Z Transform (H.4) Pairs

20161126

Based on
Complex Analysis for Mathematics and Engineering
J. Mathews

Z-Transform Pairs

$$\delta[n] \longleftrightarrow 1$$

$$u[n] \longleftrightarrow \frac{z}{z-1}$$

$$b^n \longleftrightarrow \frac{z}{z-b}$$

$$b^{n-1}u[n-1] \longleftrightarrow \frac{1}{z-b}$$

$$e^{an} \longleftrightarrow \frac{z}{z-e^a}$$

$$n \longleftrightarrow \frac{z}{(z-1)^2}$$

$$n^2 \longleftrightarrow \frac{z(z+1)}{(z-1)^3}$$

$$n b^n \longleftrightarrow \frac{bz}{(z-b)^2}$$

$$n e^{an} \longleftrightarrow \frac{ze^a}{z-e^a}$$

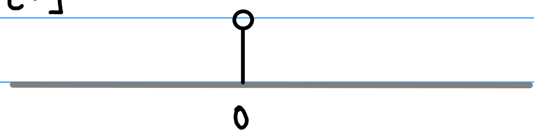
$$\sin(an) \longleftrightarrow \frac{i(-1+e^{2ia})z}{2(e^{ia}-z)(-1+e^{ia}z)}$$

$$b^n \sin(an) \longleftrightarrow \frac{ib(-1+e^{2ia})z}{2(be^{ia}-z)(-b+e^{ia}z)}$$

$$\cos(an) \longleftrightarrow \frac{(1+e^{2ia}-2e^{ia}z)z}{2(e^{ia}-z)(-1+e^{ia}z)}$$

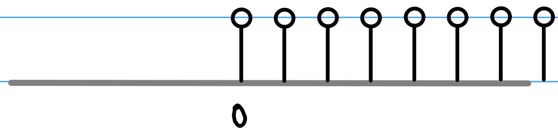
$$b^n \cos(an) \longleftrightarrow \frac{(b+be^{2ia}-2e^{ia}z)z}{2(be^{ia}-z)(-b+e^{ia}z)}$$

$x[n] = \delta[n]$



$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = 1$$

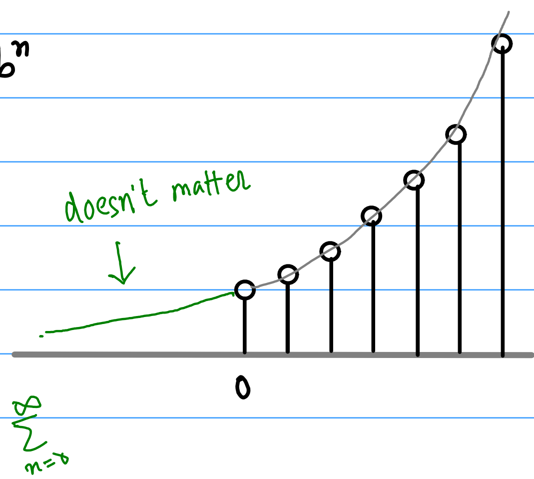
$x[n] = u[n]$



$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$= \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

$x[n] = b^n$



$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{b}{z}\right)^n$$

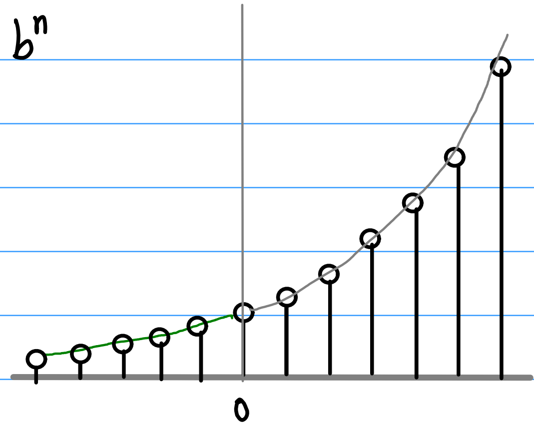
$$= \frac{1}{1 - \frac{b}{z}} = \frac{z}{z-b}$$

