Cauchy-Euler Equations (5A)

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Homogeneous Linear Equations with variable coefficients

Cauchy-Euler Equation

Second Order Linear Equations with Constant Coefficients

$$a_2 \frac{d^2 y}{d x^2} + a_1 \frac{d y}{d x} + a_0 y = g(x)$$

$$a_2 y'' + a_1 y' + a_0 y = g(x)$$

Second Order Linear Equations with Variable Coefficients

$$a_2(x) \frac{d^2 y}{d x^2} + a_1(x) \frac{d y}{d x} + a_0(x) y = g(x)$$

$$a_2(x) y'' + a_1(x) y' + a_0(x) y = g(x)$$

Cauchy-Euler Equation

$$a_2 x^2 \frac{d^2 y}{d x^2} + a_1 x \frac{d y}{d x} + a_0 y = g(x)$$

$$a_2 x^2 y'' + a_1 x y' + a_0 y = g(x)$$

Auxiliary Equation of Cauchy-Euler Equation

Homogeneous Second Order Cauchy-Euler Equation

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

try a solution $y = x^m$

$$a x^2 \frac{d^2}{d x^2}\{x^m\} + b x \frac{d}{d x}\{x^m\} + c\{x^m\} = 0$$

$$a\{m(m-1)x^m\} + b\{m x^m\} + c\{x^m\} = 0$$

$$(a m^2 + (b-a)m + c) \cdot x^m = 0$$

$$a x^2 \{x^m\}'' + b x \{x^m\}' + c\{x^m\} = 0$$

$$a\{m(m-1)x^m\} + b\{m x^m\} + c\{x^m\} = 0$$

$$(a m^2 + (b-a)m + c) \cdot x^m = 0$$

auxiliary equation

$$(a m^2 + (b-a)m + c) = 0$$

$$(a m^2 + (b-a)m + c) = 0$$

General Solution – y_h of Cauchy-Euler Equations

Homogeneous Second Order Cauchy-Euler Equation

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

try a solution $y = x^m$

auxiliary equation

$$(a m^2 + (b-a) m + c) = 0$$

$$m_1 = \{-(b-a) + \sqrt{(b-a)^2 - 4ac}\}/2a$$

$$m_2 = \{-(b-a) - \sqrt{(b-a)^2 - 4ac}\}/2a$$

(A) $(b-a)^2 - 4ac > 0$ Real, distinct m_1, m_2

(B) $(b-a)^2 - 4ac = 0$ Real, equal m_1, m_2

(C) $(b-a)^2 - 4ac < 0$ Conjugate complex m_1, m_2

$$y = C_1 x^{m_1} + C_2 x^{m_2}$$

$$y = C_1 x^{m_1} + C_2 x^{m_1} \ln x$$

$$y = C_1 x^{m_1} + C_2 x^{m_2} \quad = C_1 x^{(\alpha + i\beta)} + C_2 x^{(\alpha - i\beta)}$$

Homogeneous Second Order Cauchy-Euler Equation

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

auxiliary equation

try a solution $y = x^m$ ⇒ $$(a m^2 + (b-a) m + c) = 0$$

$$m_1 = \{-(b-a) + \sqrt{(b-a)^2 - 4ac}\}/2a \qquad y_1 = x^{m_1}$$

$$m_2 = \{-(b-a) - \sqrt{(b-a)^2 - 4ac}\}/2a \qquad y_2 = x^{m_2}$$

(A) $(b-a)^2 - 4ac > 0$ Real, distinct m_1, m_2 $\qquad y = C_1 x^{m_1} + C_2 x^{m_2}$

(B) $(b-a)^2 - 4ac = 0$ Real, equal m_1, m_2 $\qquad y = C_1 x^{m_1} + C_2 x^{m_1} \ln x$

(C) $(b-a)^2 - 4ac < 0$ $\qquad y = C_1 x^{m_1} + C_2 x^{m_2} = C_1 x^{(\alpha + i\beta)} + C_2 x^{(\alpha - i\beta)}$

Homogeneous Second Order Cauchy-Euler Equation

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

auxiliary equation

try a solution $y = x^m$ → $\left(a m^2 + (b-a) m + c\right) = 0$ $\quad (b-a)^2 - 4ac = 0$ →

$$m_1 = \{-(b-a) + \sqrt{(b-a)^2 - 4ac}\}/2a$$

$$m_2 = \{-(b-a) - \sqrt{(b-a)^2 - 4ac}\}/2a$$

→

$$m_1 = -(b-a)/2a$$

$$m_2 = -(b-a)/2a$$

$$x^{m_1} = x^{m_2} = x^{-\frac{(b-a)}{2a}}$$

(A) $(b-a)^2 - 4ac > 0$ Real, distinct m_1, m_2 $\quad y = C_1 x^{m_1} + C_2 x^{m_2}$

