Random Process Background (1C)

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Outline

Collection (1)

- ets can also contain other sets.
- For example, {Z, Q} is a set containing two infinite sets.
- {{a, b}, {c}} is a set containing two finite sets.
- sets that contain other sets.
- use the term collection to refer to a set that contains other sets,
  and use a script letter for its variable name.

https://mfleck.cs.illinois.edu/building-blocks/version-1.3/sets-of-sets.pdf

Collection (2)

- Theorem 1: If F is an arbitrary collection of open sets then U$A \in F$A is an open set.
- By "arbitrary" we mean that F can be a finite, countably infinite, or uncountably infinite collection of sets.
- Proof: Let F be an arbitrary collection of open sets and let:
- (1) S=U$A \in F$A
- We want to show that S=int(S).
- First suppose that $x \in S$. Then $x \in A$ for some set $A \in F$. Since A is an open set, there exists an $r>0$ such that $B(x,r) \subseteq A$. But $A \subseteq S$, so by extension, there exists an $r>0$ such that $B(x,r) \subseteq S$, so $x \in int(S)$ and hence $S \subseteq int(S)$.
- Now suppose that $x \in int(S)$. Then for some $r>0$ there exists a $B(x,r) \subseteq S$. Since $x \in B(x,r)$ we have that by extension, $x \in S$, so $int(S) \subseteq S$.

Collection (3)

- heorem 2: If F={A1,A2,...,An} is a finite collection of open sets then ∩i=1nAi is an open set.
- Proof: Let F={A1,A2,...,An} be a finite collection of open sets and let:
- (2) S=∩i=1nAi
- Once again, we want to show that S=int(S).
- Let x∈S. Then x∈Ai for all i∈{1,2,...,n} and so for each i there exists some ri>0 such that:
- (3) B(x,ri)⊆Aiforalli=1,2,...,n
- Let r=min{r1,r2,...,rn}. Then we have that B(x,r)⊆Ai for all i∈I, so B(x,r)⊆S. Hence there exists an r>0 such that B(x,r)⊆S so x∈S and S⊆int(S).
- Now suppose that x∈int(S). Then once again there exists an

Open set examples

- The *circle* represents the set of points (x, y)
  satisfying $x^2 + y^2 = r^2$.
  the *circle* set is its **boundary set**
- The *disk* represents the set of points (x, y)
  satisfying $x^2 + y^2 < r^2$.
  The *disk* set is an **open set**
- the union of the *circle* and *disk* sets
  is a **closed set**. (boundary set + open set)

https://en.wikipedia.org/wiki/Open_set

Open set (1)

- an **open set** is a generalization of an **open interval** in the real line.
- a **metric space** is a set along with a distance defined between any two points
- in a **metric space**, an **open set** is a set that, along with every point P, contains all points that are sufficiently near to P
    - all points whose distance to P is less than some value depending on P

https://en.wikipedia.org/wiki/Open_set

Open set (2-1)

- more generally, an **open set** is
  a member of a given collection of subsets of a given set

> - a given set
> - subsets of a given set
> - a given collection of subsets of a given set

https://en.wikipedia.org/wiki/Open_set

Open set (2-2)

- a **collection** has the following property of containing
  - a **collection** contains
    - every **union** of its members
    - every finite **intersection** of its members
    - the **empty set**
    - the **whole set** itself

https://en.wikipedia.org/wiki/Open_set

Open set (3)

- These conditions are very loose, and allow enormous flexibility in the choice of open sets.
- For example,
  - every subset can be open (the **discrete topology**)
  - no subset can be open (the **indiscrete topology**) except
    - the space itself and
    - the empty set

https://en.wikipedia.org/wiki/Open_set

Open set (4)

- A set in which such a collection is given is called a **topological space**, and the collection is called a **topology**.
  - A **set** is a collection of distinct objects.
  - Given a **set** A, we say that a is an **element** of A

    if a is one of the distinct objects in A,
    and we write $a \in A$ to denote this
  - Given two **sets** A and B, we say that A is a **subset** of B

    if every element of A is also an element of B
    write $A \subseteq B$ to denote this.

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Open set (5) Open Balls

- An **open ball** $B_r(\boldsymbol{a})$ in $\mathbb{R}^n$
  centered at $\boldsymbol{a} = (a_1, \dots a_n) \in \mathbb{R}^n$ with radius r

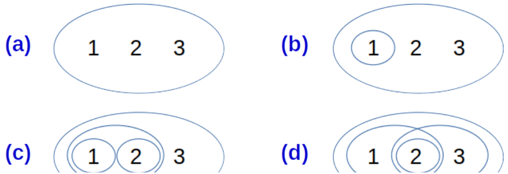
  is the set of all points $\boldsymbol{x} = (x_1, \dots x_n) \in \mathbb{R}^n$
  such that the distance between x and $\boldsymbol{a}$ is less than r
- In $\mathbb{R}^2$ an **open ball** is often called an **open disk**

We give these definitions in general, for when one is working in $\mathbb{R}^n$
since they are really not all that different to define in $\mathbb{R}^n$ than in $\mathbb{R}^2$

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

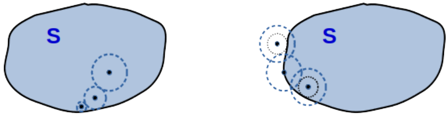
Open set (6) Interior points

- Suppose that $S \subseteq \mathbb{R}^n$
- A **point** $\boldsymbol{p} \in S$ is an **interior point** of S if there exists an open ball $B_r(\boldsymbol{p}) \subseteq S$
- Intuitively, $\boldsymbol{p}$ is an **interior point** of S if we can *squeeze* an entire open ball centered at $\boldsymbol{p}$ within S



Open set (7) Boundary points

- A point $\boldsymbol{p} \in \mathbb{R}^n$ is a **boundary point** of S if all open balls centered at $\boldsymbol{p}$ contain both points in S and points not in S
- The **boundary** of S is the set ∂S that consists of all of the **boundary points** of S.



an interior point | a boundary point

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Open set (8) Open and Closed Sets

- A set $O \subseteq \mathbb{R}^n$ is **open**
  if every point in O is an interior point.
- A set $C \subseteq \mathbb{R}^n$ is **closed**
  if it contains all of its boundary points.

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Open set (9) Bounded and Unbounded

- A set S is **bounded** if there is an open ball $B_M(0)$ such that

$$S \subseteq B.$$

intuitively, this means that we can enclose all of the set S within a large enough ball centered at the origin, $B_M(0)$

- A set that is not **bounded** is called **unbounded**

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Family of sets (1)

- a **collection** F of subsets of a given set S is called
  a **family** of **subsets** of S, or
  a **family** of **sets** over S.
- More generally,
  a **collection** of any sets whatsoever is called
  a **family** of **sets**,
  set family, or
  a **set system**

https://en.wikipedia.org/wiki/Family_of_sets

Family of sets (2)

- The term "**collection**" is used here because,
  - in some contexts,
    a **family** of **sets** may be allowed
    to contain repeated copies of any given member, and
  - in other contexts
    it may form a proper class rather than a set.

https://en.wikipedia.org/wiki/Family_of_sets

Examples of family of sets (1)

- The set of all subsets of a given set S is called the **power set** of S and is denoted by $\wp(S)$.

  The **power set** $\wp(S)$ of a given set S is a **family** of **sets** over S.

- A subset of S having k elements is called a k-subset of S.

