

# Laurent Series and z-Transform

## - Geometric Series

## Permutations B

20240909 Mon

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$a^n$  $\times$  $R(n)$ 

$a^n$	$a^{-n}$

 $\times$ 

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

(1)	$a^n u(n)$	$a^{-n} u(n)$	(2)
(3)	$a^n u(-n)$	$a^{-n} u(-n)$	(4)
(5)	$a^n u(-n-1)$	$a^{-n} u(-n-1)$	(6)
(7)	$a^n u(n-1)$	$a^{-n} u(n-1)$	(8)

## Exponent Shifting

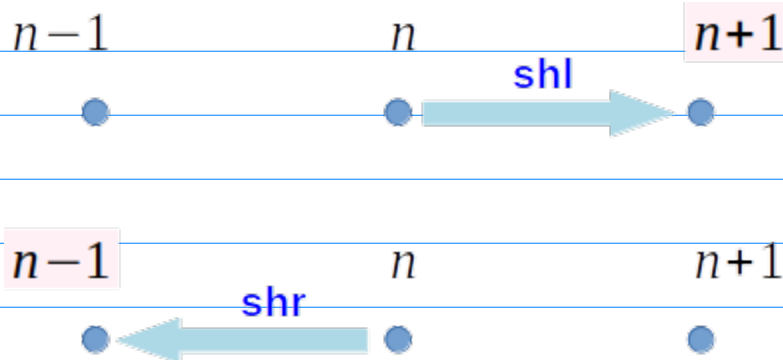
	$shl(b^n)$
$b^n$	$b^{n+1}$
$a^n$	$a^{(n+1)}$
$a^{-n}$	$a^{-(n+1)}$

	$shr(b^n)$
$b^n$	$b^{n-1}$
$a^n$	$a^{(n-1)}$
$a^{-n}$	$a^{-(n-1)}$

## Range Shifting

	$shl(R(n))$
$R(n)$	$R(n+1)$
$u(n-1)$	$u(n)$
$u(-n)$	$u(-(n+1))$

	$shr(R(n))$
$R(n)$	$R(n-1)$
$u(n)$	$u(n-1)$
$u(-(n+1))$	$u(-n)$



$$\begin{array}{ccc} b^n & \longleftrightarrow & b^{n+1} \\ R(n) & \longleftrightarrow & R(n+1) \end{array}$$

$$\begin{array}{ccc} b^n & \longleftrightarrow & b^{-(n+1)} \\ R(n) & \longleftrightarrow & R(n+1) \end{array}$$

$$\begin{array}{ccc} b^n & \longleftrightarrow & b^{n-1} \\ R(n) & \longleftrightarrow & R(n-1) \end{array}$$

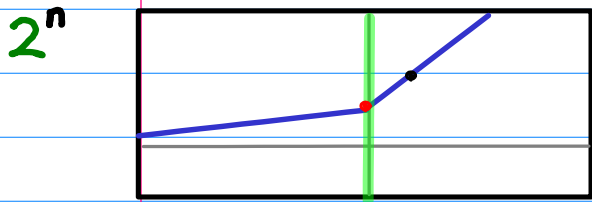
$$\begin{array}{ccc} b^n & \longleftrightarrow & b^{-(n-1)} \\ R(n) & \longleftrightarrow & R(n-1) \end{array}$$

# Exponent Shifting

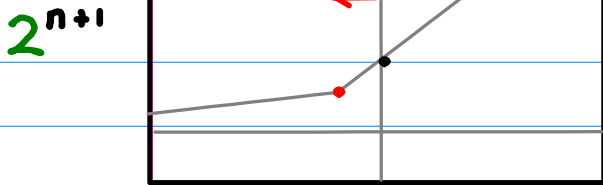
$$\text{shl}(b^n) = b^{n+1}$$

$$\text{shr}(b^n) = b^{n-1}$$

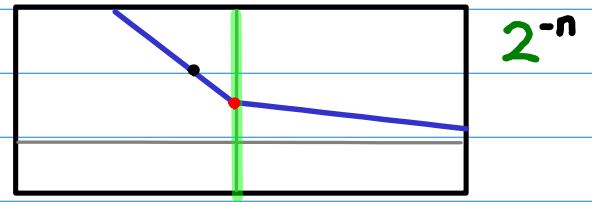
$$b^n \longleftrightarrow b^{n+1}$$



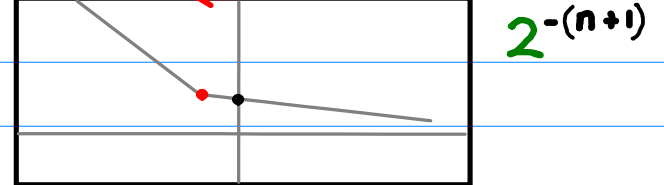
shift left  
 $n \leftarrow n+1$



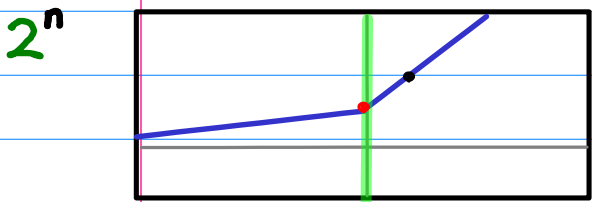
$$b^n \longleftrightarrow b^{-(n+1)}$$



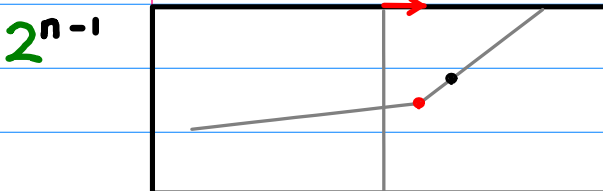
shift left  
 $n \leftarrow n+1$



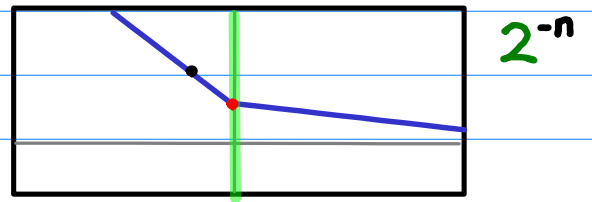
$$b^n \longleftrightarrow b^{n-1}$$



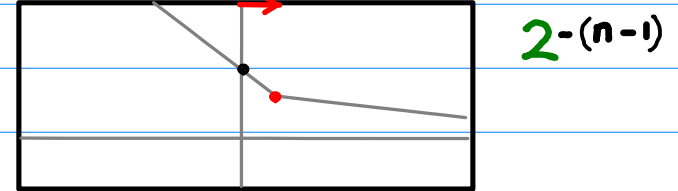
shift right  
 $n \leftarrow n-1$



$$b^n \longleftrightarrow b^{-(n-1)}$$



shift right  
 $n \leftarrow n-1$



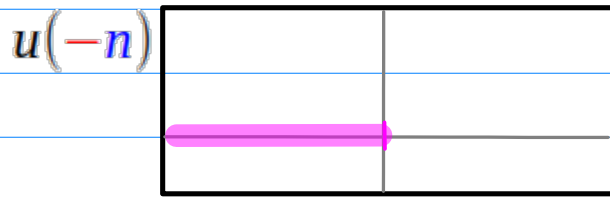
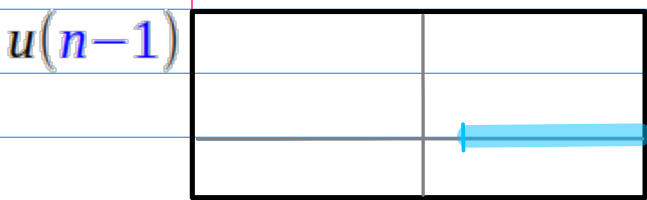
# Range Shifting

$$\text{shl}(R(n)) = R(n+1)$$

$$\text{shr}(R(n)) = R(n-1)$$

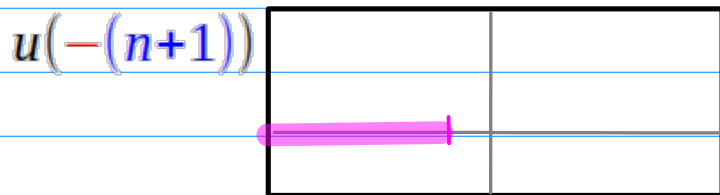
$R(n) \longleftrightarrow R(n+1)$

$R(n) \longleftrightarrow R(n+1)$



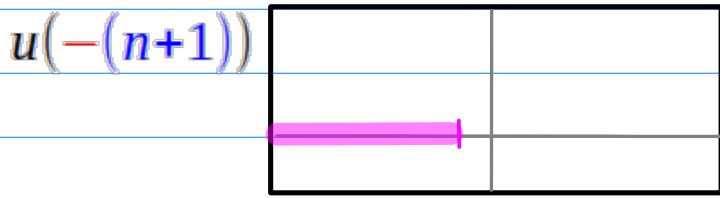
shift left  
 $n \leftarrow n+1$

shift left  
 $n \leftarrow n+1$



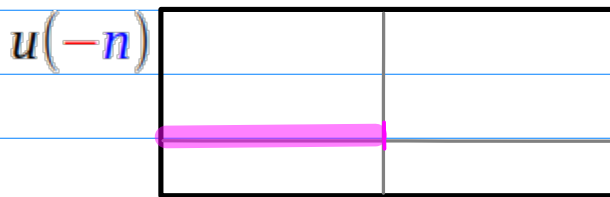
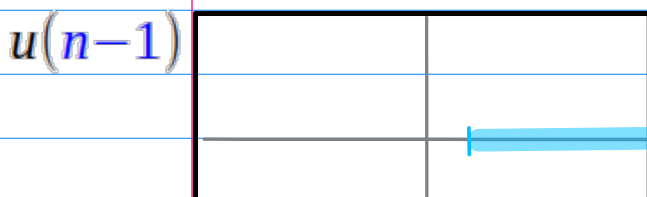
$R(n) \longleftrightarrow R(n-1)$

$R(n) \longleftrightarrow R(n-1)$



shift right  
 $n \leftarrow n-1$

shift right  
 $n \leftarrow n-1$



$a^n$  $\times$  $R(n)$ 

$a^{n+1}$	$a^{-n-1}$
$a^{n-1}$	$a^{-n+1}$

 $\times$ 

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

(1')	$a^{n+1}u(n)$	$a^{-n-1}u(n)$	(2')
(3')	$a^{n-1}u(-n)$	$a^{-n+1}u(-n)$	(4')
(5')	$a^{n+1}u(-n-1)$	$a^{-n-1}u(-n-1)$	(6')
(7')	$a^{n-1}u(n-1)$	$a^{-n+1}u(n-1)$	(8')

(1'')	$a^{n-1}u(n)$	$a^{-n+1}u(n)$	(2'')
(3'')	$a^{n+1}u(-n)$	$a^{-n-1}u(-n)$	(4'')
(5'')	$a^{n-1}u(-n-1)$	$a^{-n+1}u(-n-1)$	(6'')
(7'')	$a^{n+1}u(n-1)$	$a^{-n-1}u(n-1)$	(8'')