e^{an}

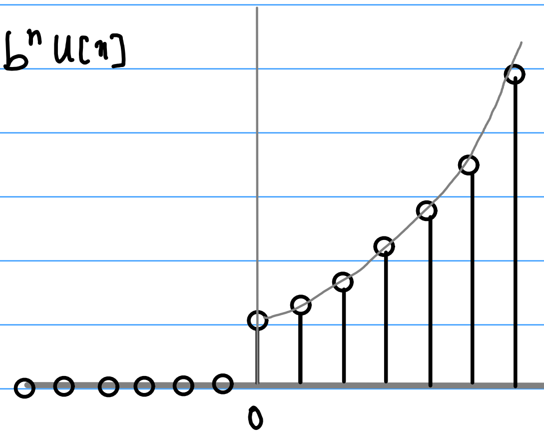
$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{e^a}{z}\right)^n$$

$$= \frac{1}{1 - \frac{e^a}{z}} = \frac{z}{z - e^a}$$

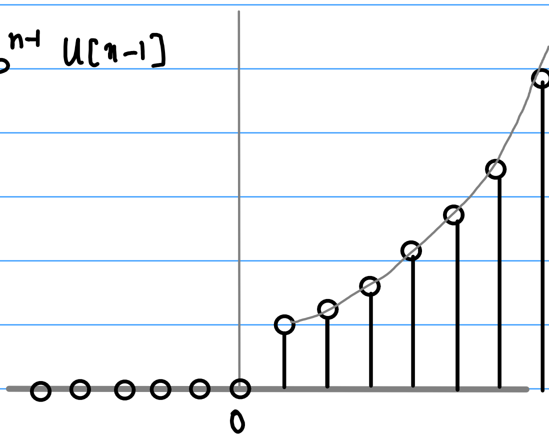
b^n



$b^n u[n]$



$x[n] = b^{n-1} u[n-1]$



$$X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$$

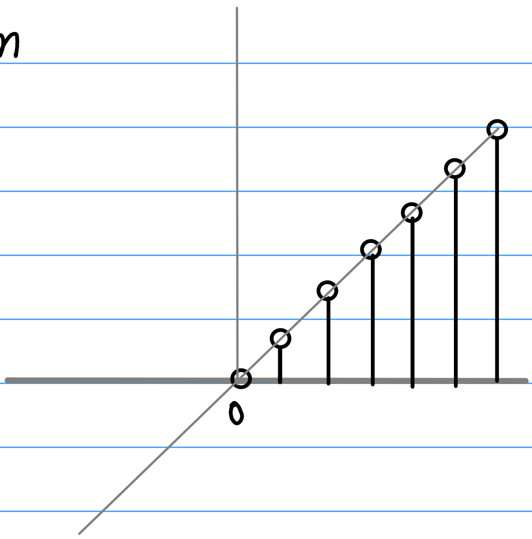
$$= \sum_{n=0}^{\infty} \left(\frac{b}{z}\right)^{n-1} u[n-1] \frac{1}{z}$$

$$= \sum_{n=1}^{\infty} \left(\frac{b}{z}\right)^{n-1} \left(\frac{1}{z}\right)$$

$$= \left(\frac{1}{z}\right) \sum_{k=0}^{\infty} \left(\frac{b}{z}\right)^{k}$$

$$= \frac{\frac{1}{z}}{1 - \frac{b}{z}} = \frac{1}{z-b}$$

$x[n] = n$



$$\sum_{n=0}^{\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$\sum_{n=0}^{\infty} n \ z^{-n} = -\sum_{n=0}^{\infty} (-n \cdot z^{-n-1})\, z =$$

$$= (-z) \sum_{n=0}^{\infty} \frac{d}{dz}\left(z^{-n}\right)$$

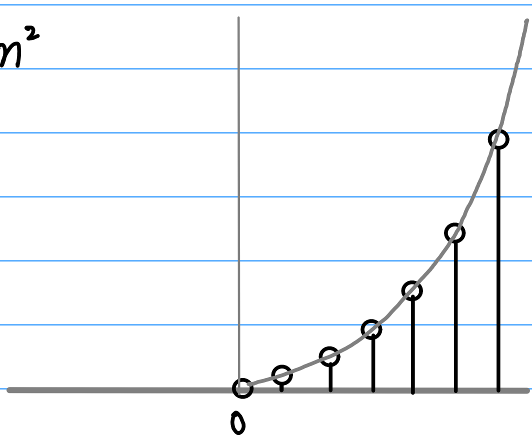
$$= (-z) \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^{-n}\right)$$

$$= (-z) \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= (-z) \frac{z-1-z}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}$$