  The k-subset $S^{(k)}$ of a set S form a **family** of **sets**.

https://en.wikipedia.org/wiki/Family_of_sets

Examples of family of sets (2)

- Let $S = \{a, b, c, 1, 2\}$.
  An example of a **family** of **sets** over S
  (in the multiset sense) is given by $F = \{A_1, A_2, A_3, A_4\}$, where
  $A_1 = \{a, b, c\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2\}$, and $A_4 = \{a, b, 1\}$.

https://en.wikipedia.org/wiki/Family_of_sets

Neighbourhood basis (1)

- A **neighbourhood basis** or **local basis** (or **neighbourhood base** or **local base**) for a point x is a filter base of the neighbourhood filter;
- this means that it is a subset $\mathscr{B} \subseteq \mathscr{N}(x)$ such that for all $V \in \mathscr{N}(x)$, there exists some $B \in \mathscr{B}$ such that $B \subseteq V$. That is, for any **neighbourhood** V we can find a **neighbourhood** B in the **neighbourhood basis** that is contained in V.

https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis

Neighbourhood basis (2)

- Equivalently, $\mathscr{B}$ is a local basis at x if and only if the neighbourhood filter $\mathscr{N}$ can be recovered from $\mathscr{B}$ in the sense that the following equality holds:

$$\mathscr{N}(x) = \{V \subseteq X \ : \ B \subseteq V \text{ for some } B \in \mathscr{B}\}$$

- A family $\mathscr{B} \subseteq \mathscr{N}(x)$ is a neighbourhood basis for x if and only if $\mathscr{B}$ is a cofinal subset of $(\mathscr{N}(x), \supseteq)$ with respect to the partial order $\supseteq$ (importantly, this partial order is the superset relation and not the subset relation).

https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis

A collection of sets around x

- In general, one refers to the family of sets containing 0, used to **approximate** 0, as a neighborhood basis;
- a member of this neighborhood basis is referred to as an **open set**.
- In fact, one may generalize these notions to an arbitrary set (X); rather than just the real numbers.
- In this case, given a point (x) of that set (X), one may define a **collection** of sets "around" (that is, containing) x, used to **approximate** x.

https://en.wikipedia.org/wiki/Open_set

Smaller sets containing x

- Of course, this collection would have to *satisfy* certain properties (known as axioms) for otherwise we may not have a *well-defined method* to measure distance.
- For example, every point in X should **approximate** x to some degree of accuracy.
- Thus X should be in this family.
- Once we begin to define "smaller" sets containing x, we tend to **approximate** x to a greater degree of accuracy.
- Bearing this in mind, one may define the remaining axioms that the family of sets about x is required to satisfy.

https://en.wikipedia.org/wiki/Open_set

Outline

Open ball (1)

- a **ball** is the solid figure bounded by a **sphere**;
  it is also called a **solid sphere**.
  - a **closed ball**
    includes the *boundary points* that constitute the sphere
  - an **open ball**
    excludes them

https://en.wikipedia.org/wiki/Ball_(mathematics)

Open ball (2)

- A **ball** in n dimensions is called a **hyperball** or **n-ball** and is bounded by a **hypersphere** or $(\boldsymbol{n}-1)$**-sphere**
- One may talk about **balls** in any topological space X, not necessarily induced by a metric.
- An n-dimensional **topological ball** of X is any subset of X which is homeomorphic to an **Euclidean n-ball**.

https://en.wikipedia.org/wiki/Ball_(mathematics)

Neighborhood (1)

- a neighbourhood is one of the basic *concepts* in a topological space.
- It is closely related to the *concepts* of open set and interior.
- Intuitively speaking, a neighbourhood of a point is
  a set of points containing that point
  where one can move some amount in any direction
  away from that point without leaving the set.

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Interior

- the interior of a subset S of a topological space X is the union of *all* subsets of S that are open in X.
- A point that is in the interior of S is an interior point of S.
- The interior of S is the complement of the closure of the complement of S.
  the closure of (boundary + exterior)
- In this sense, interior and closure are dual notions.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Exterior

- The exterior of a set S is the complement of the closure of $\underline{S}$;
  the closure of S = boundary + interior
- it consists of the points that are
  in neither the set nor its boundary.
- The interior, boundary, and exterior of a subset together partition the whole space into three blocks
- fewer when one or more of these is empty

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Interior Point (1)

- If S is a subset of a Euclidean space, then x is an interior point of S

  if there exists an open ball centered at x which is completely contained in S.

- This definition generalizes to any subset S of a metric space X with metric d:

  x is an interior point of S if there exists a real number $r > 0$, such that y is in S whenever the distance $d(x, y) < r$.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Interior Point (2)

- This definition generalizes to topological spaces by replacing "open ball" with "open set".
  - if there exists an *open ball* centered at x which is completely contained in S.
  - if x is contained in an *open subset* of X that is completely contained in S.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Interior Point (3)

- If S is a subset of a topological space X then x is an interior point of S in X

  if x is contained in an open subset of X that is completely contained in S.
- Equivalently, x is an interior point of S if S is a neighbourhood of x.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Interior of a Set (1)

- The interior of a subset S of a topological space X, can be defined in any of the following equivalent ways:
  - the largest open subset of X contained in S.
  - the union of all open sets of X contained in S.
  - the set of all interior points of S.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Interior of a Set (2)

- The interior of a subset S of a topological space X, denoted by $int_X S$ or $int S$ or S°
- If the space X is understood from context then the shorter notation $int S$ is usually preferred to $int_X S$.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Neighborhood of a point (1-1)

- If X is a topological space and p is a point in X, then a neighbourhood of p is a subset V of X that includes an open set U containing p,

$$p \in U \subseteq V \subseteq X.$$

  - X : a topological space
  - V : a subset of X
  - U : an open set containing p
  - p : a point in X
  - V : a neighbourhood of p

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Neighborhood of a point (1-2)

- This is also equivalent to the point $p \in X$ belonging to the topological interior of V in X.
- The neighbourhood V need not be an open subset of X, but when V is open in X then it is called an open neighbourhood.
- Some authors have been known to require neighbourhoods to be open, so it is important to note conventions.

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Neighborhood of a point (2)

- A set that is a neighbourhood of each of its points is open since it can be expressed as the union of open sets containing each of its points.
- A closed rectangle, as illustrated in the figure, is not a neighbourhood of all its points;
  - points on the edges or corners of the rectangle are not contained in any open set that is contained within the rectangle.
- The collection of all neighbourhoods of a point is called the neighbourhood system at the point.

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Neighborhood of a set (1-1)

- If S is a subset of a topological space X, then a neighbourhood of S is a set V that includes an open set U containing S,

$$S \subseteq U \subseteq V \subseteq X.$$

- It follows that a set V is a neighbourhood of S if and only if it is a neighbourhood of all the points in S.
- Furthermore, V is a neighbourhood of S if and only if S is a subset of the interior of V.

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Neighborhood of a set (1-2)

- A neighbourhood of S that is also an open subset of X is called an open neighbourhood of S.
- The neighbourhood of a point is just a special case of this definition.

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Neighborhood definition (1)

- the open set axioms are often taken as the definition of a topology, when they are quite *unintuitive*, though extremely useful in the long run.
- the neighbourhood definition, while somewhat *cumbersome*, has the advantage of being closely related to ideas from analysis, and has a *historical basis*

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology

Neighborhood definition (2-1)

- A neighbourhood topology on a set X
  assigns to each element $x \in X$
  a non empty set $N(x)$ of subsets of X,
  called neighbourhoods of x
- with the following properties:

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology

Neighborhood definition (2-2)

- the properties of a neighbourhood topology:
  - If N is a neighbourhood of x then $x \in X$
  - If M is a neighbourhood of x and $M \subseteq N \subseteq X$, then N is a neighbourhood of x
  - The intersection of two neighbourhoods of x is a neigbourhood of x
  - If N is a neighbourhood of x, then N contains a neighbourhood M of x such that N is a neighbourhood of each point of M.