$a^n$  $\times$  $R(n)$ 

$a^{n+1}$	$a^{-n-1}$

 $\times$ 

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

$$(1') \quad a^{n+1} u(n) \quad a^{-n-1} u(n) \quad (2')$$

$$(3') \quad a^{n-1} u(-n) \quad a^{-n+1} u(-n) \quad (4')$$

$$(5') \quad a^{n+1} u(-n-1) \quad a^{-n-1} u(-n-1) \quad (6')$$

$$(7') \quad a^{n-1} u(n-1) \quad a^{-n+1} u(n-1) \quad (8')$$

$a^n$  $\times$  $R(n)$ 

$a^{n+1}$	$a^{-n-1}$
$a^{n-1}$	$a^{-n+1}$

 $\times$ 

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

(1'')	$a^{n-1} u(n)$	$a^{-n+1} u(n)$	(2'')
(3'')	$a^{n+1} u(-n)$	$a^{-n-1} u(-n)$	(4'')
(5'')	$a^{n-1} u(-n-1)$	$a^{-n+1} u(-n-1)$	(6'')
(7'')	$a^{n+1} u(n-1)$	$a^{-n-1} u(n-1)$	(8'')

$$a^{n-i} u(n) \quad (1'')$$

$$a^{-n+i} u(n) \quad (2'')$$

$$a^{n+i} u(-n) \quad (3'')$$

$$a^{-n-i} u(-n) \quad (4'')$$

$$a^{n-i} u(-n-1) \quad (5'')$$

$$a^{-n+i} u(-n-1) \quad (6'')$$

$$a^{n+i} u(n-1) \quad (7'')$$

$$a^{-n-i} u(n-1) \quad (8'')$$

**Unshifted Sequence  $x$** **Shifted Sequence 1  $x'$** **Shifted Sequence 2  $x''$** 

(1)  $a^n u(n)$

$shr(b^n) a^{n+i} u(n) (1')$

$shr(b^n) a^{n-i} u(n) (1'')$

(2)  $a^{-n} u(n)$

$shr(b^n) a^{-n-i} u(n) (2')$

$shr(b^n) a^{-n+i} u(n) (2'')$

(3)  $a^n u(-n)$

$shr(b^n) a^{n-i} u(-n) (3')$

$shl(b^n) a^{n+i} u(-n) (3'')$

(4)  $a^{-n} u(-n)$

$shr(b^n) a^{-n+i} u(-n) (4')$

$shl(b^n) a^{-n-i} u(-n) (4'')$

(5)  $a^n u(-n-1)$

$shl(b^n) a^{n+i} u(-n-1) (5')$

$shr(b^n) a^{n-i} u(-n-1) (5'')$

(6)  $a^{-n} u(-n-1)$

$shl(b^n) a^{-n-i} u(-n-1) (6')$

$shr(b^n) a^{-n+i} u(-n-1) (6'')$

(7)  $a^n u(n-1)$

$shr(b^n) a^{n-i} u(n-1) (7')$

$shl(b^n) a^{n+i} u(n-1) (7'')$

(8)  $a^{-n} u(n-1)$

$shr(b^n) a^{-n+i} u(n-1) (8')$

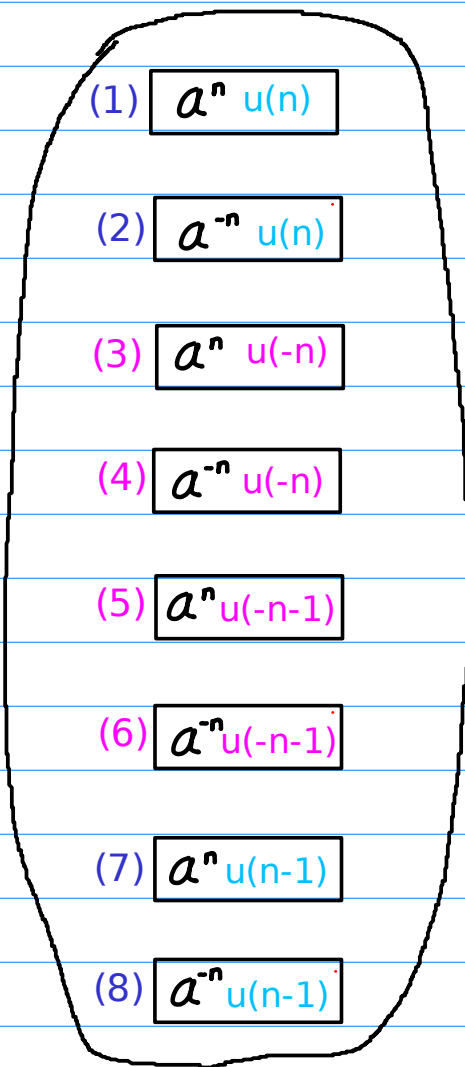
$shl(b^n) a^{-n-i} u(n-1) (8'')$

many possible permutations are possible  
but consider these two

**Unshifted  
Sequence  $x$**



**Shifted  
Sequence 1  $x'$**



$shl(b^n)$

$a^{n+i} u(n)$  (1')

$shl(b^n)$

$a^{-n-i} u(n)$  (2')

$shr(b^n)$

$a^{n-i} u(-n)$  (3')

$shr(b^n)$

$a^{-n+i} u(-n)$  (4')

$shl(b^n)$

$a^{n+i} u(-n-1)$  (5')

$shl(b^n)$

$a^{-n-i} u(-n-1)$  (6')

$shr(b^n)$

$a^{n-i} u(n-1)$  (7')

$shr(b^n)$

$a^{-n+i} u(n-1)$  (8')

**Inter-permutations over unshifted sequence and shifted sequence**

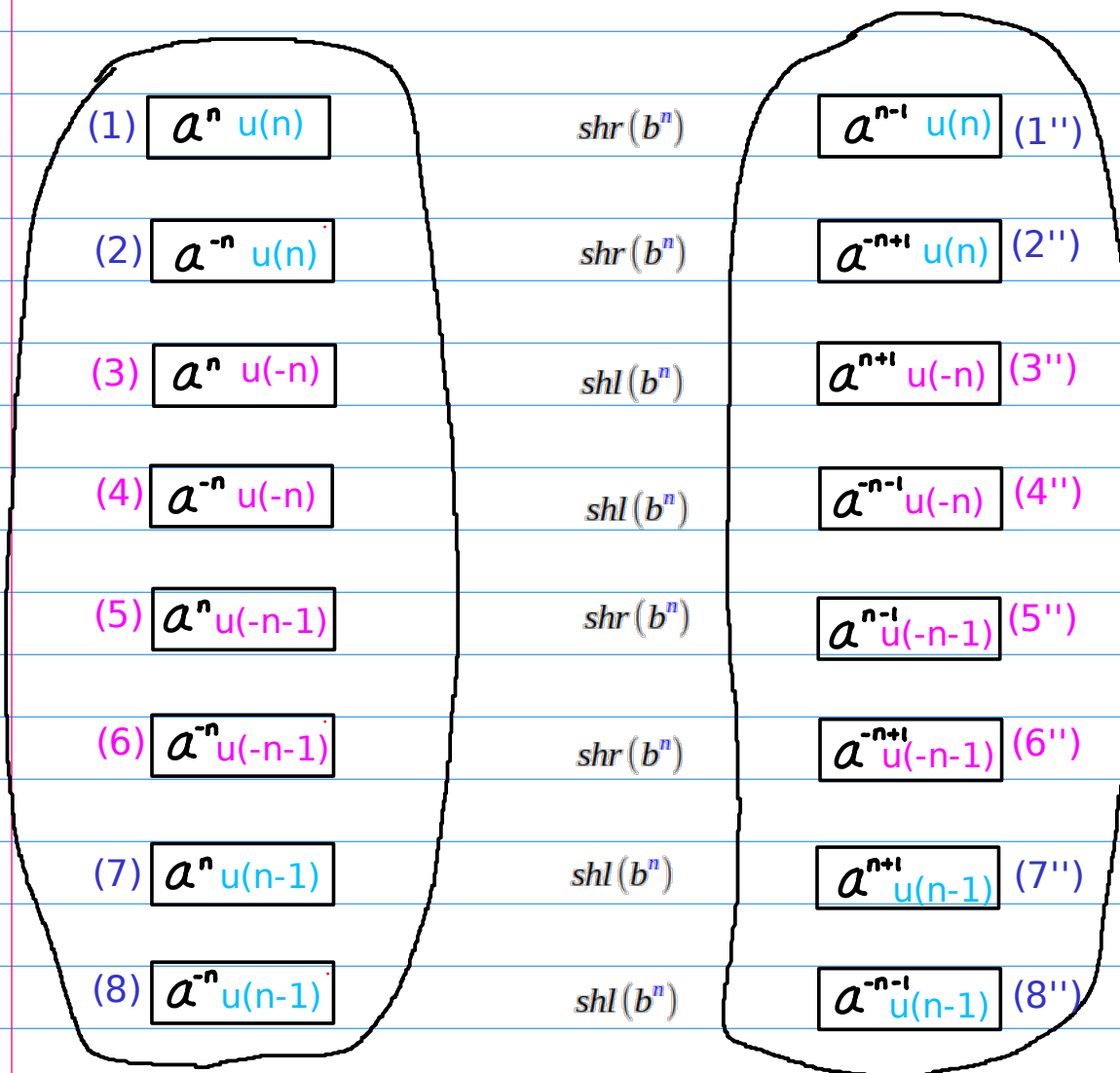
**Intra-permutations over unshifted sequence**

**Intra-permutations over shifted sequence**

Unshifted  
Sequence  $x$



Shifted  
Sequence 2  $x''$



**Inter-permutations over unshifted sequence and shifted sequence**

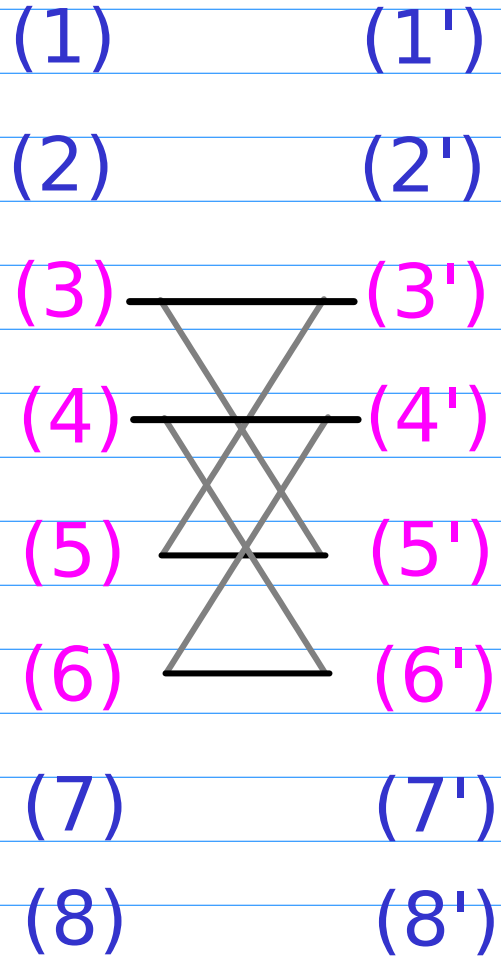
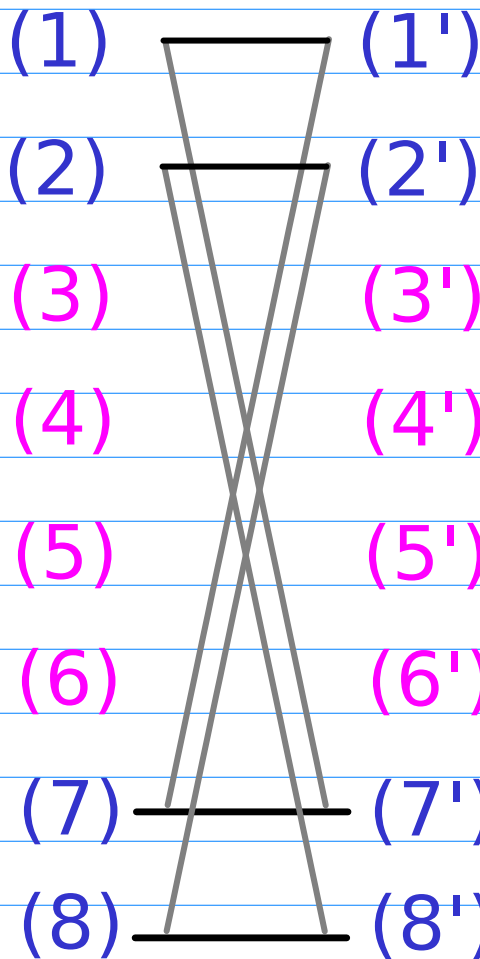
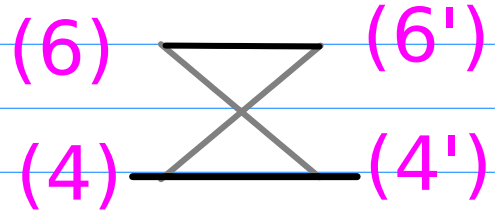
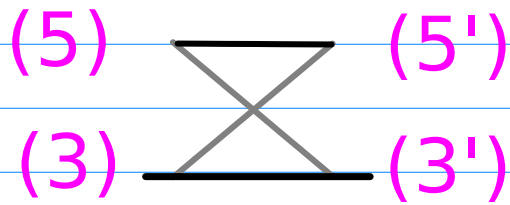
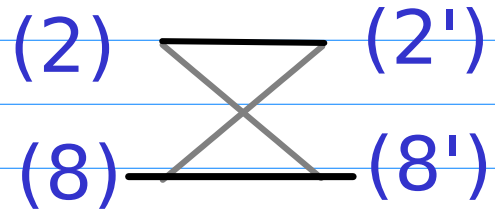
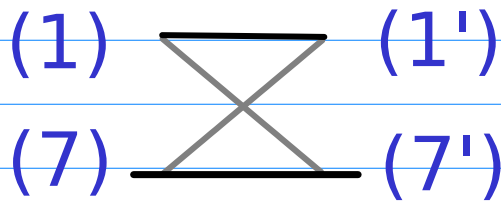
**Intra-permutations over unshifted sequence**

**Intra-permutations over shifted sequence**

(1) - (1')  
(7) - (7')  
(5) - (5')  
(3) - (3')

(2) - (2')  
(8) - (8')  
(6) - (6')  
(4) - (4')

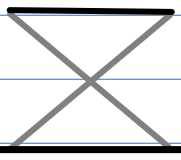

**(x)**  **(x')**  
(1)~(8) (1')~(8')

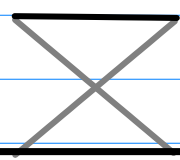



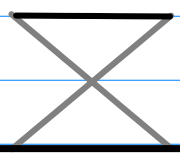

(1) - (1')  
(7) - (7')  
(5) - (5')  
(3) - (3')