$x[n] = n^2$



$$\sum_{n=0}^{\infty} n \ z^{-n} = \frac{z}{(z-1)^2}$$

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} n \ z^{-n}\right) = \frac{1}{(z-1)^2} + \frac{(-2)z}{(z-1)^3}$$

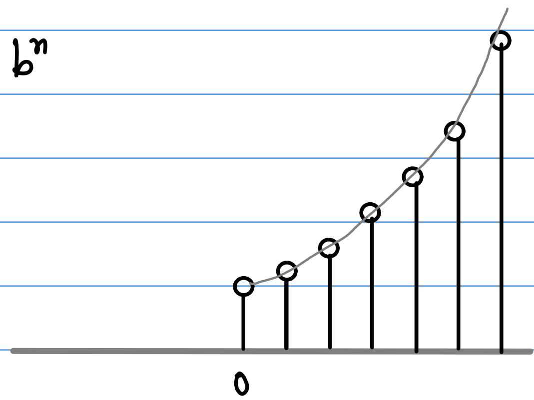
$$\sum_{n=0}^{\infty} -n^2 z^{-n-1} = \frac{1}{(z-1)^3}\left(z-1-2z\right)$$

$$\frac{-1}{z} \sum_{n=0}^{\infty} n^2 z^{-n} = \frac{-(z+1)}{(z-1)^3}$$

$$\sum_{n=0}^{\infty} n^2 z^{-n} = \frac{z(z+1)}{(z-1)^3}$$

$$n\, b^n \leftrightarrow \frac{bz}{(z-b)^2}$$

$x[n] = n\, b^n$



$$\sum_{n=0}^{\infty} b^n z^{-n} = \frac{z}{z-b}$$

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} b^n z^{-n}\right) = \frac{z-b-z}{(z-b)^2}$$

$$\frac{-1}{z}\sum_{n=0}^{\infty} n\, b^n z^{-n} = \frac{-b}{(z-b)^2}$$

$$\sum_{n=0}^{\infty} n\, b^n z^{-n} = \frac{bz}{(z-b)^2}$$

$x[n] = n\, e^{an}$

$$\sum_{n=0}^{\infty} e^{an} z^{-n} = \frac{z}{z-e^a}$$

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} e^{an} z^{-n}\right) = \frac{z-e^a-z}{(z-e^a)^2}$$

$$\frac{-1}{z}\sum_{n=0}^{\infty} n\, e^{an} z^{-n} = \frac{-e^a}{(z-e^a)^2}$$

$$\sum_{n=0}^{\infty} n\, e^{an} z^{-n} = \frac{z e^a}{(z-e^a)^2}$$

$$\sin(an) = \frac{1}{2i}\left(e^{ian} - e^{-ian}\right)$$

$$\frac{1}{2i}\left(\mathcal{Z}[e^{ian}] - \mathcal{Z}[e^{-ian}]\right)$$

$$= \frac{1}{2i}\left(\frac{z}{z-e^{ia}} - \frac{z}{z-e^{-ia}}\right)$$

$$= \frac{z}{2i}\ \frac{z-e^{-ia}-z+e^{ia}}{(z-e^{ia})(z-e^{-ia})}$$

$$= \frac{z}{2i}\ \frac{(-e^{-ia}+e^{ia})}{(z-e^{ia})(z-e^{-ia})}$$

$$= \frac{z}{2i}\ \frac{(-1 \quad +e^{2ia})}{(z-e^{ia})(e^{ia}z-1)}\ \frac{e^{-ia}}{e^{-ia}}$$