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology

Neighborhood definition (3-1)

- Then one says a function $f : X \to Y$ is continuous wrt neighbourhoods on X and Y if for each $x \in X$ and neighbourhood N of $f(x)$ there is a neighbourhood M of x such that $f(M) \subseteq N$.
- The open set definition of continuity is then justified as being equivalent to this definition in terms of neighbourhoods.

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology

Neighborhood definition (3-2)

- One also says a set U in X is open
  if U is a neighbourhood of all of its points.
  THEN one can develop the open set axioms
  and show that one can recover the neighbourhoods.

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology

Outline

Class (1)

- a **class** is a collection of sets
  (or sometimes other mathematical objects)
  that can be unambiguously defined
  by a property that all its members share.
- **Classes** act as a way to have set-like collections
  while differing from sets so as to avoid **Russell's paradox**

https://en.wikipedia.org/wiki/Class_(set_theory)

Class (2)

- A **class** *that is not a set* is called a **proper class**, and
- a **class** *that is a set* is sometimes called a **small class**.
- the followings are **proper classes** in many formal systems
  - the **class** of all sets
  - the **class** of all ordinal numbers
  - the **class** of all cardinal numbers

https://en.wikipedia.org/wiki/Class_(set_theory)

Class (3)

- consider "the set of all sets with property X."
- especially when dealing with categories,
  since the objects of a concrete category
  are all sets with certain additional structure.
- However, **if** we're not *careful* about this
  we can get into serious *trouble* –

https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects

Class (4)

- let X be the set of all sets which do not contain *themselves*
- Since X is a set, we can ask whether X is an element of *itself*.
- But then we run into a paradox – **if** X contains *itself* as an element, **then** it does not, and vice versa.

https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects

Class (5)

- In order to avoid this paradox,
  we cannot consider the collection of all sets
  to be itself a set.
- This means we have to *throw out*
  the whole "the set of all sets with property X" construction.
  But we wanted that.
- So the way we get around it is to say that
  there's something called a **class**,
  which is like a set but not a set.

https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects

Class (6)

- Then we can talk about
  "the **class** X of all sets with property Y."
- Since X is not a set,
  it can't be an element of itself, and we're fine.
- Of course, if we need to talk about the collection of all classes,
  we need to create another name that goes another step back,
  and so forth.

https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects

Class Examples (1)

- The **collection** of all algebraic structures of a given type will usually be a **proper class**.
  (a **class** *that is not a set* is called a **proper class**)
  - the **class** of all groups
  - the **class** of all vector spaces
  - and many others.
- Within set theory, many **collections** of sets turn out to be proper classes.

https://en.wikipedia.org/wiki/Class_(set_theory)

Class Examples (2)

- One way to *prove* that a **class** is **proper** is to place it in bijection with the class of all ordinal numbers.
  - **Cardinal numbers** indicate an amount how many of something we have: one, two, three, four, five.
  - **Ordinal numbers** indicate position in a series: first, second, third, fourth, fifth.

https://en.wikipedia.org/wiki/Class_(set_theory)
https://editarians.com/cardinals-ordinals/

Class Paradoxes (1)

- The **paradoxes** of naive set theory can be explained in terms of the *inconsistent tacit assumption* that "all classes are sets".
- These paradoxes do not arise with classes because there is no notion of classes containing classes.
- Otherwise, one could, for example, define a class of all classes that do not contain themselves, which would lead to a Russell paradox for classes.

https://en.wikipedia.org/wiki/Class_(set_theory)

Class Paradoxes (2)

- With a rigorous foundation, these **paradoxes** instead *suggest proofs* that certain classes are proper (i.e., that they are not sets).
  - **Russell's paradox** *suggests* a *proof* that the class of all sets which do not contain themselves is proper
  - the **Burali-Forti paradox** *suggests* that the class of all ordinal numbers is proper.

https://en.wikipedia.org/wiki/Class_(set_theory)

Russell's Paradox (1)

- According to the unrestricted comprehension principle,
  for any sufficiently well-defined property,
  there is the set of all
  and only the objects that have that property.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (2)

- Let R be the set of all sets ($R = \{x \mid x \notin x\}$) that are not members of themselves ($R \notin R$).
  - ***if*** R is not a member of itself ($R \notin R$), ***then*** its definition (the set of all sets) entails that it is a member of itself ($R \in R$);
  - yet, ***if*** it is a member of itself ($R \in R$), ***then*** it is not a member of itself ($R \notin R$), since it is the set of all sets that are not members of themselves ($R \notin R$)
- the resulting contradiction is **Russell's paradox**.
- Let $R = \{x \mid x \notin x\}$, then $R \in R \iff R \notin R$

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (3)

- *Most sets* commonly encountered are not members of themselves.
- For example, consider the set of all squares in a plane.
- This set is not itself a square in the plane, thus it is not a member of itself.
- Let us call a set "**normal**" if it is not a member of itself, and "**abnormal**" if it is a member of itself.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (4)

- Clearly *every* set must be either normal or abnormal.
- The set of squares in the plane is normal.
- In contrast, the complementary set
  that contains everything which is not a square in the plane
  is itself not a square in the plane,
  and so it is one of its own members
  and is therefore abnormal.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (5)

- Now we consider the set of all normal sets, R, and try to determine whether R is normal or abnormal.
  - ***If*** R were normal,
    it would be contained
    in the set of all normal sets (itself),
    and therefore be abnormal;
  - on the other hand ***if*** R were abnormal,
    it would <u>not</u> be contained
    in the set of all normal sets (itself),
    and therefore be normal.
- This leads to the conclusion that
  R is neither normal nor abnormal: **Russell's paradox.**

https://en.wikipedia.org/wiki/Russell%27s_paradox

Outline

Binary Relation (1)

- a **binary relation** associates elements of one set, called the domain, with elements of another set, called the codomain.
- A **binary relation** over sets X and Y is a new set of ordered pairs (x, y) consisting of elements x from X and y from Y.

https://en.wikipedia.org/wiki/Binary_relationelation

Binary Relation (2)

- It is a generalization of a unary function.
- It encodes the common concept of relation:
- an element x is related to an element y,
  if and only if the pair (x, y) belongs
  to the set of ordered pairs that defines the binary relation.
- A binary relation is the most studied special case $n = 2$
  of an n-ary relation over sets $X_1, ..., X_n$,
  which is a subset of the Cartesian product $X_1 \times \cdots \times X_n$.

https://en.wikipedia.org/wiki/Binary_relationelation

Homogeneous Relation

- a **homogeneous relation** (also called endorelation) on a set X is a binary relation between X and itself, i.e.
  it is a subset of the Cartesian product $X \times X$.
- This is commonly phrased as "a **relation** on X"
  or "a (**binary**) **relation** over X".
- An example of a **homogeneous relation** is the relation of kinship, where the relation is between people.

https://en.wikipedia.org/wiki/Homogeneous_relation

Partially Ordered Set (1-1)

- a **partial order** on a set is an arrangement such that, for certain pairs of elements, one precedes the other.
- The word **partial** is used to indicate
  that not every pair of elements needs to be comparable;
  that is, there may be pairs for which neither element precedes the other.
- **Partial orders** thus generalize **total orders**, in which every pair is comparable.