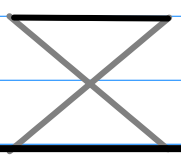

(2) - (2')  
(8) - (8')  
(6) - (6')  
(4) - (4')

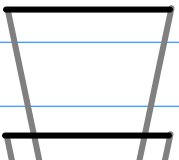
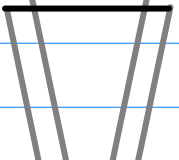
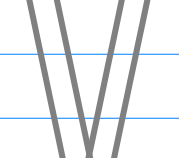
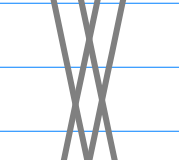
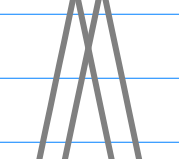



**(x)**  **(x'')**  
(1)~(8) (1')~(8')





(1)  (1'')  
(7)  (7'')

(2)  (2'')  
(8)  (8'')

(5)  (5'')  
(3)  (3'')

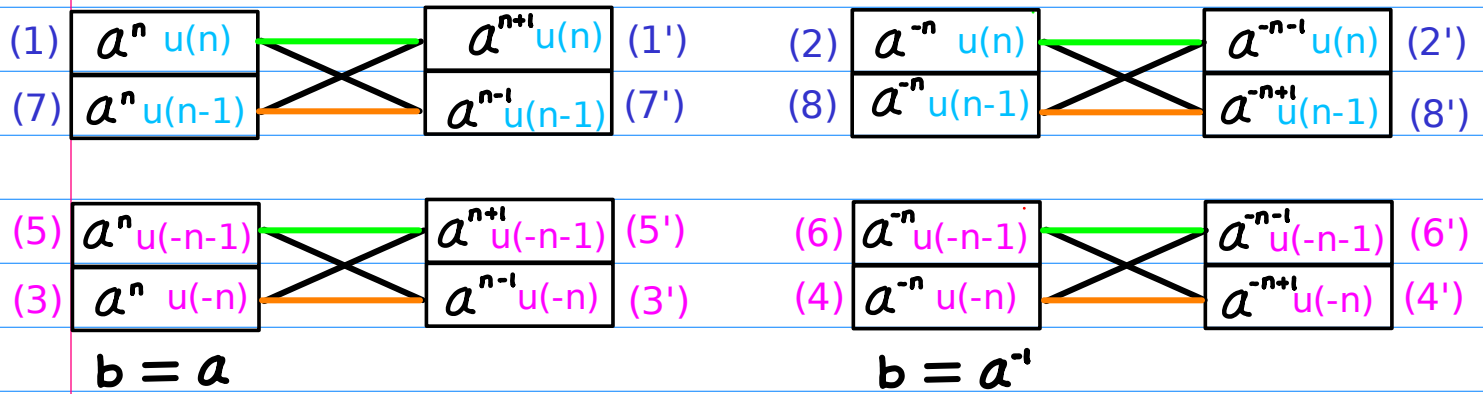
(6)  (6'')  
(4)  (4'')

(1)  (1'')  
(2)  (2'')  
(3)  (3'')  
(4)  (4'')  
(5)  (5'')  
(6)  (6'')  
(7)  (7'')  
(8)  (8'')

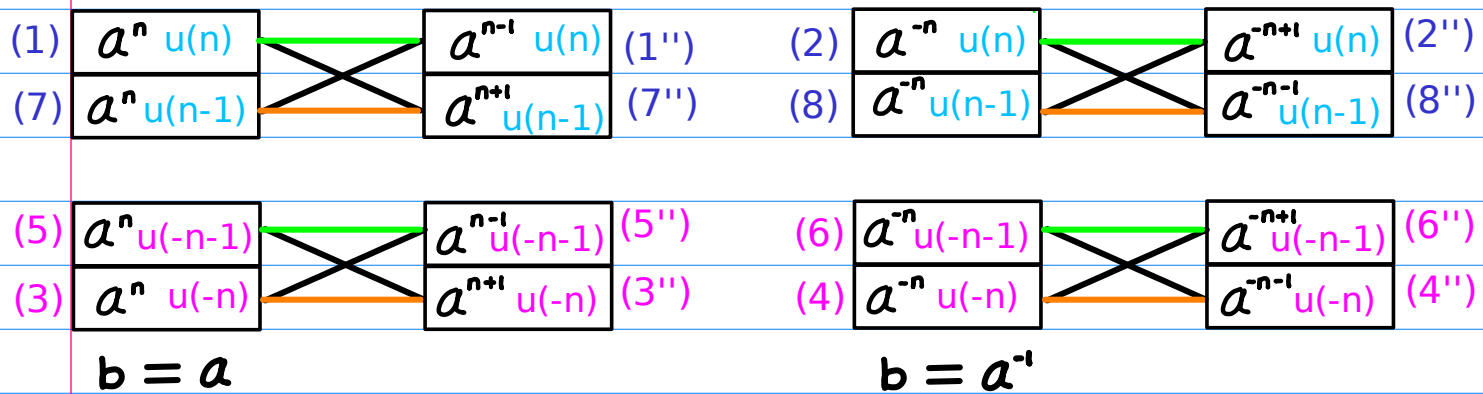
(1) (1'')  
(2) (2'')  
(3)  (3'')  
(4)  (4'')  
(5)  (5'')  
(6)  (6'')  
(7) (7'')  
(8) (8'')



## A Shifting Shifted Sequence 1



## B Shifting Shifted Sequence 2



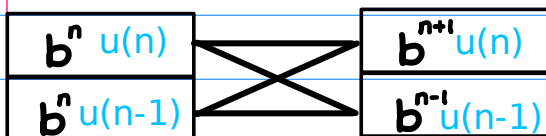
## A Shifting Shifted Sequence 1

(1) Exponent Shifting  
(2) Range Shifting

## B Shifting Shifted Sequence 2

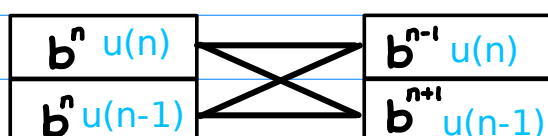
(1) Exponent Shifting  
(2) Range Shifting

(1)/(2)  
(7)/(8)



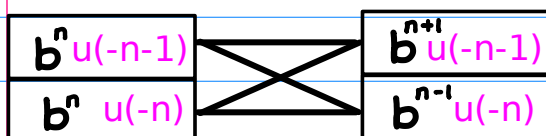
(1')/(2')  
(7')/(8')

(1)/(2)  
(7)/(8)



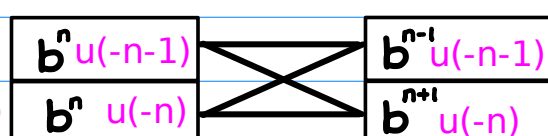
(1'')/(2'')  
(7'')/(8'')

(5)/(6)  
(3)/(4)



(5')/(6')  
(3')/(4')

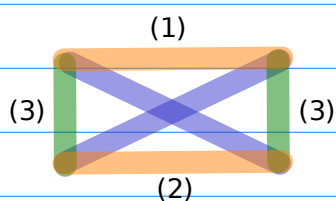
(5)/(6)  
(3)/(4)



(5'')/(6'')  
(3'')/(4'')

$$b = a/a^i$$

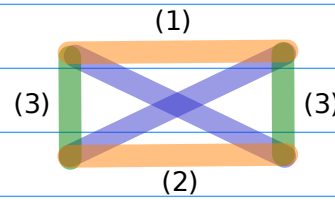
$$b = a/a^i$$



$$b^n \xrightarrow{(1)} b^{n+1}$$

$$b^n \xrightarrow{(2)} b^{n-1}$$

$$R(n) \xrightarrow{(3)} R(n-1)$$



$$b^n \xrightarrow{(1)} b^{n-1}$$

$$b^n \xrightarrow{(2)} b^{n+1}$$

$$R(n) \xrightarrow{(3)} R(n-1)$$

### Exponent Shifting

$b^n$	$b^{sh(n)}$	
$a^n$	$a^{(n+1)}$	$shl(b^n)$
$a^n$	$a^{(n-1)}$	$shr(b^n)$
$a^{-n}$	$a^{-(n+1)}$	$shl(b^n)$
$a^{-n}$	$a^{-(n-1)}$	$shr(b^n)$

### Exponent Shifting

$b^n$	$b^{sh(n)}$	
$a^n$	$a^{(n-1)}$	$shr(b^n)$
$a^n$	$a^{(n+1)}$	$shl(b^n)$
$a^{-n}$	$a^{-(n-1)}$	$shr(b^n)$
$a^{-n}$	$a^{-(n+1)}$	$shl(b^n)$

### Range Shifting

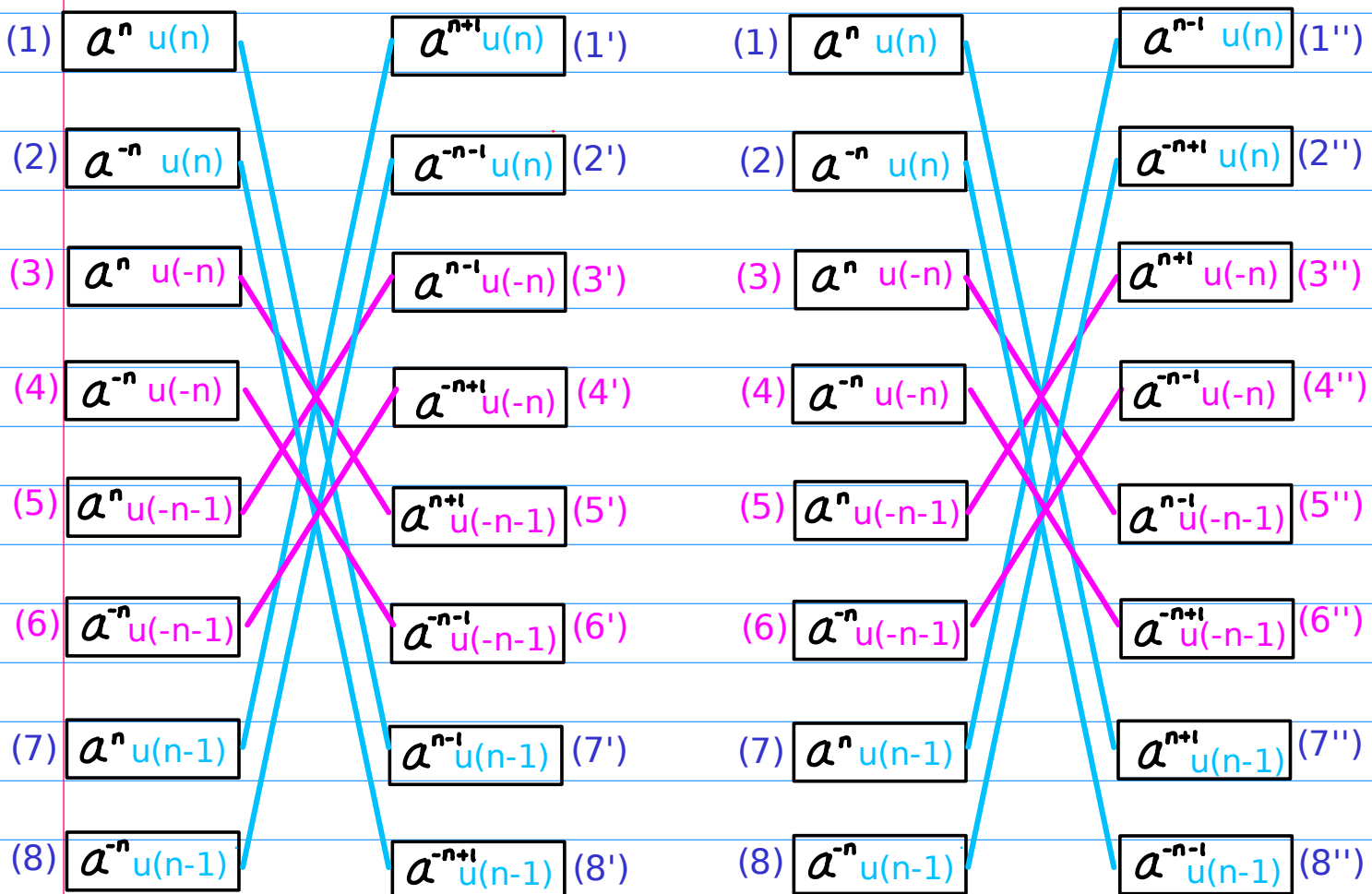
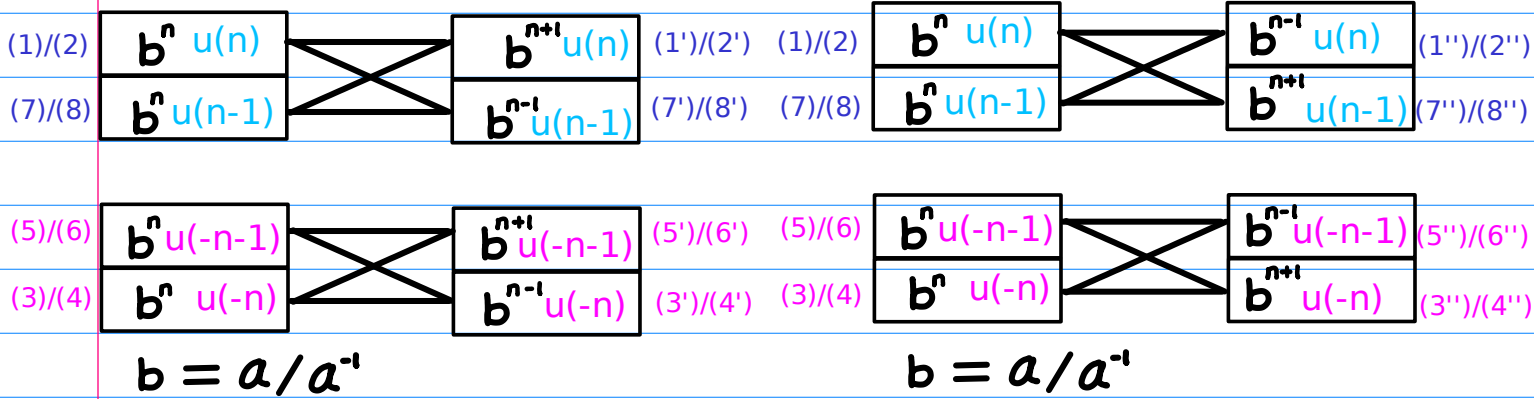
$R(n)$	$R(sh(n))$	
$u(n)$	$u(n-1)$	$shr(R(n))$
$u(-(n+1))$	$u(-n)$	$shr(R(n))$

