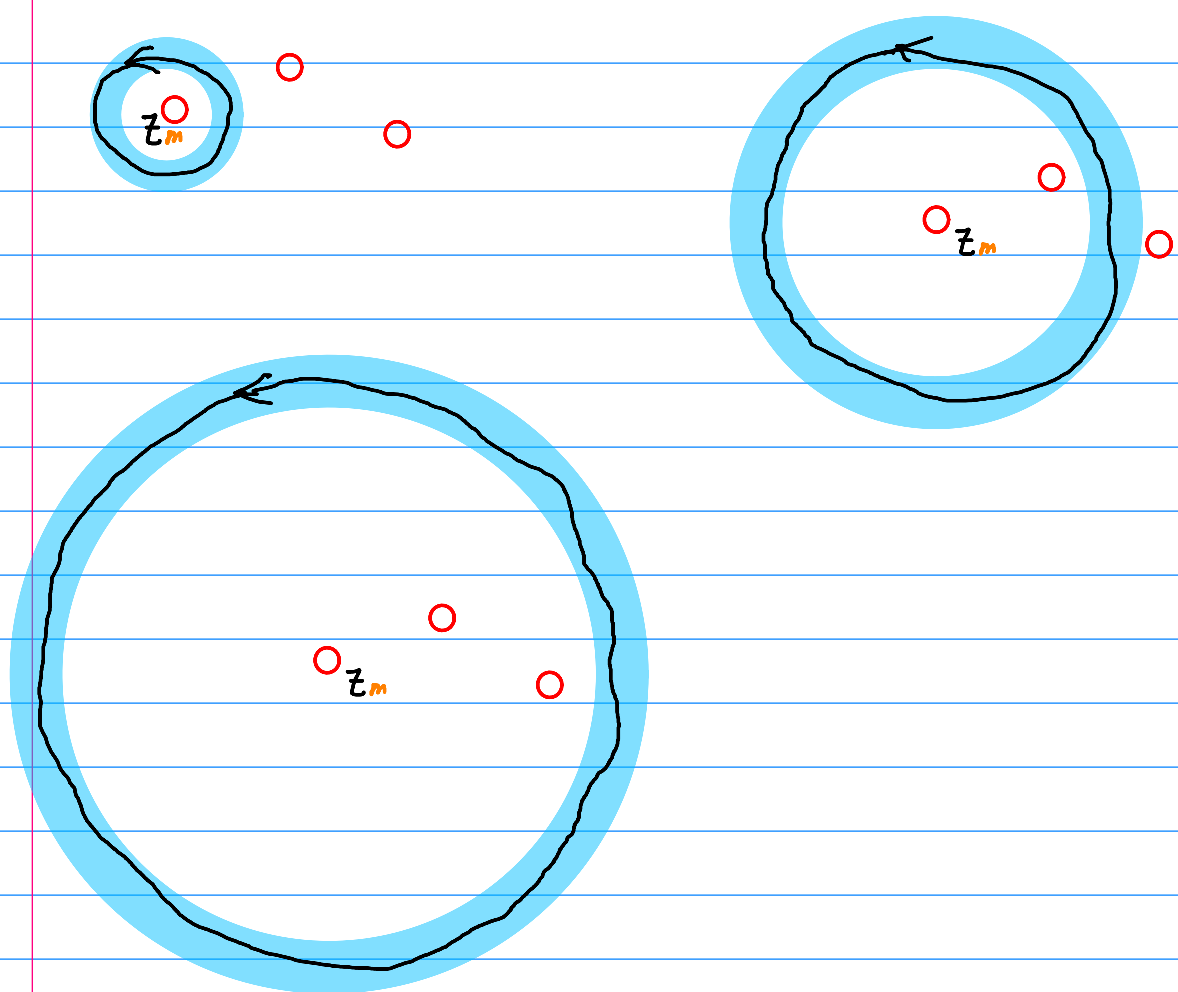
over powers of $(z - z_m)$.

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}}\, dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$





$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z'-z_m)^{n+1}} dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

analytic at z_m

$n_1 \geq 0$		Taylor Series
general n_1	$z_m = 0$	MacLaurin Series

singular at z_m

general n_1		Laurent Series
general n_1	$z_m = 0$	z - Transform

$$\boxed{\begin{aligned} f(z) &= \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n \\ a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z'-z_m)^{n+1}}\, dz' \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}},\ z_k\right) \end{aligned}}$$

$$\boxed{z_m = 0} \qquad \boxed{a_{-n}^{\{0\}} = h(n)} \qquad \boxed{n \to -n}$$

$$\boxed{\begin{aligned} H(z) &= \sum_{n=n_1}^{\infty} h(-n)\, z^n \\ h(n) &= \frac{1}{2\pi i} \oint_c \frac{H(z')}{z'^{\,n+1}}\, dz' \\ &= \sum_k \mathrm{Res}\left(\frac{H(z)}{z^{n+1}},\ z_k\right) \end{aligned}}$$

$$\boxed{\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} \\ h(n) &= \frac{1}{2\pi i} \oint_c H(z')\, z'^{\,n-1}\, dz' \\ &= \sum_k \mathrm{Res}\left(H(z)\, z^{n-1},\ z_k\right) \end{aligned}}$$