https://en.wikipedia.org/wiki/Partially_ordered_set

Partially Ordered Set (1-2)

- Formally, a **partial order** is a homogeneous binary relation that is reflexive, transitive and antisymmetric.
- A **partially ordered set** (**poset** for short) is a set on which a **partial order** is defined.
- A reflexive, weak, or non-strict **partial order**, commonly referred to as simply as a **partial order**, is a homogeneous relation $\leq$ on a set P that is reflexive, antisymmetric, and transitive.

https://en.wikipedia.org/wiki/Partially_ordered_set

Partially Ordered Set (2)

- a homogeneous relation $\leq$ on a set P that is reflexive, antisymmetric, and transitive.
- That is, for all $a, b, c \in P$, it must satisfy:
  - Reflexivity:
    $a \leq a$, i.e. every element is related to itself.
  - Antisymmetry:
    if $a \leq b$ and $b \leq a$ then $a = b$,
    i.e. no two distinct elements precede each other.
  - Transitivity:
    if $a \leq b$ and $b \leq c$ then $a \leq c$.
- A non-strict **partial order** is also known as an antisymmetric preorder.

https://en.wikipedia.org/wiki/Partially_ordered_set

Filter in Set Theory (1-1)

- A **filter** on a set may be thought of as representing a "collection of *large* subsets", one intuitive example being the **neighborhood filter**.
- keep *large* grains excluding *small* impurities

https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

Filter in Set Theory (1-2)

- When you put a filter in your sink,
  the idea is that you filter out the *big* chunks of food,
  and let the water and the *smaller* chunks go through
  (which can, in principle, be washed through the pipes) .
- You filter out the *larger parts*.
- A filter filters out the *larger* sets.
- It is a way to say "these sets are *'large'*"

https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

Filter in Set Theory (1-3)

- a **filter** on a set X is a family $\mathscr{B}$ of subsets such that:

1. $X \in \mathscr{B}$ and $\emptyset \notin \mathscr{B}$
2. if $A \in \mathscr{B}$ and $B \in \mathscr{B}$,
   then $A \cap B \in \mathscr{B}$
3. If $A, B \subset X, A \in \mathscr{B}$, and $A \subset B$,
   then $B \in \mathscr{B}$

https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

Filter in Set Theory (1-4)

- The set of "everything" is definitely *large*

$X \in \mathscr{B}$

- and "nothing" is definitely not;

$\emptyset \notin \mathscr{B}$

- if something is *larger* than a *large* set, then it is also *large*;

If $A, B \subset X, A \in \mathscr{B}$, and $A \subset B$, then $B \in \mathscr{B}$

- and two *large* sets *intersect* on a *large* set.

If $A \in \mathscr{B}$ and $B \in \mathscr{B}$, then $A \cap B \in \mathscr{B}$

https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

Filter in Set Theory (1-5)

- you can think about this as
  - being **co-finite**,
  - or being of **measure 1** on the **unit interval**,
  - or having a **dense open subset** (again on the unit interval).
- These are examples of ways
  where a set can be thought of as "almost everything".
  and that is the idea behind a filter.

https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

Co-finite

- a **cofinite** subset of a set X is
  a subset A whose complement in X is a finite set.
- a subset A contains all but *finitely many* elements of X
- If the complement is not finite, but is countable,
  then one says the set is **cocountable**.
- These arise naturally when generalizing structures
  on finite sets to infinite sets,
  particularly on infinite products,
  as in the product topology or direct sum.
- This use of the prefix "**co**" to describe a property
  possessed by a set's complement
  is *consistent* with its use in other terms such as "comeagre set".

https://en.wikipedia.org/wiki/Cofiniteness

Unit interval

- the **unit interval** is the closed interval [0,1], that is, the set of all real numbers that are greater than or equal to 0 and less than or equal to 1.
- It is often denoted **I** (capital letter I).
- In addition to its role in real analysis, the **unit interval** is used to study homotopy theory in the field of topology.
- the term "**unit interval**" is sometimes applied to the other shapes that an interval from 0 to 1 could take: (0,1], [0,1), and (0,1).
- However, the notation **I** is most commonly reserved for the closed interval [0,1].

https://en.wikipedia.org/wiki/Unit_interval

Dense set

- In topology, a subset A of a topological space X is said to be **dense** in X if *every* point of X either belongs to A or else is arbitrarily "close" to a member of A
  - for instance, the rational numbers are a **dense** subset of the real numbers because every real number either is a rational number or has a rational number arbitrarily close to it (see **Diophantine approximation**).
- Formally, A is **dense** in X if the *smallest* closed subset of X containing A is X itself.
- The **density** of a topological space X is the least cardinality of a **dense subset** of X.

https://en.wikipedia.org/wiki/Dense_set

Proper Subset

- a set A is a subset of a set B
  if all elements of A are also elements of B;
- B is then a superset of A.
- It is possible for A and B to be equal;
- if they are unequal, then A is a proper subset of B.
- The relationship of one set being a subset of another
  is called inclusion (or sometimes containment).
- A is a subset of B may also be expressed
  as B includes (or contains) A or A is included (or contained) in B.
- A k-subset is a subset with k elements.

https://en.wikipedia.org/wiki/Subset

Outline

Proper Filter (1-1)

- Fix a partially ordered set (poset) P.
- Intuitively, a filter F is a subset of P whose members are elements large enough to satisfy some *criterion*.
- For instance, if $x \in P$, then the set of elements above x is a filter, called the principal filter at x.

https://en.wikipedia.org/wiki/Filter_(mathematics)

Proper Filter (1-2)

- If $x$and y are incomparable elements of P,
  then neither the principal filter at x nor y
  is contained in the other
  - two elements x and y of a set P are said to be **comparable** with respect to a binary relation $\leq$
    if at least one of $x \leq y$ or $y \leq x$ is true.
    They are called **incomparable** if they are not **comparable**.
  - Hasse diagram of the natural numbers,
    partially ordered by "$x \leq y$ if x divides y".
    The numbers 4 and 6 are **incomparable**,
    since neither divides the other.

https://en.wikipedia.org/wiki/Filter_(mathematics)
https://en.wikipedia.org/wiki/Comparability

Proper Filter (1-3)

- Similarly, a filter on a set S contains
  those subsets that are sufficiently large
  to contain some given *thing*.
- For example, if S is the *real line* and $x \in S$,
  then the family of sets including x *in their interior* is
  a filter, called the neighborhood filter at x.
- The *thing* in this case is *slightly larger* than x,
  but it still does not contain any other specific point of the line.

https://en.wikipedia.org/wiki/Filter_(mathematics)

Proper Filter (2)

- The above considerations motivate
  the upward closure requirement in the definition below:
  "*large enough*" objects can always be made *larger*.
- To understand the other two conditions,
  reverse the roles and instead consider F
  as a "*locating scheme*" to find x.
- In this interpretation, one searches in some space X,
  and expects F to describe those subsets of X that contain the goal.
- The goal must be located somewhere;
  thus the empty set $\varnothing$ can never be in F.
- And if two subsets both contain the goal,
  then should "zoom in" to their common region.

https://en.wikipedia.org/wiki/Filter_(mathematics)

Proper Filter (3)

- An ultrafilter describes a "perfect locating scheme" where each scheme component gives new information (either "look here" or "look elsewhere").
- Compactness is the property that "every search is fruitful," or, to put it another way, "every locating scheme ends in a search result."
- A common use for a filter is to define properties that are satisfied by "generic" elements of some topological space.
- This application generalizes the "locating scheme" to find points that might be hard to write down explicitly.