### Range Shifting

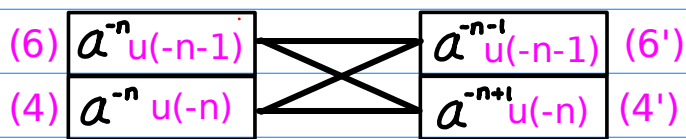
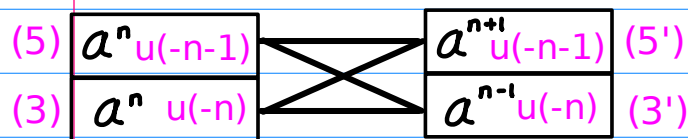
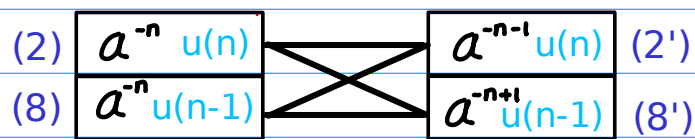
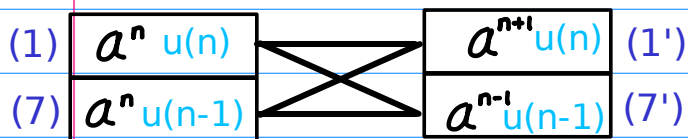
$R(n)$	$R(sh(n))$	
$u(n)$	$u(n-1)$	$shr(R(n))$
$u(-(n+1))$	$u(-n)$	$shr(R(n))$

## A Shifting Shifted Sequence 1

## B Shifting Shifted Sequence 2

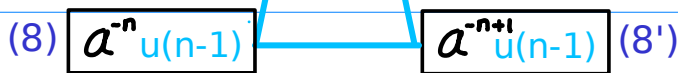
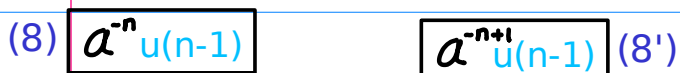
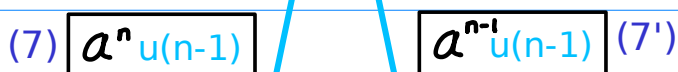
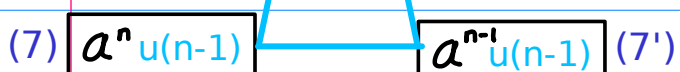
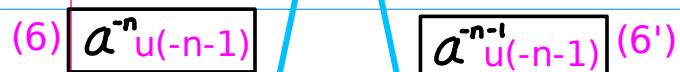
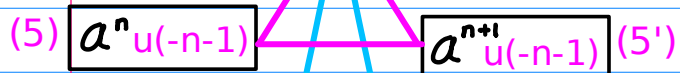
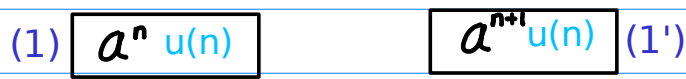
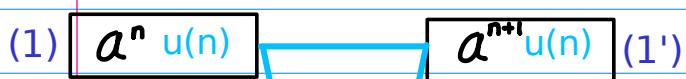


# A Shifting Shifted Sequence 1

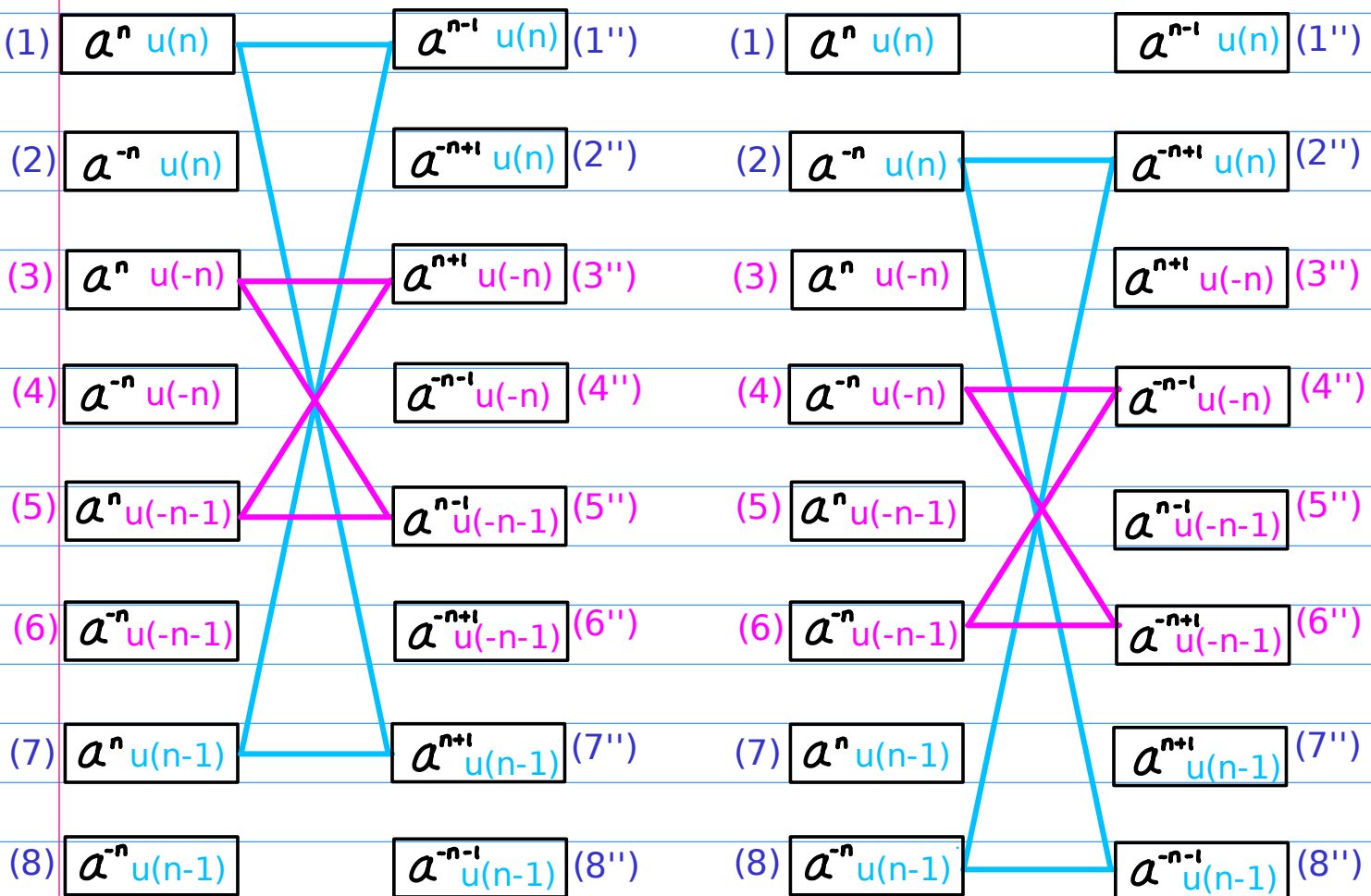
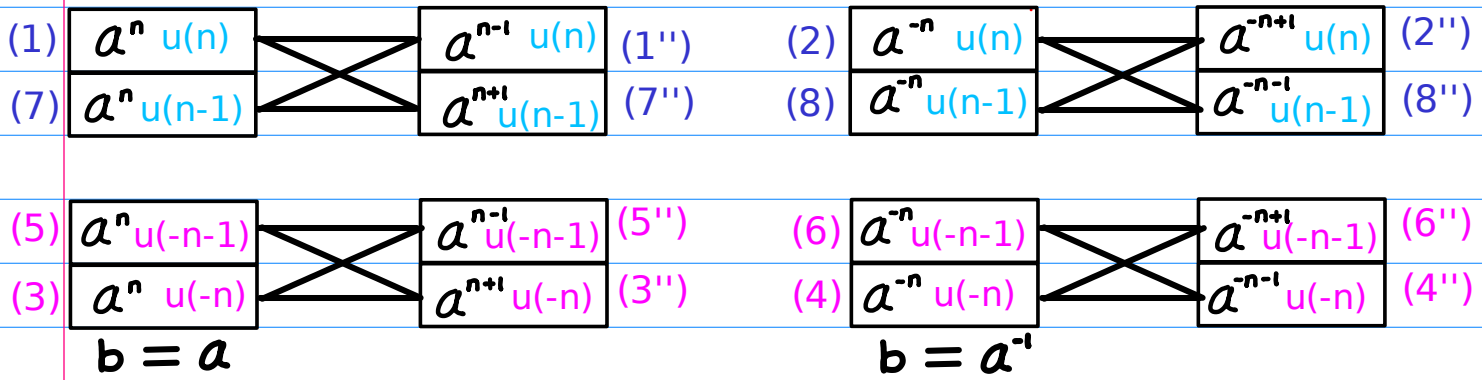


$b = a$

$b = a^{-1}$



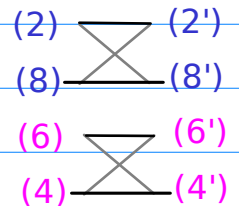
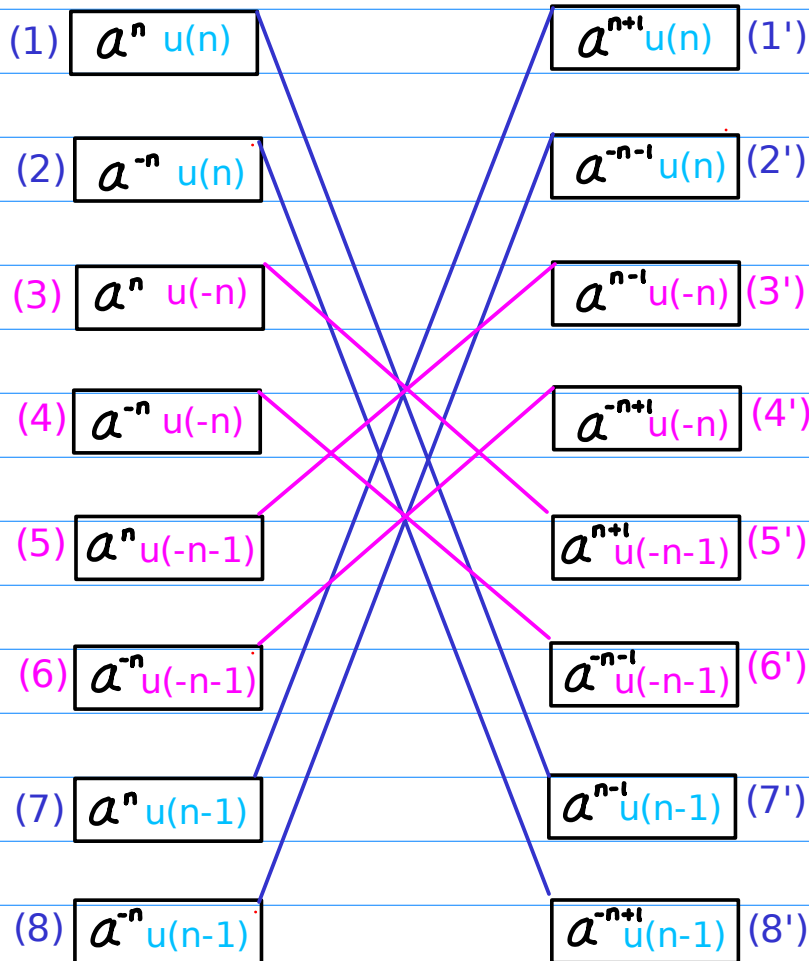
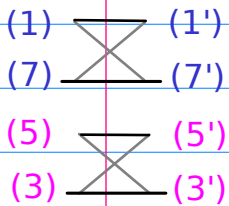
## B Shifting Shifted Sequence 2