$$= \frac{-i}{2}\ \frac{(-1+e^{2ia})\,z}{(z-e^{ia})(-1+e^{ia}z)}$$

$$= \boxed{\frac{i}{2}\ \frac{(-1+e^{2ia})\,z}{(e^{ia}-z)(-1+e^{ia}z)}}$$

$$= \frac{z\left(\frac{-e^{-ia}+e^{ia}}{2i}\right)}{(z-e^{ia})(z-e^{-ia})}$$

$$= \boxed{\frac{z\sin(a)}{(z-e^{ia})(z-e^{-ia})}}$$

$$\cos(an) = \frac{1}{2}\left(e^{ian} + e^{-ian}\right)$$

$$\frac{1}{2}\left(Z[e^{ian}] + Z[e^{-ian}]\right)$$

$$= \frac{1}{2}\left(\frac{z}{z - e^{ia}} + \frac{z}{z - e^{-ia}}\right)$$

$$= \frac{z}{2}\,\frac{z - e^{-ia} + z - e^{ia}}{(z - e^{ia})(z - e^{-ia})}$$

$$= \frac{z}{2}\,\frac{(-e^{-ia} - e^{ia} + 2z)}{(z - e^{ia})(z - e^{-ia})}$$

$$= \frac{z}{2}\,\frac{(-1 - e^{2ia} + 2e^{ia}z)}{(z - e^{ia})(e^{ia}z - 1)}\,\frac{e^{-ia}}{e^{-ia}}$$

$$= \boxed{\frac{1}{2}\,\frac{(1 + e^{2ia} + 2e^{ia}z)\,z}{(e^{ia} - z)(-1 + e^{ia}z)}}$$

$$= \frac{z}{2}\,\frac{2\left(-\frac{e^{-ia} + e^{ia}}{2} + z\right)}{(z - e^{ia})(z - e^{-ia})}$$

$$= \boxed{\frac{z\,(z - \cos a)}{(z - e^{ia})(z - e^{-ia})}}$$

$$b^n \sin(an) = \frac{b^n}{2i}\left(e^{ian} - e^{-ian}\right)$$

$$\frac{1}{2i}\left(\mathcal{Z}[b^n e^{ian}] - \mathcal{Z}[b^n e^{-ian}]\right)$$

$$= \frac{1}{2i}\left(\frac{z}{z-be^{ia}} - \frac{z}{z-be^{-ia}}\right)$$

$$= \frac{z}{2i}\,\frac{z-be^{-ia}-z+be^{ia}}{(z-be^{ia})(z-be^{-ia})}$$

$$= \frac{z}{2i}\,\frac{b(-e^{-ia}+e^{ia})}{(z-be^{ia})(z-be^{-ia})}$$

$$= \frac{z}{2i}\,\frac{b(-e^{-ia}+e^{ia})}{(z-be^{ia})(z-be^{-ia})}\,\frac{e^{+ia}}{e^{+ia}}$$

$$= \frac{-i}{2}\,\frac{b(-1+e^{2ia})z}{(z-be^{ia})(-b+e^{ia}z)}$$

$$= \boxed{\frac{i}{2}\,\frac{b(-1+e^{2ia})z}{(be^{ia}-z)(-b+e^{ia}z)}}$$

$$b^n \cos(an) = \frac{b^n}{2}\left(e^{ian} + e^{-ian}\right)$$

$$\frac{1}{2}\left(\mathcal{Z}[b^n e^{ian}] + \mathcal{Z}[b^n e^{-ian}]\right)$$

$$= \frac{1}{2}\left(\frac{z}{z - be^{ia}} + \frac{z}{z - be^{-ia}}\right)$$

$$= \frac{z}{2}\,\frac{-be^{-ia} - be^{ia} + 2z}{(z - be^{ia})(z - be^{-ia})}$$

$$= \frac{-z}{2}\,\frac{(be^{-ia} + be^{ia} - 2z)}{(z - be^{ia})(z - be^{-ia})}$$

$$= \frac{-z}{2}\,\frac{(be^{-ia} + be^{ia} - 2z)}{(z - be^{ia})(z - be^{-ia})}\,\frac{e^{+ia}}{e^{+ia}}$$

$$= \frac{-z}{2}\,\frac{(b + be^{2ia} - 2e^{ia}z)}{(z - be^{ia})(e^{ia}z - b)}$$

$$= \boxed{\frac{1}{2}\,\frac{(b + be^{2ia} - 2e^{ia}z)z}{(be^{ia} - z)(-b + e^{ia}z)}}$$