https://en.wikipedia.org/wiki/Filter_(mathematics)

Neighborhood Filter (1-1)

- Let X be a set;
- the elements of X are usually called points
- We allow X to be empty.
- Let $\mathcal{N}$ be a function
  assigning to each x (point) in X
  a non-empty collection $\mathcal{N}(x)$ of subsets of X.
- The elements of $\mathcal{N}(x)$ will be called
  neighbourhoods of x with respect to $\mathcal{N}$
  (or, simply, neighbourhoods of x).

https://en.wikipedia.org/wiki/Topological_space

Neighborhood Filter (1-2)

- Let X be a set;
- $\mathcal{N}$: a function assigning to each point x in X
- $\mathcal{N}(x)$: a non-empty collection of subsets of X.
- The elements of $\mathcal{N}(x)$
  - subsets of X
  - neighbourhoods of x with respect to $\mathcal{N}$

https://en.wikipedia.org/wiki/Topological_space

Neighborhood Filter (1-3)

- The function $\mathscr{N}$ is called
  a neighbourhood topology
  if *some axioms* are satisfied;
- then X with $\mathscr{N}$ is called
  a topological space – $(X, \mathscr{N})$

https://en.wikipedia.org/wiki/Topological_space

Neighborhood Filter (1-4)

- If $(X, \mathscr{T})$ is a topological space and $p \in X$,
  a neighbourhood of p is a subset V of X,
  in which $p \in U \subseteq V$, and U is open.
- We say that V is a $\mathscr{T}-$ neighbourhood of $x \in X$
  or that V is a neighborhood of x
- The set of all neighbourhoods of $x \in X$, denoted $\mathscr{N}_{\mathscr{X}}$
  is called the neighbourhood filter of x

https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter

Neighborhood Filter (1-4)

- An example of Neighborhood Filters on a Topological space.
- Let $X = \{a, b, c\}$ and let $\mathscr{T} = \{\varnothing, \{a\}, \{b\}, \{b, c\}, \{a, b\}, X\}$
- Let
  $\mathscr{N}_a = \{\{a\}, \{a, b\}, \{a, c\}, X\}$
  $\mathscr{N}_b = \{\{b\}, \{a, b\}, \{b, c\}, X\}$
  $\mathscr{N}_c = \{\{b, c\}, X\}$.
- In this example a, c is a neighborhood of a but not of c.
- Thus a set does not have to be a neighborhood of all of its points.

https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter

Neighborhood Filter (2)

- One can specify a topology in more than four different ways.
  1. The standard definition specifies the open sets, what we usually call a "topology."
  2. to specify the close sets - this is of course only a *trivial difference*.
  3. to specify a closure operation on subsets of your space
  4. to specify a neighborhood filter for every point satisfying the natural axiom that every neighborhood of x is a neighborhood of every point of one of its subsets

  - So in this sense neighborhood filters tell you everything they possibly could about a topological space.

https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter

Neighborhood Filter (3-1)

- Probably the best way to think about the neighborhood filter of x: is that it contains all information regarding convergence to x.
- In the first topological spaces one encounters, convergence is usually of sequences.
- But this isn't enough to describe the topology in arbitrary spaces, for instance the infinite-dimensional spaces of functional analysis.
- It becomes important to speak of convergence of nets, or of filters.

https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter

Neighborhood Filter (3-2)

- A filter on X is just a nontrivial subset of the powerset of X closed under finite intersection and superset, and a filter converges to a point x if and only if it contains the neighborhood filter of x.
- In contrast to the case with sequences, this is enough to specify a topology: in fact it's enough to describe how ultrafilters, that is, maximal filters, converge.
- So in this sense the neighborhood filter encapsulates the viewpoint that topology generalizes the study of convergent sequences.

https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter

Neighborhood Filter (4-1)

- in a sense the neighborhood filter describes
  the smallest neighborhood of a point
  - except that there is no smallest neighborhood!
- That's true, at least, in many of the most interesting spaces,
  and is the main reason to worry about a whole filter of neighborhoods
  - if there were a smallest neighborhood then in any hypothesis
  requiring something to hold on a sufficiently small neighborhood of x
  we could just pick the smallest neighborhood.

https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter

Neighborhood Filter (4-2)

- But the smallest neighborhood of a point must be contained in the intersection of all its neighborhoods, and in, say, a Hausdorff space the intersection of all neighborhoods of x is x, which is not a neighborhood of x when x is not isolated.
- So the filter functions as a virtual smallest neighborhood of x: it doesn't converge to a neighborhood of x, so we can't think about its limit, but functionally we do just that.

https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter

Ultrafilter (1)

- an **ultrafilter** on a given partially ordered set (or "poset") P is a certain subset of P, namely a maximal filter on P; that is, a proper filter on P that cannot be enlarged to a bigger proper filter on P.
- If X is an arbitrary set, its power set $\mathscr{P}(X)$, ordered by set inclusion, is always a Boolean algebra and hence a poset, and ultrafilters on $\mathscr{P}(X)$ are usually called ultrafilter on the set X.

https://en.wikipedia.org/wiki/Ultrafilter

Ultrafilter (2)

- In order theory, an **ultrafilter** is
  a subset of a partially ordered set
  that is maximal among all proper filters.
- This implies that any filter that properly contains an ultrafilter
  has to be equal to the whole poset.

https://en.wikipedia.org/wiki/Ultrafilter

Ultrafilter (3)

- An ultrafilter on a set X may be considered as a finitely additive measure on X.
- In this view, every subset of X is

  either considered "*almost everything*" (has measure 1) or

  "*almost nothing*" (has measure 0),
  depending on whether it belongs to the given ultrafilter or not

https://en.wikipedia.org/wiki/Ultrafilter

Ultrafilter (4)

- Formally, if P is a set, partially ordered by $\leq$ then
- a subset $F \subseteq P$ is called a filter on P
  if F is nonempty, for every $x, y \in F$,
  there exists some element $z \in F$
  such that $z \leq x$ and $z \leq y$,
  and for every $x \in F$ and $y \in P$,
  $x \leq y$ implies that y is in F too;
- a proper subset U of P is called
  an ultrafilter on P if U is a filter on P,
  and there is no proper filter F on P
  that properly extends U
  (that is, such that U is a proper subset of F).

https://en.wikipedia.org/wiki/Ultrafilter

Outline

Filter Examples (1)

- Let $X = 1, 2, 3$
  Choose some element from X say $F = 1, 1, 2, 1, 3, 1, 2, 3$
- Then every intersection of an element of F
  with another element in F is in F again.
  Examples: $1 \bigcap 1, 2, 3 = 1 \quad 1, 2 \bigcap 1, 2, 3 = 1, 2$
  $1, 3 \bigcap 1, 2, 3 = 1, 3 \quad 1, 2, 3 \bigcap 1, 2, 3 = 1, 2, 3$
- Also the original $X = 1, 2, 3$ is also in F.
  Here $F = 1, 1, 2, 1, 3, 1, 2, 3$ is called the filter on $X = 1, 2, 3$

https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory

Filter Examples (2)

- .Suppose we have the collection $G = 1, 1, 2, 1, 3, 2, 3, 1, 2, 3$
- Then we have $1, 3 \bigcap 2, 3 = 3$ but 3 isn't in G.
  So this G is not called a filter.
- Now with $F = 1, 1, 2, 1, 3, 1, 2, 3$
  can we put as any other element in it
  so that after placing the extra element it is still a filter?
  Probably not in this case.
  So on $X = 1, 2, 3$, $\quad F = 1, 1, 2, 1, 3, 1, 2, 3$ is an Ultrafilter.