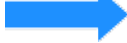
# A Shifting Shifted Sequence 1

Unshifted Sequence  $x$   $\rightarrow$  Shifted Sequence 1  $x'$

(1)	$\frac{1}{1-2z^{-1}} 2^n u(n)$	(2)	$\frac{1}{1-0.5z^{-1}} 2^{-n} u(n)$	(1')	$\frac{2}{1-2z^{-1}} 2^{n+1} u(n)$	(2')	$\frac{0.5}{1-0.5z^{-1}} 2^{-n-1} u(n)$
(3)	$\frac{1}{1-0.5z^{-1}} 2^n u(-n)$	(4)	$\frac{1}{1-2z^{-1}} 2^{-n} u(-n)$	(3')	$\frac{0.5}{1-0.5z^{-1}} 2^{n-1} u(-n)$	(4')	$\frac{2}{1-2z^{-1}} 2^{-n+1} u(-n)$
(5)	$\frac{0.5z^{-1}}{1-0.5z^{-1}} 2^n u(-n-1)$	(6)	$\frac{2z^{-1}}{1-2z^{-1}} 2^{-n} u(-n-1)$	(5')	$\frac{z^{-1}}{-0.5z^{-1}} 2^{n+1} u(-n-1)$	(6')	$\frac{z^{-1}}{1-2z^{-1}} 2^{-n-1} u(-n-1)$
(7)	$\frac{2z}{1-2z} 2^n u(n-1)$	(8)	$\frac{0.5z}{1-0.5z} 2^{-n} u(n-1)$	(7')	$\frac{z}{1-2z} 2^{n-1} u(n-1)$	(8')	$\frac{z}{1-0.5z} 2^{-n+1} u(n-1)$



# B Shifting Shifted Sequence 2

Unshifted Sequence  $x$   Shifted Sequence 2  $x''$

(1)  $\frac{1}{1-2z} 2^n u(n)$

(2)  $\frac{1}{1-0.5z} 2^{-n} u(n)$

(1'')  $2^{n-1} u(n)$

(2'')  $2^{-n+1} u(n)$

(3)  $\frac{1}{1-0.5z^2} 2^n u(-n)$

(4)  $\frac{1}{1-2z^2} 2^{-n} u(-n)$

(3'')  $2^{n+1} u(-n)$

(4'')  $2^{-n-1} u(-n)$

(5)  $\frac{0.5z^2}{1-0.5z^2} 2^n u(-n-1)$

(6)  $\frac{2z^2}{1-2z^2} 2^{-n} u(-n-1)$

(5'')  $2^{n-1} u(-n-1)$

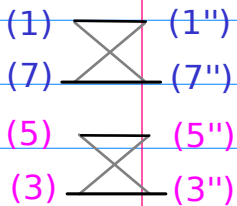
(6'')  $2^{-n+1} u(-n-1)$

(7)  $\frac{2z}{1-2z} 2^n u(n-1)$

(8)  $\frac{0.5z}{1-0.5z} 2^{-n} u(n-1)$

(7'')  $2^{n+1} u(n-1)$

(8'')  $2^{-n-1} u(n-1)$



(1)  $a^n u(n)$

(2)  $a^{-n} u(n)$

(3)  $a^n u(-n)$

(4)  $a^{-n} u(-n)$

(5)  $a^n u(-n-1)$

(6)  $a^n u(-n-1)$

(7)  $a^n u(n-1)$

(8)  $a^{-n} u(n-1)$

$a^{n-1} u(n)$  (1'')

$a^{-n+1} u(n)$  (2'')

$a^{n+1} u(-n)$  (3'')

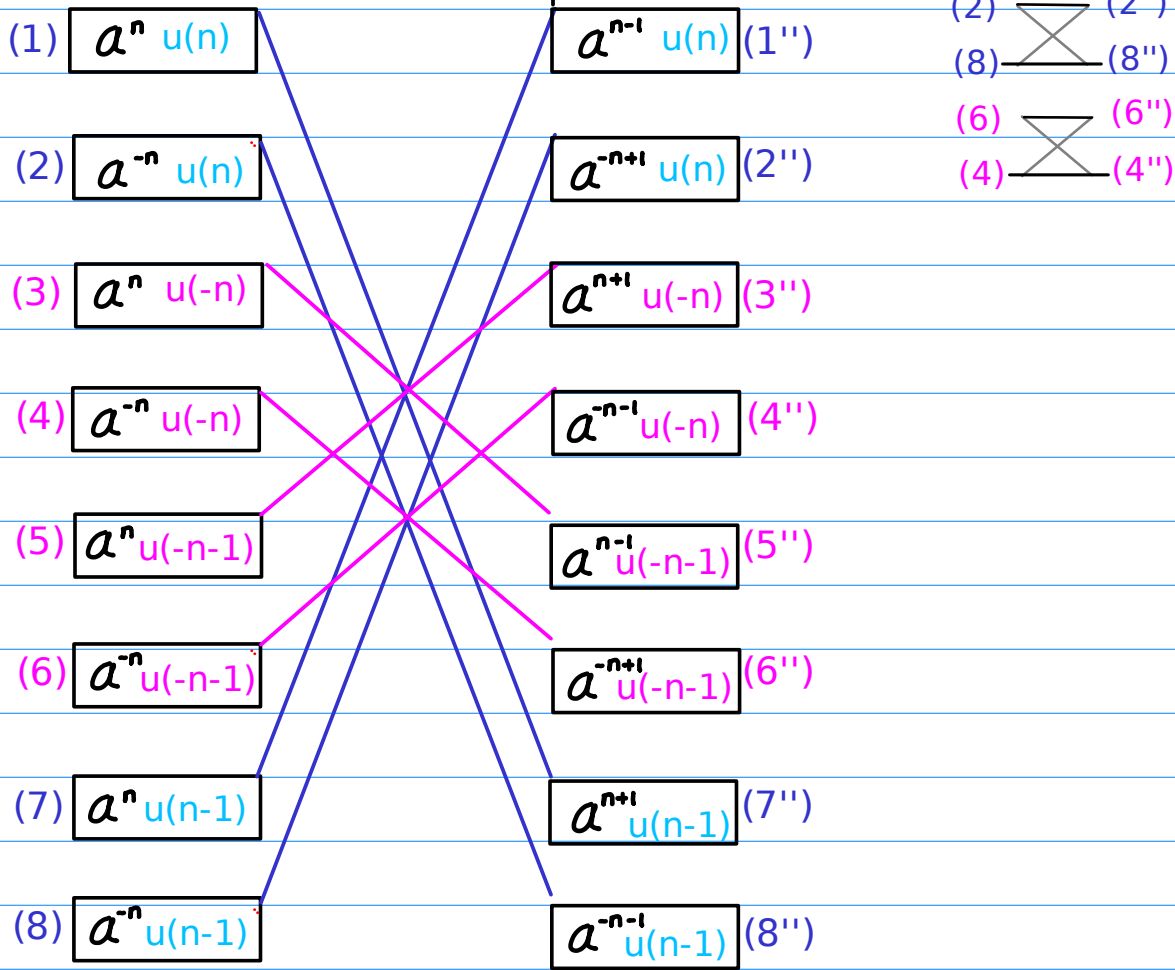
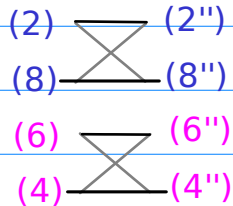
$a^{-n-1} u(-n)$  (4'')

$a^{n-1} u(-n-1)$  (5'')

$a^{-n+1} u(-n-1)$  (6'')

$a^{n+1} u(n-1)$  (7'')

$a^{-n-1} u(n-1)$  (8'')



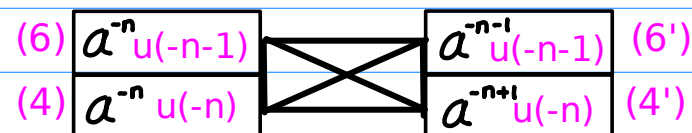
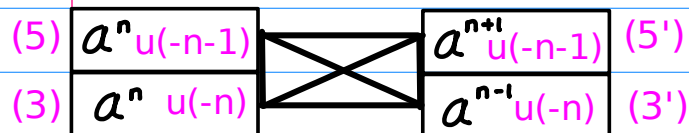
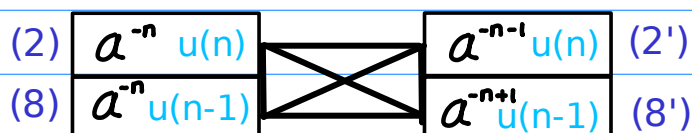
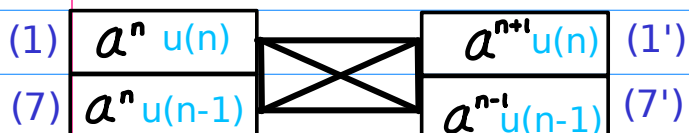
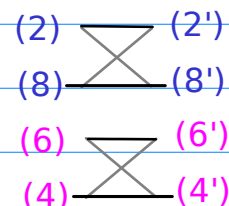
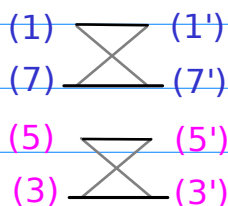
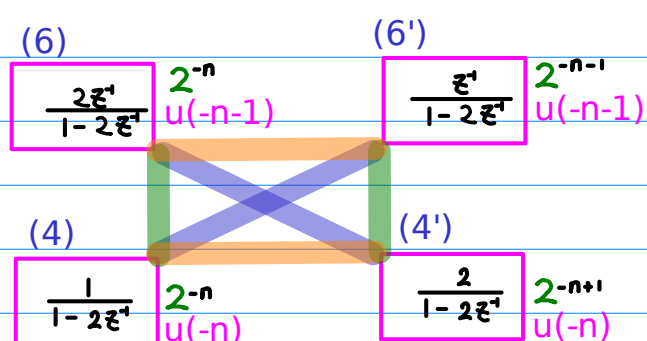
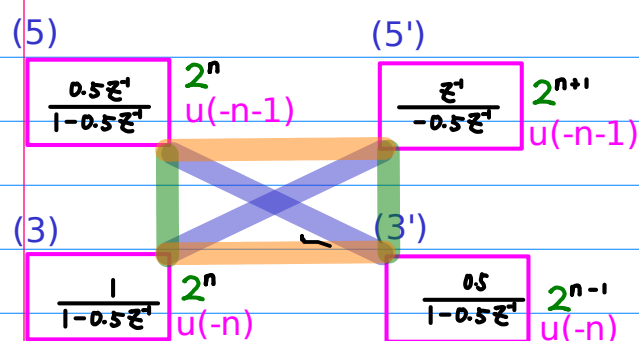
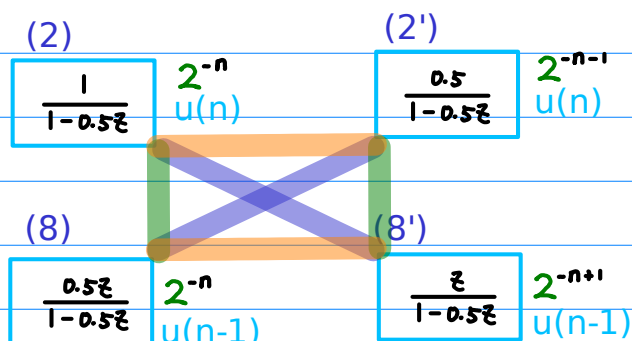
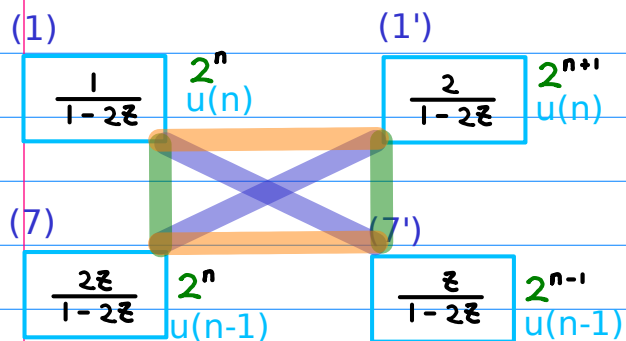
# Inter-permutation

## A Shifting Shifted Sequence 1

(x)  $\longrightarrow$  (x')

(1)~(8)

(1')~(8')