(B) $(b-a)^2 - 4ac = 0$ Real, equal m_1, m_2 $\quad y = C_1 x^{m_1} + C_2 x^{m_1} \ln x$

(C) $(b-a)^2 - 4ac < 0$ $\quad y = C_1 x^{m_1} + C_2 x^{m_2} = C_1 x^{(\alpha + i\beta)} + C_2 x^{(\alpha - i\beta)}$

Homogeneous Second Order Cauchy-Euler Equation

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

try a solution $y = x^m$ → auxiliary equation

$$(a m^2 + (b-a) m + c) = 0$$

$$(b-a)^2 - 4ac = 0$$

$$m_1 = \{-(b-a) + \sqrt{4ac-(b-a)^2}\, i\}/2a \quad \Rightarrow \quad m_1 = \alpha + i\beta$$

$$m_2 = \{-(b-a) - \sqrt{4ac-(b-a)^2}\, i\}/2a \quad \Rightarrow \quad m_2 = \alpha - i\beta$$

$$y_1 = x^{m_1} = x^{\alpha + i\beta}$$

$$y_2 = x^{m_2} = x^{\alpha - i\beta}$$

(A) $(b-a)^2 - 4ac > 0$ Real, distinct m_1, m_2 $\qquad y = C_1 x^{m_1} + C_2 x^{m_2}$

(B) $(b-a)^2 - 4ac = 0$ Real, equal m_1, m_2 $\qquad y = C_1 x^{m_1} + C_2 x^{m_1} \ln x$

(C) $(b-a)^2 - 4ac < 0$ $\qquad y = C_1 x^{m_1} + C_2 x^{m_2} = C_1 x^{(\alpha + i\beta)} + C_2 x^{(\alpha - i\beta)}$

Homogeneous Second Order Cauchy-Euler Equation

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

$$m_1 = \{-(b-a) + \sqrt{4ac-(b-a)^2}\, i\}/2a \quad \Rightarrow \quad m_1 = \alpha + i\beta$$

$$m_2 = \{-(b-a) - \sqrt{4ac-(b-a)^2}\, i\}/2a \quad \Rightarrow \quad m_2 = \alpha - i\beta$$

$$y_1 = x^{m_1} = x^{\alpha + i\beta}$$

$$y_2 = x^{m_2} = x^{\alpha - i\beta}$$

$$\begin{cases} x^{m_1} = x^{\alpha} \cdot x^{+i\beta} = x^{\alpha} \cdot e^{+i\beta \ln x} \\ x^{m_2} = x^{\alpha} \cdot x^{-i\beta} = x^{\alpha} \cdot e^{-i\beta \ln x} \end{cases}$$

$$x = e^{\ln x}$$

$$x^{i\beta} = e^{+i\beta \ln x}$$

$$y = C_1 x^{m_1} + C_2 x^{m_2} = C_1 x^{\alpha} \cdot e^{+i\beta \ln x} + C_2 x^{\alpha} \cdot e^{+i\beta \ln x}$$

Pick **two** homogeneous solution

$$y_1 = x^{\alpha}\{e^{+i\beta \ln x} + e^{-i\beta \ln x}\}/2 = x^{\alpha} \cos(\beta \ln x) \qquad (C_1 = +1/2, \quad C_2 = +1/2)$$

$$y_2 = x^{\alpha}\{e^{+i\beta \ln x} - e^{-i\beta \ln x}\}/2i = x^{\alpha} \sin(\beta \ln x) \qquad (C_1 = +1/2i, \quad C_2 = -1/2i)$$

$$y = C_3 x^{\alpha} \cos(\beta \ln x) + C_4 x^{\alpha} \sin(\beta \ln x) = x^{\alpha}(C_3 \cos(\beta \ln x) + C_4 \sin(\beta \ln x))$$