https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory

Filter Examples (3)

- If we have started say with $H = 1, 1, 2, 1, 2, 3$
  this is still a filter on $X = 1, 2, 3$
  but we can still add $1, 3$
  and it will still be classified as filter.
- So on $X = 1, 2, 3$
  $F = 1, 1, 2, 1, 3, 1, 2, 3$ is an Ultrafilter
  but $H = 1, 1, 2, 1, 2, 3$ is a filter but not an Ultrafilter.

https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory

Filter Examples (4)

- Now suppose we have $X = 1, 2, 3, 4$
  Let $F = 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$
- Every in intersection of element of F is in F again.
  We have as examples $1, 4 \bigcap 1, 4 = 1, 4 \quad 1, 4 \bigcap 1, 2, 4 = 1, 4$
  $1, 4 \bigcap 1, 3, 4 = 1, 4 \quad 1, 2, 4 \bigcap 1, 2, 4 = 1, 2, 4 \quad 1, 2, 4 \bigcap 1, 3, 4 = 1, 4$
  $1, 3, 4 \bigcap 1, 3, 4 = 1, 3, 4 \quad 1, 2, 3, 4 \bigcap 1, 2, 3, 4 = 1, 2, 3, 4$
- Also $X = 1, 2, 3, 4$ is also in $F = 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$
  and the null element $\varnothing =$ is not in F.

https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory

Filter Examples (5)

- We call F a filter but not an Ultrafilter on $X = 1, 2, 3, 4$
- We can still add element in it and it will still be a filter
  for instance by adding the element 1 from $X = 1, 2, 3, 4$
  we can have the filter $F = 1, 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$
- This is an Ultrafilter on $X = 1, 2, 3, 4$
  as we cannot add any further element from $X = 1, 2, 3, 4$
  that satisfies closures on intersection.

https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory

Filter Examples (6)

- There is another collection of sets taken from $X = 1, 2, 3, 4$
- which is the powerset
  $P =, 1, 2, 3, 4, 1, 2, 1, 3, 1, 4, 2, 3, 2, 4, 3, 4, 1, 2, 3, 1, 2, 4, 1, 3, 4, 2, 3, 4, 1, 2, 3, 4$
- This contain the null element $\varnothing =$ so we cannot call this as Ultrafilter.
- This is not a proper filter according to the article in Wikipedia.
- In the powerset every intersection of element is again in the powerset again but it contains t
- he null element $\varnothing =$ and isn't classified as proper filter.

https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory

Filter Examples (7)

- There is another collection of sets taken from $X = 1, 2, 3, 4$
- which is the powerset
  $P =, 1, 2, 3, 4, 1, 2, 1, 3, 1, 4, 2, 3, 2, 4, 3, 4, 1, 2, 3, 1, 2, 4, 1, 3, 4, 2, 3, 4, 1, 2, 3, 4$
- This contain the null element $\varnothing =$ so we cannot call this as Ultrafilter.
- This is not a proper filter according to the article in Wikipedia.
- In the powerset every intersection of element is again in the powerset again but it contains t
- he null element $\varnothing =$ and isn't classified as proper filter.

https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

Outline

Topology (1)

- **topology**
  from the Greek words
  τόπος, 'place, location',
  and λόγος, 'study'

https://en.wikipedia.org/wiki/Topology

Topology (2)

- **topology** is concerned with
  the *properties* of a geometric object that are *preserved*
  - under continuous deformations such as
    - stretching
    - twisting
    - crumpling
    - bending
  - that is, *without*
    - closing holes
    - opening holes
    - tearing
    - gluing
    - passing through itself

https://en.wikipedia.org/wiki/Topology

Topology (3)

- There are several *equivalent* definitions of a **topology**,

  the most commonly used of which is
  the definition through open sets,

  which is easier than the others to manipulate.

  https://en.wikipedia.org/wiki/Borel_set

Topological space (1)

- a **topological space** is, roughly speaking,

  a geometrical space
  in which closeness is defined

  but cannot necessarily be measured
  by a numeric distance.

  https://en.wikipedia.org/wiki/Borel_set

Topological space (2)

- More specifically, a **topological space** is
  - a set whose elements are called points,
  - along with an additional structure called a topology,
- which can be defined as
  - a set of neighbourhoods for each point
  - that satisfy some axioms
    formalizing the concept of closeness.

https://en.wikipedia.org/wiki/Borel_set

Topological space (3)

- A **topological space** is
  the most general type of a mathematical space
  that allows for the definition of
  - limits
  - continuity
  - connectedness
- Although very general,
  the concept of **topological spaces** is fundamental,
  and used in virtually every branch of modern mathematics.
- The study of **topological spaces** in their own right
  is called point-set topology or general topology.

https://en.wikipedia.org/wiki/Topological_space

Topological space (4)

- Common types of **topological spaces** include
  - **Euclidean spaces** : a set of points satisfying certain relationships, expressible in terms of distance and angles.
  - **metric spaces** : a set together with a notion of distance between points. The distance is measured by a function called a metric or distance function.
  - **manifolds** : a topological space that *locally* resembles Euclidean space near each point. More precisely, an n-manifold is a topological space with the property that each point has a neighborhood that is homeomorphic to an open subset of n-dimensional Euclidean space.

https://en.wikipedia.org/wiki/Topological_space

Discrete Topology

- a **discrete space** is a topological space,

  in which the points form a discontinuous sequence,
  meaning they are isolated from each other in a certain sense.
- The **discrete topology** is
  the finest topology that can be given on a set.
  - every subset is open
  - every singleton subset is an open set

https://en.wikipedia.org/wiki/Discrete_space

Singletone

- a **singleton**, also known as a **unit set** or **one-point set**, is a set with exactly one element.
- for example, the set $\{0\}$ is a **singleton** whose single element is 0

https://en.wikipedia.org/wiki/Discrete_space

Indiscrete Space (1)

- a **topological space** with the **trivial topology** is one where the only open sets are the empty set and the entire space.
- Such spaces are commonly called **indiscrete**, **anti-discrete**, **concrete** or **codiscrete**.

  - every subset can be open (the **discrete topology**), or
  - no subset can be open (the **indiscrete topology**) except the space itself and the empty set .

https://en.wikipedia.org/wiki/Discrete_space

Indiscrete Space (2)

- Intuitively, this has the consequence that all points of the space are "lumped together" and cannot be distinguished by topological means (not topologically distinguishable points)
- Every **indiscrete space** is a **pseudometric space** in which the distance between any two points is zero.

https://en.wikipedia.org/wiki/Discrete_space

T_0 Space

- a topological space X is a T_0 **space** or
  if for every pair of distinct points of X,
  at least one of them has a neighborhood
  not containing the other.
- In a T_0 **space**, all points are topologically distinguishable.
- This condition, called the T_0 **condition**,
  is the weakest of the separation axioms.
- Nearly all topological spaces *normally*
  studied in mathematics are T_0 **space**.

https://en.wikipedia.org/wiki/Kolmogorov_space

Topologically distinguishable points

- Intuitively, an open set provides a *method* to *distinguish* two points.
- two points in a topological space,
  there exists an open set
  - containing one point but
  - not containing the other (distinct) point
  - the two points are **topologically distinguishable**.

https://en.wikipedia.org/wiki/Open_set

Topologically distinguishable points

- Intuitively, an open set provides a *method* to *distinguish* two points.
- two points in a topological space,
  there exists an open set
  - containing one point but
  - not containing the other (distinct) point
  - the two points are **topologically distinguishable**.

https://en.wikipedia.org/wiki/Open_set

Metric spaces

- In this manner, one may speak of whether two points, or more generally two subsets, of a topological space are "**near**" without concretely defining a distance.
- Therefore, **topological spaces** may be seen as a generalization of **spaces** equipped with a notion of distance, which are called **metric spaces**.

https://en.wikipedia.org/wiki/Open_set

Outline

Why called a discrete topology? (1)

- the **discrete topology** is the finest topology
  - it cannot be subdivided further.
- if you think of the elements of the set
  as indivisible "discrete" atoms,
  each one appears as a singleton set.
- can effectively "see" the individual points
  in the topology itself.