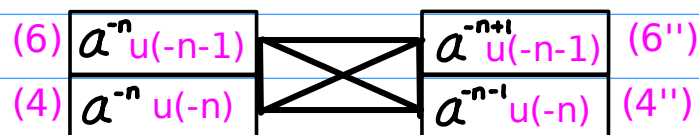
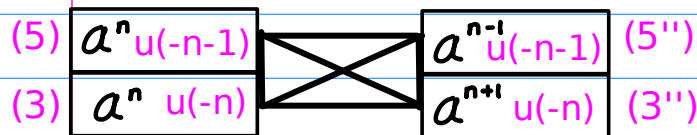
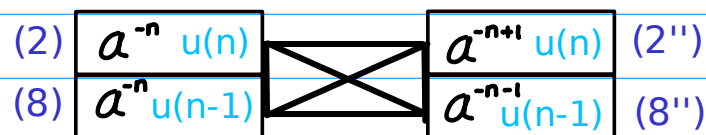
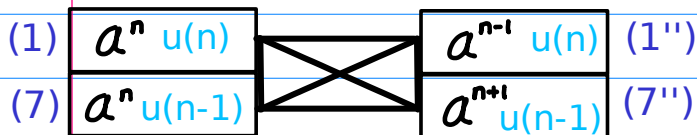
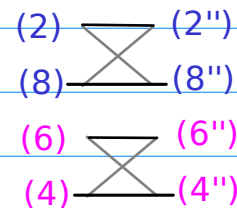
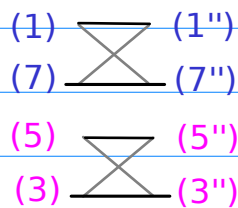
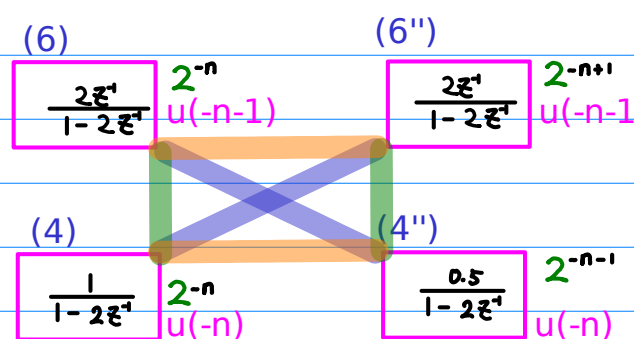
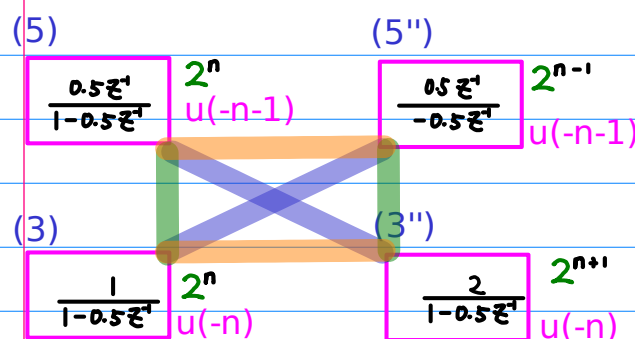
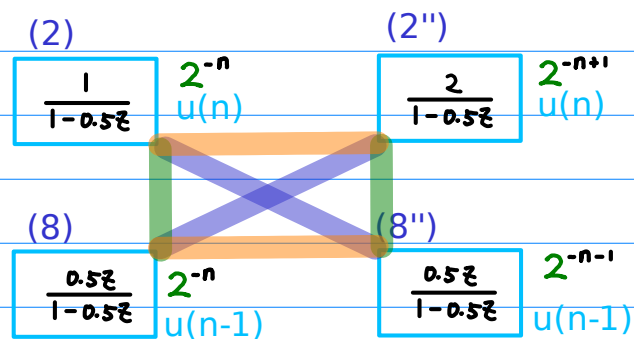
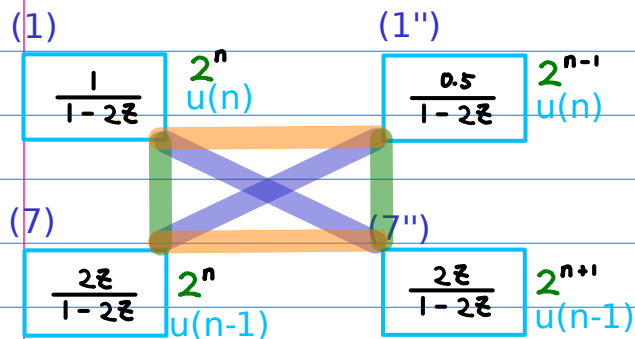


# Inter-permutation

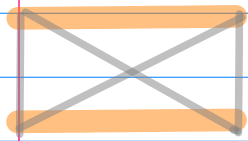
## B Shifting Shifted Sequence 2

(x)  $\longrightarrow$  (x'')

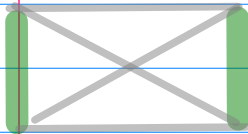
(1)~(8) (1'')~(8'')



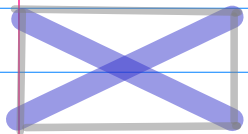
# Decomposing Shift Operations



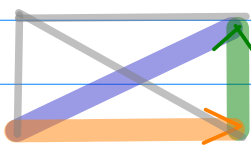
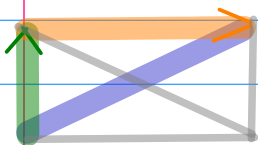
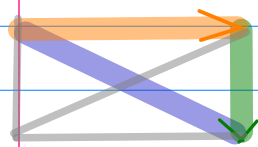
( exponent shift, identity )



( identity, range shift )



( exponent shift, range shift )  
= ( exponent shift, identity )  
+ ( identity, range shift )



**A Shifting Shifted Sequence 1**  
**(1) Exponent Shifting**  
**(2) Range Shifting**

$$\begin{aligned}(\text{SR}, \text{id}) + (\text{id}, \text{SR}) &= (\text{SR}, \text{SR}) \\ (\text{SL}, \text{id}) + (\text{id}, \text{SL}) &= (\text{SL}, \text{SL})\end{aligned}$$

**B Shifting Shifted Sequence 2**  
**(1) Exponent Shifting, Flipping**  
**(2) Range Shifting**

$$\begin{aligned}(\text{SR}, \text{id}) + (\text{id}, \text{SL}) &= (\text{SR}, \text{SL}) \\ (\text{SL}, \text{id}) + (\text{id}, \text{SR}) &= (\text{SL}, \text{SR})\end{aligned}$$

# A Shifting Shifted Sequence 1

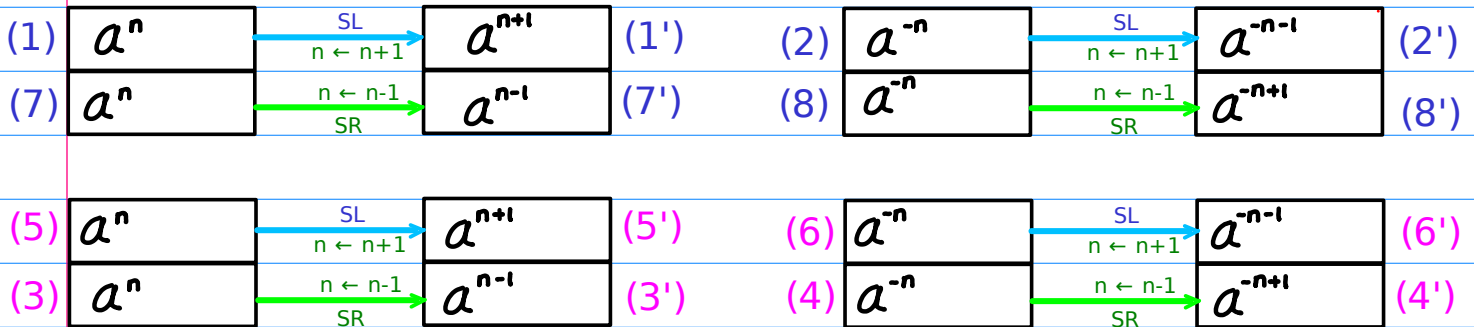
(1) Exponent Shifting

(2) Range Shifting

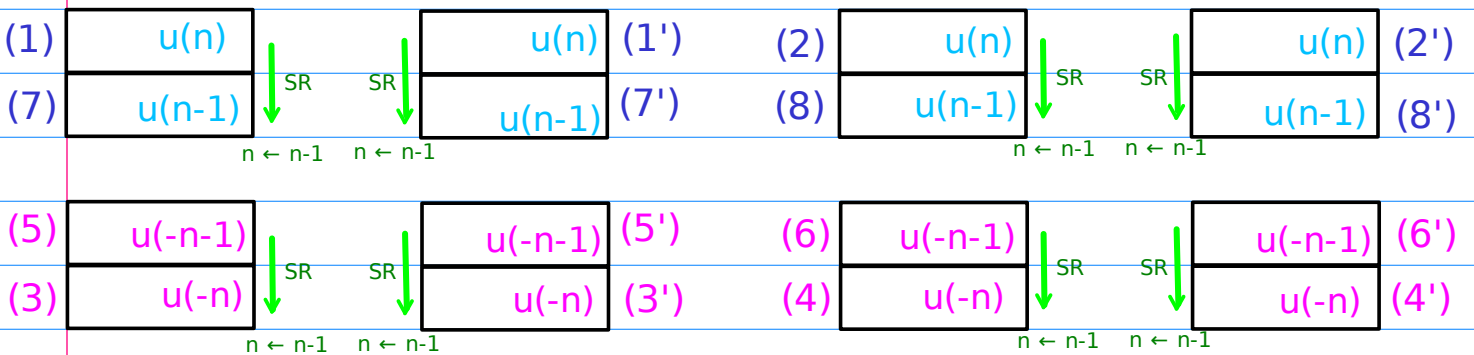
(SR, id) or (SL, id)

(id, SR) or (id, SL)

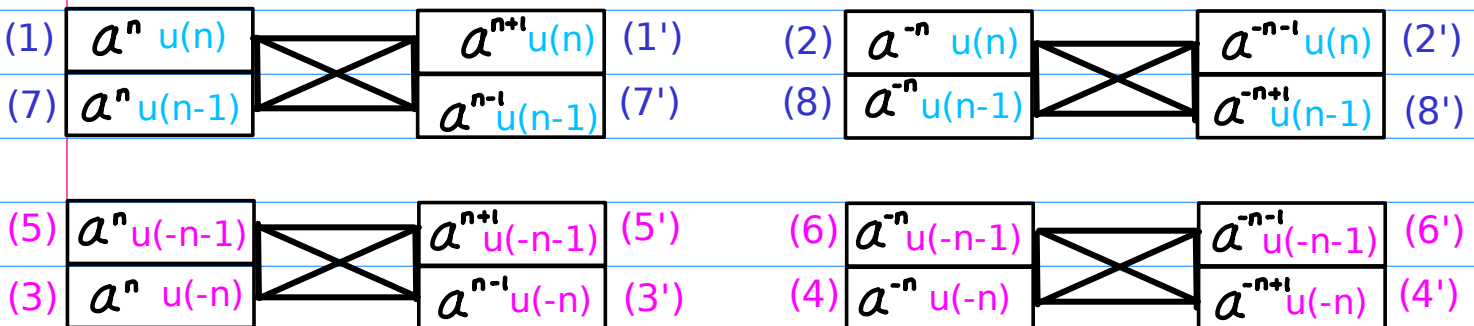
## Exponent Shifts : (SR, id) or (SL, id)