Properties of z-transform

$$\mathcal{Z}[x_n] = \mathcal{Z}[x[n]] = X(z)$$

$$\mathcal{Z}[y_n] = \mathcal{Z}[y[n]] = Y(z)$$

Linearity

$$\mathcal{Z}[c_1 x_n + c_2 y_n] = \mathcal{Z}[c_1 x[n] + c_2 y[n]] = c_1 X(z) + c_2 Y(z)$$

Delay shift

$$\mathcal{Z}[x[n-N]u[n-N]] = X(z)\, z^{-N}$$

Advance shift

$$\mathcal{Z}[x[n+N]] = z^{N}\left(X(z) - x[0] - x[1]z^{-1} - x[2]z^{-2} - \cdots - x[N-1]z^{-N+1}\right)$$

Multiplication by n

$$\mathcal{Z}[n\,x[n]] = -z \frac{d}{dz} X(z)$$

$$n \longleftrightarrow \frac{z}{(z-1)^2}$$

$$n^2 \longleftrightarrow \frac{z(z+1)}{(z-1)^3}$$

$$n b^n \longleftrightarrow \frac{bz}{(z-b)^2}$$

$$n b^{n-1} \longleftrightarrow \frac{z}{(z-b)^2}$$

$$a^n \longleftrightarrow \frac{z}{(z-a)}$$

$$n a^{n-1} \longleftrightarrow \frac{z}{(z-a)^2}$$

$$\frac{n(n-1)}{2!} a^{n-2} \longleftrightarrow \frac{z}{(z-a)^3}$$

$a_n = 0 \quad (n < 0)$

Ambardar
p 579

$$a^n \longleftrightarrow \frac{z}{(z-a)}$$

$$na^{n-1} \longleftrightarrow \frac{z}{(z-a)^2}$$

$$\frac{n(n-1)}{2!} a^{n-2} \longleftrightarrow \frac{z}{(z-a)^3}$$

$$\sum_{n=0}^{\infty} a^n z^{-n} = \frac{z}{z-a}$$

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} a^n z^{-n}\right) = \frac{z-a-z}{(z-a)^2}$$

$$-\sum_{n=0}^{\infty} n a^n z^{-n-1} = \frac{-a}{(z-a)^2}$$

$$\sum_{n=0}^{\infty} \boxed{na^n} z^{-n} = \frac{az}{(z-a)^2}$$

$$\sum_{n=0}^{\infty} a^n z^{-n} = \frac{z}{z-a}$$

$$\frac{d}{da}\left(\sum_{n=0}^{\infty} a^n z^{-n}\right) = -\frac{z}{(z-a)^2}$$

$$-\sum_{n=0}^{\infty} n a^{n-1} z^{-n} = \frac{-z}{(z-a)^2}$$

$$\sum_{n=0}^{\infty} \boxed{na^{n-1}} z^{-n} = \frac{z}{(z-a)^2}$$

$$\boxed{na^n \longleftrightarrow \frac{az}{(z-a)^2}} \rightarrow \boxed{na^{n-1} \longleftrightarrow \frac{z}{(z-a)^2}}$$

$$\sum_{n=0}^{\infty} \boxed{na^{n-1}} z^{-n} = \frac{z}{(z-a)^2}$$

$$\frac{d}{da}\left(\sum_{n=0}^{\infty} \boxed{na^{n-1}} z^{-n}\right) = \frac{d}{da}\frac{z}{(z-a)^2}$$

$$\sum_{n=0}^{\infty} n(n-1)a^{n-2} z^{-n} = \frac{-2 \cdot z \cdot (-1)}{(z-a)^3}$$

$$\sum_{n=0}^{\infty} \boxed{\frac{n(n-1)}{2!} a^{n-2}} z^{-n} = \frac{z}{(z-a)^3}$$

$$\sum_{n=0}^{\infty} \boxed{\frac{n(n-1)\cdots(n-k+1)}{k!} a^{n-k}} z^{-n} = \frac{z}{(z-a)^{k+1}}$$

$$a^n \longleftrightarrow \frac{z}{z-a}$$

$$n a^{n-1} \longleftrightarrow \frac{z}{(z-a)^2}$$

$$\frac{1}{2!} n(n-1) a^{n-2} \longleftrightarrow \frac{z}{(z-a)^3}$$

$$\frac{1}{k!} n(n-1)\cdots(n-k+1) a^{n-k} \longleftrightarrow \frac{z}{(z-a)^{k+1}}$$