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (2)

- the **indiscrete topology** consists only of X itself and $\varnothing$.
- This topology obscures everything about *how many points* were in the original set.
- It fully agglomerates the points of the set together.

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (3)

- helpful to think of topologies
  as obscuring or blurring together
  the *underlying points* of the set.
- topologies are all about nearness relations:
  points in an open set are in the vicinity of one another.
- topologically indistinguishable points
  points that never appear alone in an open set,
    - they are so close as to be identical,
      from the perspective of the topology,

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (4)

- the discrete topology
  - has no indistinguishable points.
  - obscures nothing about the underlying set.
  - each point in the set is
    - clearly highlighted
    - distinguishable
    - recoverable as an open singleton set in the topology.

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (5)

- If you think of topologies that can arise from metrics, the discrete topology arises from metrics such as

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

  - This metric "*shatters*" the points X, *isolating* each one within its own unit ball.
    - In such a space, the only convergent sequences are the ones that are eventually constant;
    - you can't find points arbitrarily close to any other points.
    - because points are isolated in this way,
    - it makes sense to call the space "discrete".

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (6-1)

- Every function from a discrete space is automatically continuous.
- for this reason, the discrete topology is the one that best "represents" X in topological space.
- the nature of a set is characterized by its functions,
- the nature of a topological space is characterized by its continuous functions.

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (6-2)

- So, note that if T is any topological space, there's a natural bijective correspondence between functions $f : X \to set(T)$ and continuous morphisms $g : discrete(X) \to T$.
- For every function on X, you can find a continuous function on $discrete(X)$, and given any continuous function on $discrete(X)$, you can uniquely recover a function on X
- The discrete topology best represents the structure of the set X which, as you say, is discretized into individual points.

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Why called a discrete topology? (7-1)

- Throughout abstract algebra, isomorphisms describe which structures are "the same".
- A topological isomorphism (a homeomorphism) between two topologies says that they are essentially the same topology.
- An isomorphism of sets is just a bijection;
- it says that the sets contain the same number of elements.

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (7-3)

- Continuing the discussion of functions above,
  two discrete topologies are topologically isomorphic (homeomorphic) if and only if their underlying sets are isomorphic as sets (bijective).
- Put casually, this means that the discrete-topology-creating process maintains the similarity and differences between the underlying sets: discrete topologies are the same if and only if their underlying sets are the same.

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Why called a discrete topology? (8)

- This is all the more important
  when we realize that sets are the same
  when they have the same number of points.
- Hence discrete topologies are the same
  when (and only when) their underlying sets
  have "discrete points" in the same quantity.
- You can count the points in a discrete topology through isomorphisms,
  and the discrete topology is the only topology for which this is possible.

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Topological Space Definition by Neighbourhood (1)

- This axiomatization is due to Felix Hausdorff. Let X be a (possibly empty) set.
- The elements of X are usually called points,
  though they can be any mathematical object.
  Let $\mathscr{N}$ be a function assigning to each x (point) in X
  a non-empty collection $\mathscr{N}(x)$ of subsets of X.
- The elements of $\mathscr{N}(x)$ will be called neighbourhoods of x with respect to $\mathscr{N}$
  (or, simply, neighbourhoods of x).
- The function $\mathscr{N}$ is called a neighbourhood topology
  if the axioms below are satisfied;
  and then X with $\mathscr{N}$ is called a topological space.

https://en.wikipedia.org/wiki/Topological_space

Topological Space Definition by Neighbourhood (2)

- If N is a neighbourhood of x (i.e., $N \in \mathscr{N}(x)$), then $x \in N$.
- In other words, each point of the set X belongs to every one of its neighbourhoods with respect to $\mathscr{N}$.
- If N is a subset of X and includes a neighbourhood of x, then N is a neighbourhood of x.
- I.e., every superset of a neighbourhood of a point $x \in X$ is again a neighbourhood of x.
- The intersection of two neighbourhoods of x is a neighbourhood of x.
- Any neighbourhood N of x includes a neighbourhood M of x such that N is a neighbourhood of each point of M.

https://en.wikipedia.org/wiki/Topological_space

Topological Space Definition by Neighbourhood (3)

- The first three axioms for neighbourhoods have a clear meaning.
- The fourth axiom has a very important use in the structure of the theory,
  that of linking together the neighbourhoods of different points of X.
- A standard example of such a system of neighbourhoods is
  for the real line $\mathbb{R}$, where a subset N of $\mathbb{R}$ is defined
  to be a neighbourhood of a real number x
  if it includes an open interval containing x.

https://en.wikipedia.org/wiki/Topological_space

Topological Space Definition by Neighbourhood (4)

- Given such a structure, a subset U of X is defined to be open if U is a neighbourhood of all points in U.
- The open sets then satisfy the axioms given below in the next definition of a topological space.
- Conversely, when given the open sets of a topological space, the neighbourhoods satisfying the above axioms can be recovered by defining N to be a neighbourhood of x if N includes an open set U such that $x \in U$.

https://en.wikipedia.org/wiki/Topological_space

Continuous Functions (1)

- In category theory, one of the fundamental categories is Top, which denotes the category of topological spaces whose objects are topological spaces and whose morphisms are continuous functions.
- The attempt to classify the objects of this category (up to homeomorphism) by invariants has motivated areas of research, such as homotopy theory, homology theory, and K-theory.

https://en.wikipedia.org/wiki/Topological_space

Continuous Functions (2)

- A function $f : X \to Y$ between topological spaces
  is called continuous if for every $x \in X$ and every neighbourhood N of $f(x)$
  there is a neighbourhood M of x such that $f(M) \subseteq N$.
- This relates easily to the usual definition in analysis.
- Equivalently, f is continuous
  if the inverse image of every open set is open.
- This is an attempt to capture the intuition
  that there are no "jumps" or "separations" in the function.

https://en.wikipedia.org/wiki/Topological_space

Continuous Functions (3)

- A homeomorphism is a bijection that is continuous and whose inverse is also continuous.
- Two spaces are called homeomorphic if there exists a homeomorphism between them.
- From the standpoint of topology, homeomorphic spaces are essentially identical

https://en.wikipedia.org/wiki/Topological_space

Characterization (1-1)

- a characterization of an object is a set of conditions that, while different from the definition of the object, is logically equivalent to it.
- "Property P characterizes object X"
  - not only does X have property P
  - but that object X is the only thing that has property P
  - i.e., P is a defining property of object X

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (1-2)

- Similarly, a set of properties P is said to characterize object X, when these properties distinguish object X from all other objects.
- Even though a characterization identifies an object in a unique way, several characterizations can exist for a single object.
- Common mathematical expressions
  for a characterization of object X in terms of a set of properties P include "a set of properties P is necessary and sufficient for object X",
  and "object X holds if and only if a set of properties P".