## Range Shifts : (id, SR) or (id, SL)



## Exponent & Range Permutations



## Decomposition

$$(EP, RP) = (EP, id) + (id, RP)$$

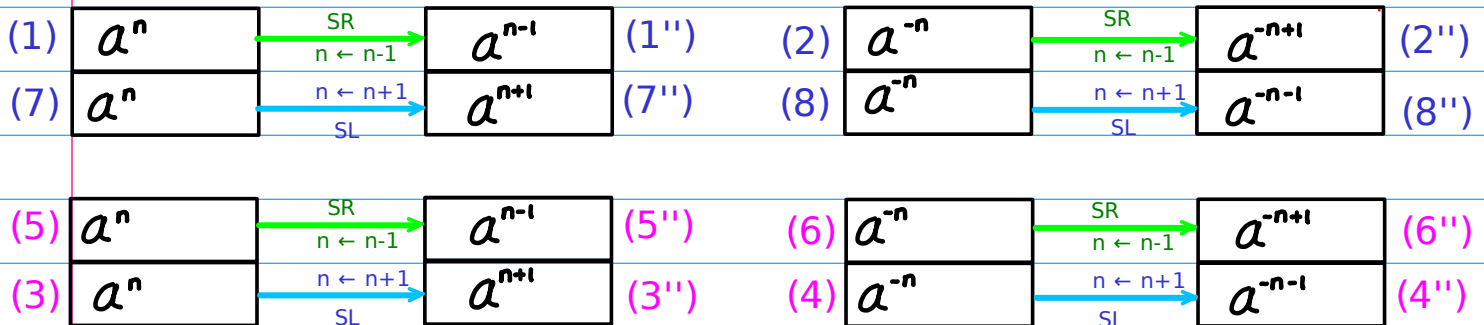
EP : Exponent Permutations

RP : Range Permutations

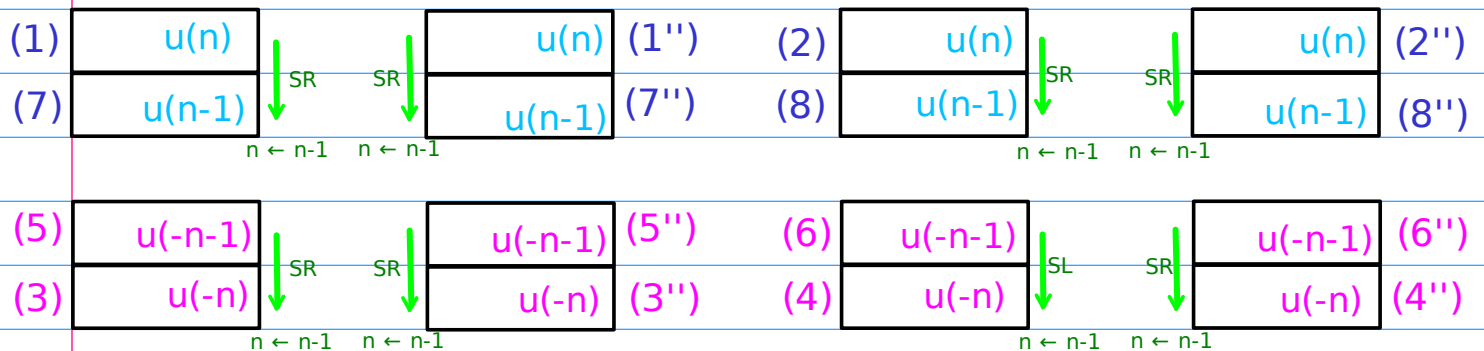
## B Shifting Shifted Sequence 2

- (1) Exponent Shifting, Flipping (SR, id) or (SL, id)  
 (2) Range Shifting (id, SR) or (id, SL)

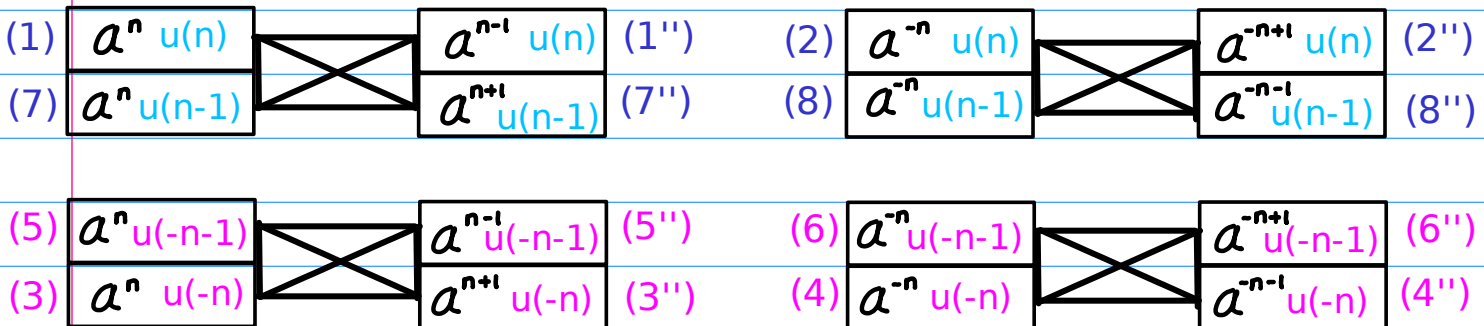
### Exponent Shifts : (SR, id) or (SL, id)



### Range Shifts : (id, SR) or (id, SL)



### Exponent & Range Permutations



### Decomposition

$$(EP, RP) = (EP, id) + (id, RP)$$

EP : Exponent Permutations

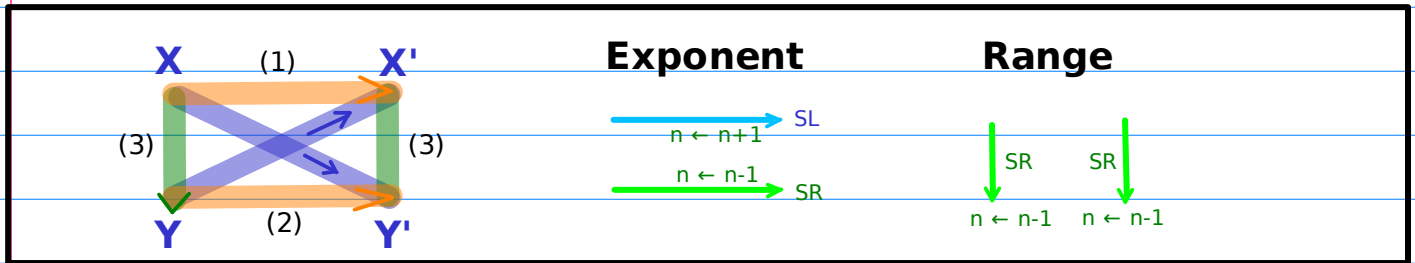
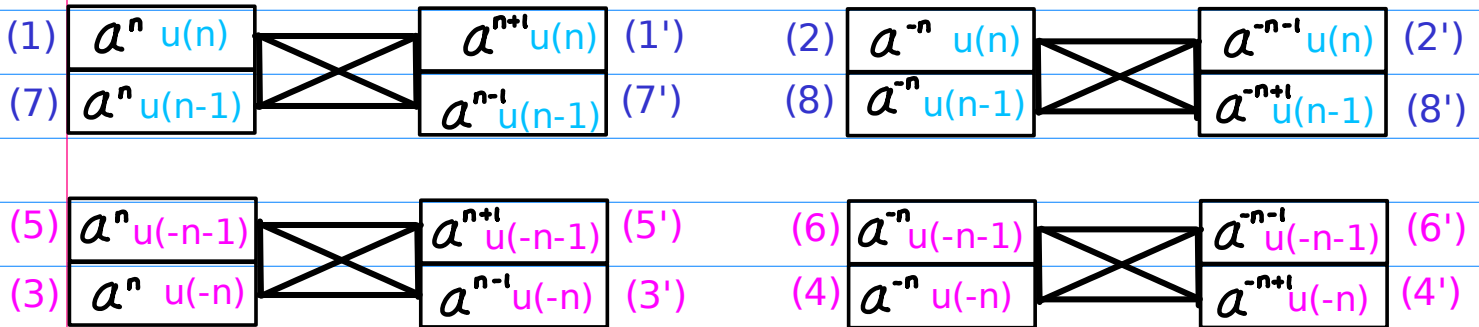
RP : Range Permutations

### A Shifting Shifted Sequence 1

- (1) Exponent Shifting
- (2) Range Shifting

$$(SR, id) + (id, SR) = (SR, SR)$$

$$(SL, id) + (id, SL) = (SL, SL)$$

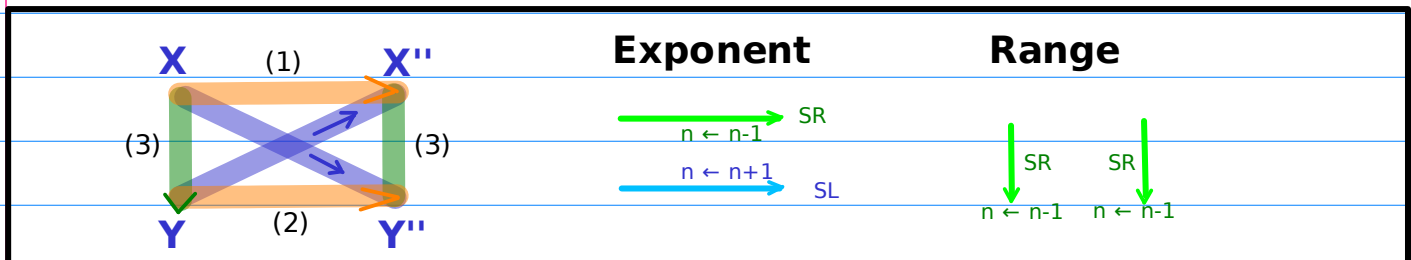
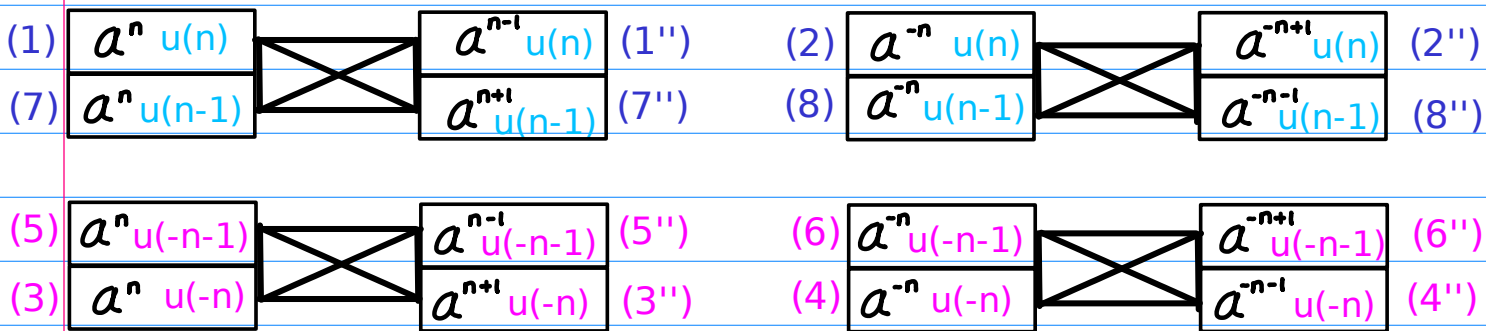


### B Shifting Shifted Sequence 2

- (1) Exponent Shifting
- (2) Range Shifting

$$(SR, id) + (id, SL) = (SR, SL)$$

$$(SL, id) + (id, SR) = (SL, SR)$$



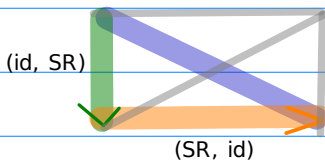
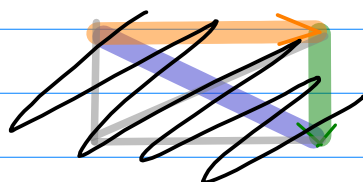
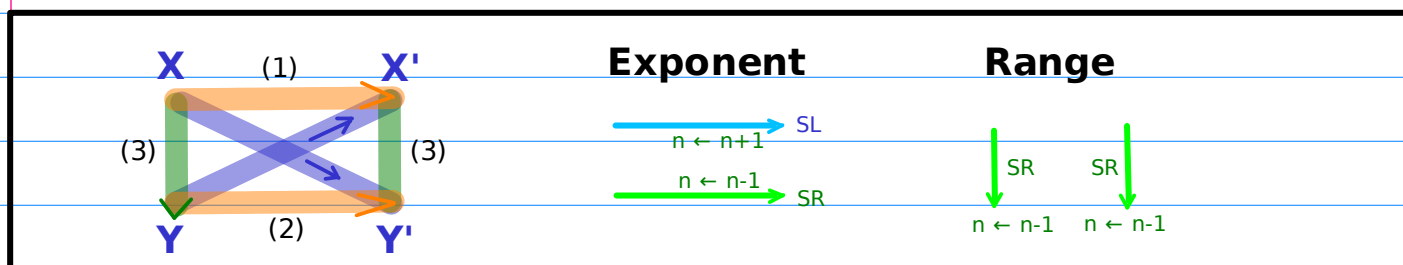
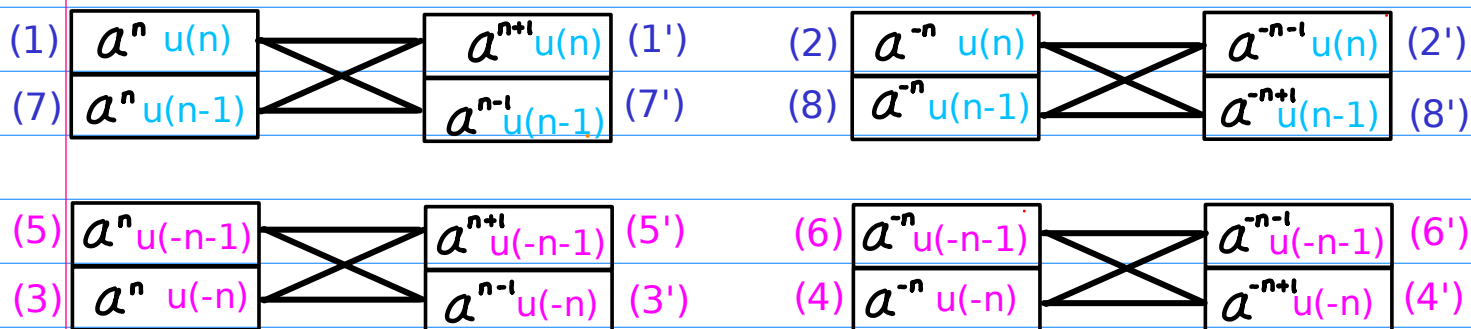
# A Shifting Shifted Sequence 1