We know one solution $y_1(x)$

Suppose the other solution $y_2(x) = u(x) y_1(x)$

$$a \frac{d^2 y}{d x^2} + b \frac{d y}{d x} + c y = 0$$

$$a y'' + b y' + c y = 0$$

$$u = c_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx + c_2$$

$$y_2 = y_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx$$

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

$$u = c_1 \int \frac{e^{-(bx/ax^2)x}}{(x^m)^2} dx + c_2$$

$$= c_1 \int \frac{e^{-(b/a)}}{x^{-2m_1}} dx$$

$$= c_1 \int \frac{e^{-(b/a)}}{x^{-\frac{(b-a)}{a}}} dx$$

$$y_1 = x^{-\frac{(b-a)}{2a}}$$

$$y_1^2 = x^{-\frac{(b-a)}{a}}$$

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$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

$$\begin{cases} a x^2 y_2'' + b x y_2' + c y_2 = 0 \\ a x^2 y_1'' + b x y_1' + c y_1 = 0 \end{cases}$$

$$\begin{cases} y_2 = u y_1 \\ y_2' = u' y_1 + u y_1' \\ y_2'' = u'' y_1 + 2u' y_1' + u y_1'' \end{cases}$$

$$\Rightarrow \begin{cases} a x^2 [u'' y_1 + 2u' y_1' + u y_1''] + b x [u' y_1 + u y_1'] + c u y_1 = 0 \\ u[a x^2 y_1'' + b x y_1' + c y_1] + a x^2 [u'' y_1 + 2u' y_1'] + b x [u' y_1] = 0 \end{cases}$$

Condition for $y_2(t)$ to be a solution

$$y_2(x) = u(x) y_1(x) \quad \Rightarrow \quad a x u'' y_1 + u'[2 a x y_1' + b y_1] = 0$$

We know one solution $y_1(x)$

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

Suppose the other solution $y_2(x) = u(x) y_1(x)$

$$a x^2 y'' + b x y' + c y = 0$$

$$a x^2 y_2'' + b x y_2' + c y_2 = 0 \quad \Rightarrow \quad a x u'' y_1 + u'[2 a x y_1' + b y_1] = 0$$

$$w(x) = u'(x) \quad \Rightarrow \quad a x w' y_1 + w[2 a x y_1' + b y_1] = 0$$

$$w' y_1 = -w[2 y_1' + \frac{b}{a x} y_1]$$

$$\frac{w'}{w} = -2 \frac{y_1'}{y_1} - \frac{b}{a x}$$

$$\frac{1}{w} \frac{dw}{dx} = -2 \frac{1}{y_1} \frac{dy_1}{dx} - \frac{b}{a x}$$

$$\int \frac{1}{w} \frac{dw}{dx} dx = -\int 2 \frac{1}{y_1} \frac{dy_1}{dx} dx - \int \frac{b}{a x} dx$$

$$\ln|w| = -\ln|y_1|^2 - \frac{b}{a} \ln x + c$$

$$\ln|w| + \ln|y_1|^2 = -\frac{b}{a} \ln x + c$$

$$\ln|w y_1^2| = -\frac{b}{a} \ln x + c$$

$$|w y_1^2| = C x^{-(b/a)}$$

$$w y_1^2 = c_1 x^{-(b/a)}$$

$$w = c_1 x^{-(b/a)} / y_1^2$$

$$u' = c_1 x^{-(b/a)} / y_1^2$$

$$u = c_1 \int \frac{x^{-(b/a)}}{y_1^2} dx + c_2$$

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We know one solution $y_1(x)$

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

Suppose the other solution $y_2(x) = u(x) y_1(x)$

$$a x^2 y'' + b x y' + c y = 0$$

$$a x^2 y_2'' + b x y_2' + c y_2 = 0 \quad \Rightarrow \quad a x u'' y_1 + u'[2 a x y_1' + b y_1] = 0$$

$$u = c_1 \int \frac{x^{-(b/a)}}{y_1^2} dx + c_2 \quad \Rightarrow \quad y_2 = y_1 \int \frac{x^{-(b/a)}}{y_1^2} dx$$

$$u = c_1 \int x^{-(b/a)} \cdot x^{-2m_1} dx + c_2$$

$$= \int \frac{x^{-(b/a)}}{x^{-((b-a)/a)}} dx + c_2$$

$$= \int \left(\frac{x^{-(b/a)}}{x^{-(b/a)} x^{(a/a)}} \right) dx + c_2$$

$$= c_1 \ln x + c_2$$

$$y_2 = c_1 y_1 \int \frac{x^{-(b/a)}}{y_1^2} dx + c_2 y_1 \qquad (c_1 = 1, \; c_2 = 0)$$

$$y_2 = y_1 \ln x$$

$$y_1 = x^{-\frac{(b-a)}{2a}}$$

$$y_1^2 = x^{-\frac{(b-a)}{a}}$$

General Solutions for the repeated roots case

$$y_2 = y_1 \int \frac{x^{-(b/a)}}{y_1^2} dx$$

$$m_1 = \{-(b-a) + \sqrt{(b-a)^2 - 4ac}\}/2a$$
$$m_2 = \{-(b-a) - \sqrt{(b-a)^2 - 4ac}\}/2a$$