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (2-1)

- It is also common to find statements
  such as "Property Q characterizes object Y up to isomorphism".
- The first type of statement says in different words
  that the extension of P is a singleton set,
  while the second says that the extension of Q
  is a single equivalence class
  (for isomorphism, in the given example
  — depending on how up to is being used,
  some other equivalence relation might be involved).

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (2-2)

- A reference on mathematical terminology notes that characteristic originates from the Greek term kharax, "a pointed stake":
- From Greek kharax came kharakhter, an instrument used to mark or engrave an object.
- Once an object was marked, it became distinctive, so the character of something came to mean its distinctive nature.
- The Late Greek suffix -istikos converted the noun character into the adjective characteristic, which, in addition to maintaining its adjectival meaning, later became a noun as well.

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (3-1)

- Just as in chemistry, the characteristic property of a material will serve to identify a sample,
  or in the study of materials, structures
  and properties will determine characterization,
  in mathematics there is a continual effort
  to express properties that will distinguish
  a desired feature in a theory or system.
- Characterization is not unique to mathematics,
  but since the science is abstract,
  much of the activity can be described as "characterization".

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (3-2)

- For instance, in Mathematical Reviews, as of 2018, more than 24,000 articles contain the word in the article title, and 93,600 somewhere in the review.
- In an arbitrary context of objects and features, characterizations have been expressed via the heterogeneous relation aRb, meaning that object a has feature b.
- For example, b may mean abstract or concrete.
- The objects can be considered the extensions of the world, while the features are expression of the intensions.
- A continuing program of characterization of various objects leads to their categorization.

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (4-1)

- A rational number, generally defined as a ratio of two integers, can be characterized as a number with finite or repeating decimal expansion.
- A parallelogram is a quadrilateral whose opposing sides are parallel.
  - one of its characterizations is that its diagonals bisect each other.
  - this means that the diagonals in all parallelograms bisect each other
  - conversely, that any quadrilateral whose diagonals bisect each other must be a parallelogram.

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (4-2)

- "Among probability distributions on the interval from 0 to ∞ on the real line, memorylessness characterizes the exponential distributions."
- This statement means that the exponential distributions are the only probability distributions that are memoryless, provided that the distribution is continuous

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (4-3)

- "According to Bohr–Mollerup theorem,
among all functions f such that $f(1) = 1$ and$xf(x) = f(x+1)$ for $x > 0$,
log-convexity characterizes the gamma function."
This means that among all such functions,
the gamma function is the only one that is log-convex.
- The circle is characterized as a manifold by being one-dimensional, compact and connected; here the characterization,
as a smooth manifold, is up to diffeomorphism.

https://en.wikipedia.org/wiki/Characterization_(mathematics)

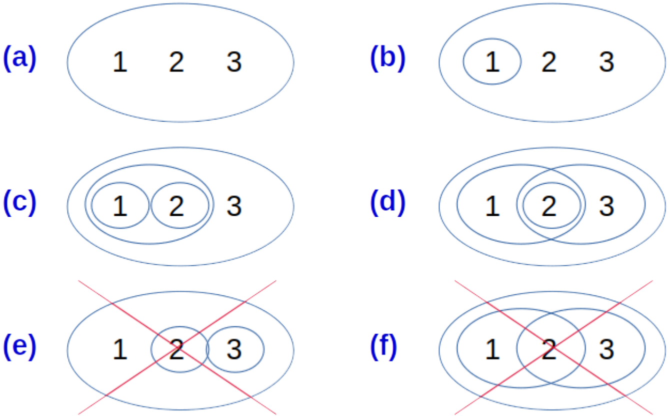
Outline

Examples of topoloy (1)

- Let τ be denoted with the circles,
  here are four examples **(a)**, **(b)**, **(c)**, **(d)**, and
  two non-examples **(e)**, **(f)** of topologies
  on the three-point set $\{1, 2, 3\}$.
- **(e)** is not a topology
  because the union of $\{2\}$ and $\{3\}$
  [i.e. $\{2, 3\}$] is missing;
- **(f)** is not a topology
  because the intersection of $\{1, 2\}$ and $\{2, 3\}$
  [i.e. $\{2\}$], is missing.

https://en.wikipedia.org/wiki/Topological_space

Examples of topoloy (2)



Every union of (c)

(c) is a topology $\{\{\},\{1\},\{2\},\{1,2\},\{1,2,3\}\}$

every union of (c)

$\cup$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{2\}$	$\{2\}$	$\{1,2\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$

https://en.wikipedia.org/wiki/Topological_space

Every intersection of (c)

(c) is a topology $\{\{\},\{1\},\{2\},\{1,2\},\{1,2,3\}\}$

every intersection of (c)

$\cap$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$
$\{1\}$	$\{\}$	$\{1\}$	$\{\}$	$\{1\}$	$\{1\}$
$\{2\}$	$\{\}$	$\{\}$	$\{2\}$	$\{2\}$	$\{2\}$
$\{1,2\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2\}$
$\{1,2,3\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$

https://en.wikipedia.org/wiki/Topological_space

Every union of (f)

(f) is not a topology $\{\{\},\{1,2\},\{2,3\},\{1,2,3\}\}$

every union of (f)

$\cup$	$\{\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3\}$
$\{2,3\}$	$\{2,3\}$	$\{1,2,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$

https://en.wikipedia.org/wiki/Topological_space

Every intersection of (f)

(f) is not a topology $\{\{\},\{1,2\},\{2,3\},\{1,2,3\}\}$

every intersection of (f)

$\cap$	$\{\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$
$\{1,2\}$	$\{\}$	$\{1,2\}$	$\{2\}$	$\{1,2\}$
$\{2,3\}$	$\{\}$	$\{2\}$	$\{2,3\}$	$\{2,3\}$
$\{1,2,3\}$	$\{\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$

https://en.wikipedia.org/wiki/Topological_space

Examples of topoloy (3)

- Given $X = \{1, 2, 3, 4\}$,
  the *trivial* or *indiscrete* topology on X is
  the family $\tau = \{\{\}, \{1, 2, 3, 4\}\} = \{\varnothing, X\}$
  consisting of only the two subsets of X
  required by the axioms
  forms a topology of X.

https://en.wikipedia.org/wiki/Topological_space

Examples of topoloy (4)

- Given $X = \{1,2,3,4\}$,
  the family $\tau = \{\{\},\{2\},\{1,2\},\{2,3\},\{1,2,3\},\{1,2,3,4\}\}$
  $= \{\varnothing,\{2\},\{1,2\},\{2,3\},\{1,2,3\},X\}$
  of six subsets of X forms another topology of X.

https://en.wikipedia.org/wiki/Topological_space

Examples of topoloy (5)

- Given $X = \{1,2,3,4\}$,
  the *discrete* topology on X is
  the power set of X, which is the family $\tau = \wp(X)$
  consisting of *all possible* subsets of X.
  the family

$$\tau = \quad \{\{\},\{1\},\{2\},\{3\},\{4\} \\ \{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}, \\ \{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}$$

  - In this case the topological space (X,τ)
    is called a *discrete* space.

https://en.wikipedia.org/wiki/Topological_space

Examples of topoloy (6)

- Given $X = \mathbb{Z}$, the set of integers,
  the family τ of
  all finite subsets of the integers
  plus $\mathbb{Z}$ itself
  is not a topology,
  because (for example)
  the union of *all finite* sets
  not containing zero is not finite
  but is also not all of $\mathbb{Z}$,
  and so it cannot be in τ .

https://en.wikipedia.org/wiki/Topological_space