C is in the same region of analyticity of $f(z)$

typically a circle centered on z_m

z_k within C : singularities of $\dfrac{f(z)}{(z-z_m)^{n+1}}$

C is in the same region of analyticity of $H(z)$

typically a circle centered on z_m

generally a circle centered on the origin

may enclose any or all singularities of $H(z)$

often the unit circle

z_k within C : singularities of $H(z)\, z^{n-1}$

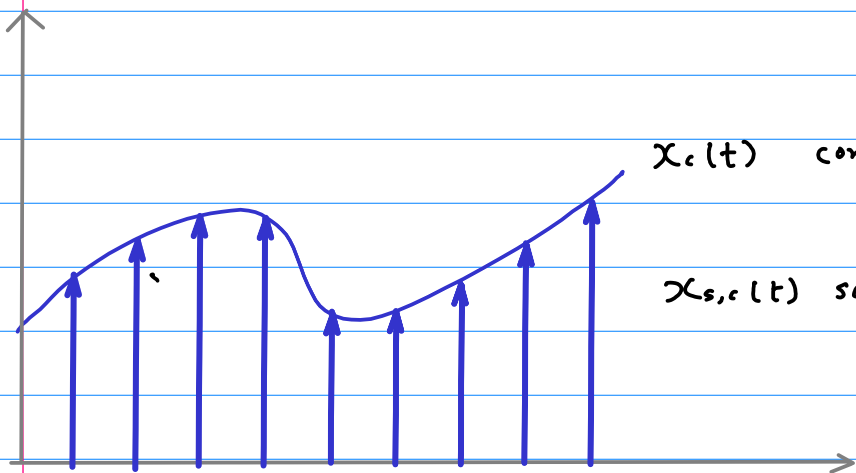
$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} \qquad z \in R.O.C$$

$$h(n) = \frac{1}{2\pi i} \oint_C H(z')\, z'^{\,n-1}\, dz' \qquad C \text{ in } R.O.C.$$

$$= \sum_k \mathrm{Res}\left(H(z)\, z^{n-1},\, z_k\right)$$

1. a power series representation of a function $f(z)$ of a complex variable z

2. a transform $H(z)$ of a sequence of 1



$$x_{s,c}(t) = \sum_{n=-\infty}^{+\infty} x(n)\, \delta_c(t - n\Delta t)$$

$$X_{s,c}(s) = \mathcal{L}\{x_{s,c}(t)\} = \int_{-\infty}^{\infty} \boxed{\sum_{n=-\infty}^{+\infty} x(n)\, \delta_c(t - n\Delta t)}\, e^{-st}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} x(n) \int_{-\infty}^{\infty} \delta_c(t - n\Delta t)\, e^{-st}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-sn\Delta t} \qquad e^{s\Delta t} \triangleq z$$

$$X_{s,c}(s) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \Big|_{z = e^{s\Delta t}}$$

$$X_{s,c}(s) = X(z)\Big|_{z = e^{s\Delta t}}$$

$$X_{s.c}(s) = \mathcal{L}\{x_{s.c}(t)\} = X(z)\Big|_{z=e^{s\Delta t}}$$

$x_{s.c}(t)$ an impulse train

whose coefficients are given by $x[n] = x_c(n\Delta t)$

Z - transform : a special Laurent series

$\boxed{z_m = 0}$ $\boxed{a_{-n}^{\{0\}} = h(n)}$ $\boxed{n \rightarrow -n}$

$$\boxed{\begin{aligned} f(z) &= \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n \\ a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} dz' \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right) \end{aligned}}$$

Time Reversal ← Laplace Transform

the transform functions

$X(s) = \int$ over negative powers e^{-st} for $t > 0$

$X(z) = \int$ over negative powers z^{-n} for $n > 0$

the time expansion functions

$x(t) = \int$ over negative powers e^{-st} for $t > 0$

$x[n] = \int$ over negative powers z^{-n} for $n > 0$

Time Reversal ← z^{-1} : unit delay , char eq (modes in z^k)

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Stable System : $h[n]$ must be asbsolutely summable

$$|e^{j\omega n}| = 1$$

$$|z^n| \qquad z = 1$$

$$\infty > M_h > \sum_{n=-\infty}^{\infty} |h[n]| \qquad \text{asolutely summable}$$

$$= \sum_{n=-\infty}^{+\infty} |h[n]\, e^{-j\omega n}|$$

$$\geq \left| \sum_{n=-\infty}^{+\infty} h[n]\, e^{-j\omega n} \right|$$

$$= \left| \, H(z) \big|_{z=e^{j\omega}} \right|$$

$$\infty > \left| \, H(z) \big|_{z=e^{j\omega}} \right|$$

a stable system,

$H(z)$ must converge on the unit circle $|z| = 1$

ROC (Region of Convergence) must include the unit circle

regardless of causality of $h[n]$

$$H(z)\big|_{|z|=1} = H(e^{j\hat{\omega}})$$ DTFT of $h[n]$

discrete all stable sequence must have convergent DTFTs

continuous all stable signal must have convergent CTFTs

$C \leftarrow$ unit circle $z = e^{j\hat{\omega}}$

ZT^{-1} $DTFT^{-1}$ identical formulas

$h[n]$ causal $h[n] \neq 0$ $n \in [0, \infty)$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]\, z^{-n}$$