$$\Rightarrow \quad (b-a)^2 - 4ac = 0$$

$$m_1 = m_2 = -(b-a)/2a$$
$$x^{m_1} = x^{m_2} = x^{-\frac{(b-a)}{2a}}$$

$$y_1(x) = x^{-\frac{(b-a)}{2a}}$$

$$y_1^2 = x^{-\frac{(b-a)}{a}}$$

$$y_2 = x^{-\frac{b}{2a}} \int \frac{x^{-(b/a)}}{x^{-(b/a)} x} dx = e^{-\frac{b}{2a}x} \int \frac{1}{x} dx$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$$\Rightarrow$$

$$y_1(x) = x^{-\frac{(b-a)}{2a}}$$

$$y_2(x) = x^{-\frac{(b-a)}{2a}} \ln x$$

$$y(x) = c_1 y_1(x) + c_2 y_1(x) \ln x$$

Constant v.s. Non-constant Coefficients

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

(A) $b^2 - 4ac > 0$ ➡ $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

(B) $b^2 - 4ac = 0$ ➡ $y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$

(C) $b^2 - 4ac < 0$ ➡ $y = C_1 e^{\alpha x} e^{+i\beta x} + C_2 e^{\alpha x} e^{-i\beta x} = e^{\alpha x}(C_3 \cos(\beta x) + C_4 \sin(\beta x))$

Homogeneous Second Order Cauchy-Euler Equation

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

$$ax^2 y'' + bxy' + cy = 0$$

$$x = e^{\ln x}$$
$$x^{i\beta} = e^{+i\beta \ln x}$$

(A) $(b-a)^2 - 4ac > 0$ ➡ $y = C_1 x^{m_1} + C_2 x^{m_2}$

(B) $(b-a)^2 - 4ac = 0$ ➡ $y = C_1 x^{m_1} + C_2 x^{m_1} \ln x$

(C) $(b-a)^2 - 4ac < 0$ ➡ $y = C_1 x^{\alpha} \cdot e^{+i\beta \ln x} + C_2 x^{\alpha} \cdot e^{+i\beta \ln x} = x^{\alpha}(C_3 \cos(\beta \ln x) + C_4 \sin(\beta \ln x))$

A Unifying View

Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

(A) $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

(B) $y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$

(C) $y = C_1 e^{\alpha x} e^{+i\beta x} + C_2 e^{\alpha x} e^{-i\beta x}$

$= e^{\alpha x}\left(C_3 \cos(\beta x) + C_4 \sin(\beta x)\right)$

x

Non-constant Coefficients

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

$x = e^{\ln x}$

$x^{i\beta} = e^{+i\beta \ln x}$

(A) $y = C_1 x^{m_1} + C_2 x^{m_2}$

(B) $y = C_1 x^{m_1} + C_2 x^{m_1} \ln x$

(C) $y = C_1 x^{\alpha} \cdot e^{+i\beta \ln x} + C_2 x^{\alpha} \cdot e^{+i\beta \ln x}$

$= x^{\alpha}\left(C_3 \cos(\beta \ln x) + C_4 \sin(\beta \ln x)\right)$

(A) $y = C_1 e^{m_1 \ln x} + C_2 e^{m_2 \ln x}$

(B) $y = C_1 e^{m_1 \ln x} + C_2 e^{m_1 \ln x} \ln x$

(C) $y = C_1 e^{\alpha \ln x} \cdot e^{+i\beta \ln x} + C_2 e^{\alpha \ln x} \cdot e^{+i\beta \ln x}$

$= e^{\alpha \ln x}\left(C_3 \cos(\beta \ln x) + C_4 \sin(\beta \ln x)\right)$

$\ln x$

Another Solution y_2 from y_1 (revisited)

We know one solution $y_1(x)$

$$a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$u = c_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx + c_2$$

$$ax^2\frac{d^2 y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

$$u = c_1 \int \frac{e^{-(bx/ax^2)x}}{(x^{m_1})^2} dx$$

$$= c_1 \int e^{-(b/a)} \cdot x^{-2m_1} dx$$

Suppose the other solution $y_2(x) = u(x) y_1(x)$

$$ay'' + by' + cy = 0$$

$$y_2 = y_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx$$

$$ax^2 y'' + bxy' + cy = 0$$

$$y_2 = y_1 \int \frac{e^{-(b/a)\ln x}}{y_1^2} dx$$

$$= y_1 \int \frac{x^{-(b/a)}}{y_1^2} dx$$

$$x$$

$$y_1 = x^{-\frac{(b-a)}{2a}}$$

$$y_1^2 = x^{-\frac{(b-a)}{a}}$$

$$\ln x$$

References

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