(1) Exponent Shifting

(2) Range Shifting

$$(SR, id) + (id, SR) = (SR, SR)$$

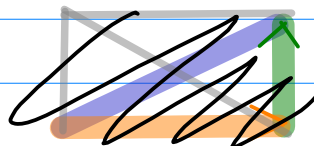
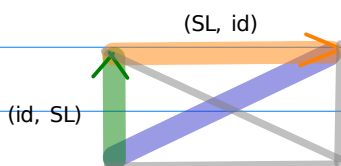
$$(SL, id) + (id, SL) = (SL, SL)$$



(SR, id) shift right exponent

(id, SR) shift right range

(SR, SR)



(SL, id) shift left exponent

(id, SL) shift left range

(SL, SL)

$$(SR, id) + (id, SR) = (SR, SR)$$

$$(SL, id) + (id, SL) = (SL, SL)$$

## B Shifting Shifted Sequence 2

(1) Exponent Shifting

(2) Range Shifting

$$(SR, id) + (id, SL) = (SR, SL)$$

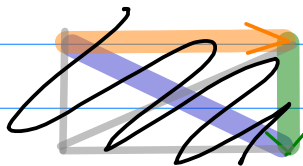
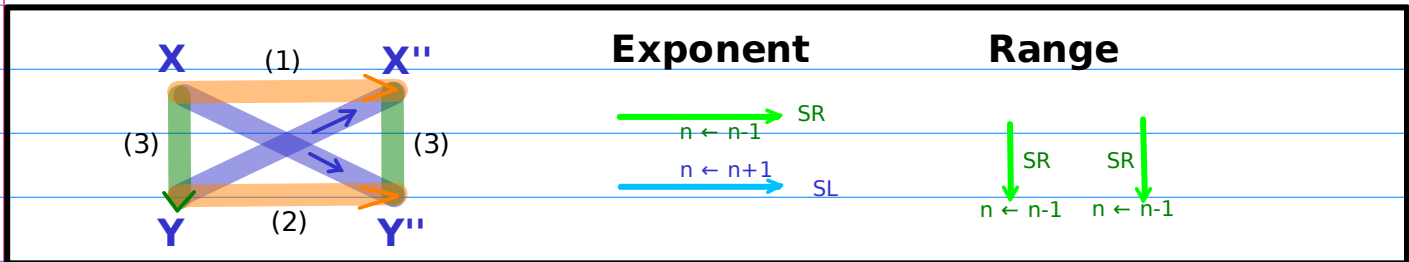
$$(SL, id) + (id, SR) = (SL, SR)$$

$$\begin{array}{|c|} \hline (1) \quad a^n u(n) \\ \hline (7) \quad a^n u(n-1) \\ \hline \end{array} \begin{array}{|c|} \hline \times \\ \hline \end{array} \begin{array}{|c|} \hline a^{n-1} u(n) \quad (1'') \\ \hline a^{n+1} u(n-1) \quad (7'') \\ \hline \end{array}$$

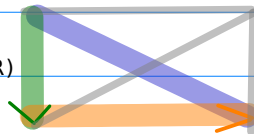
$$\begin{array}{|c|} \hline (2) \quad a^{-n} u(n) \\ \hline (8) \quad a^{-n} u(n-1) \\ \hline \end{array} \begin{array}{|c|} \hline \times \\ \hline \end{array} \begin{array}{|c|} \hline a^{-n+1} u(n) \quad (2'') \\ \hline a^{-n-1} u(n-1) \quad (8'') \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (5) \quad a^n u(-n-1) \\ \hline (3) \quad a^n u(-n) \\ \hline \end{array} \begin{array}{|c|} \hline \times \\ \hline \end{array} \begin{array}{|c|} \hline a^{n-1} u(-n-1) \quad (5'') \\ \hline a^{n+1} u(-n) \quad (3'') \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (6) \quad a^{-n} u(-n-1) \\ \hline (4) \quad a^{-n} u(-n) \\ \hline \end{array} \begin{array}{|c|} \hline \times \\ \hline \end{array} \begin{array}{|c|} \hline a^{-n+1} u(-n-1) \quad (6'') \\ \hline a^{-n-1} u(-n) \quad (4'') \\ \hline \end{array}$$



(id, SR)

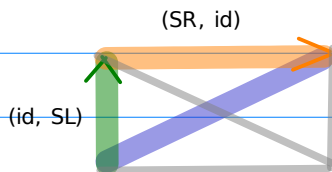


(SL, id)

(SL, id) shift right exponent

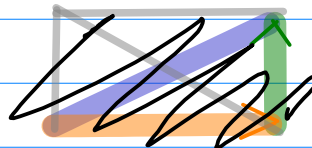
(id, SR) shift right range

(SL, SR)



(SR, id)

(id, SL)



(SR, id) shift left exponent

(id, SL) shift left range

(SR, SL)

$$(SL, id) + (id, SR) = (SL, SR)$$

$$(SR, id) + (id, SL) = (SR, SL)$$

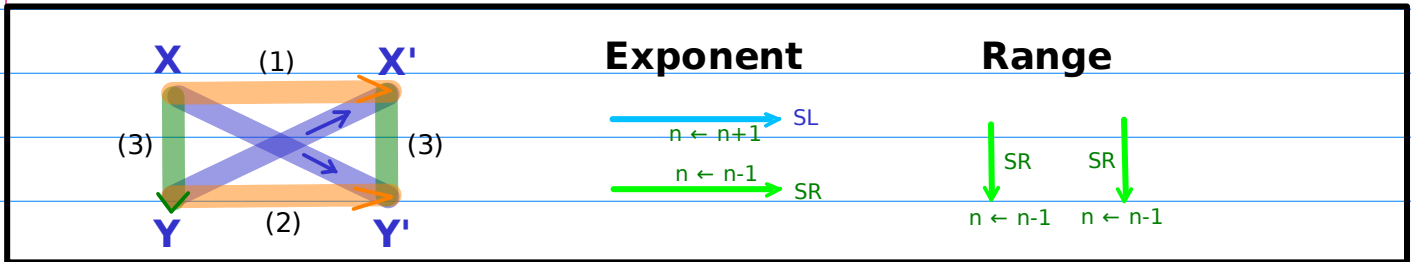




# Butterfly Relations

A Shifting Shifted Sequence 1

Unshifted Sequence  $x$   $\rightarrow$  Shifted Sequence 1  $x'$



(\*) unit starting

(1)  $\xrightarrow{*a \leftarrow (SL, id)}$  (1') **SL**

$/z \leftarrow (SL, SL)$

C.R. starting

(7)  $\xrightarrow{/a \rightarrow (SR, id)}$  (7') **SR**

$*z \Rightarrow (SR, SR)$

(\*) C.R. starting

(5)  $\xrightarrow{*a \leftarrow (SL, id)}$  (5') **SL**

$/z \leftarrow (SL, SL)$

unit starting

(3)  $\xrightarrow{/a \rightarrow (SR, id)}$  (3') **SR**

$*z \Rightarrow (SR, SR)$

(\*) unit starting

(2)  $\xrightarrow{/a \leftarrow (SL, id)}$  (2') **SL**

$/z \leftarrow (SL, SL)$

C.R. starting

(8)  $\xrightarrow{*a \rightarrow (SR, id)}$  (8') **SR**

$*z \Rightarrow (SR, SR)$

(\*) C.R. starting

(6)  $\xrightarrow{/a \leftarrow (SL, id)}$  (6') **SL**

$/z \leftarrow (SL, SL)$

unit starting

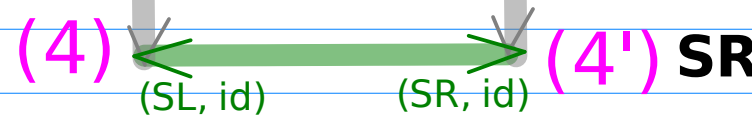
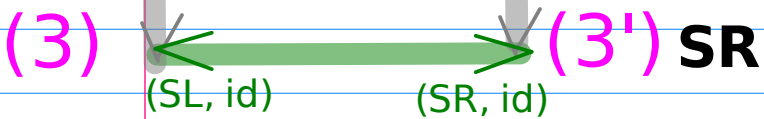
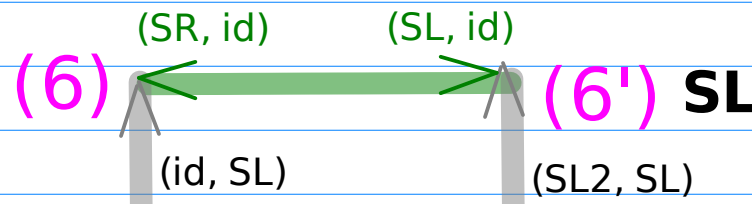
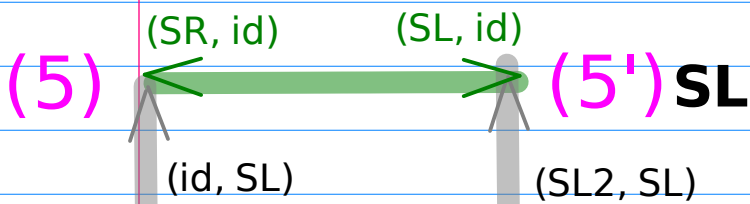
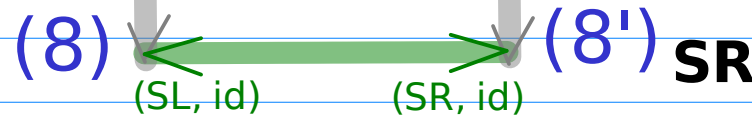
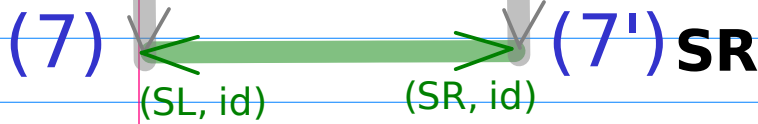
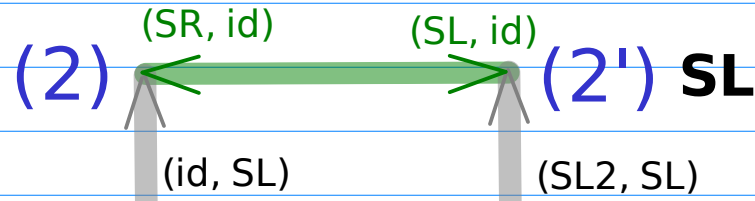
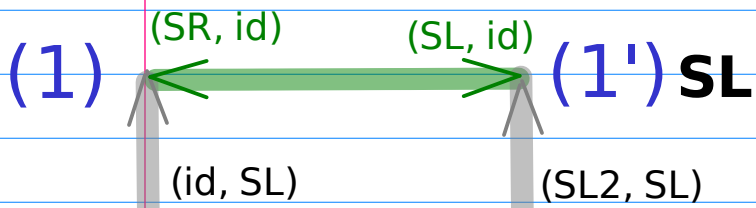
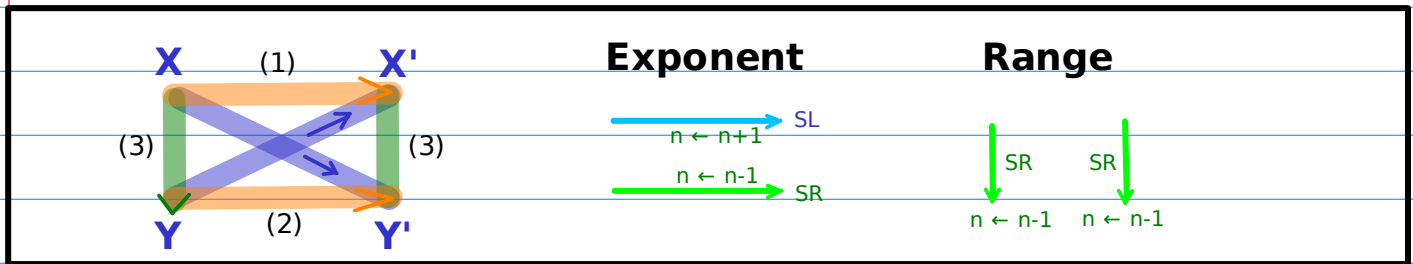
(4)  $\xrightarrow{*a \rightarrow (SR, id)}$  (4') **SR**

$*z \Rightarrow (SR, SR)$

# Butterfly Relations

A Shifting Shifted Sequence 1

Unshifted Sequence  $x$   $\rightarrow$  Shifted Sequence 1  $x'$

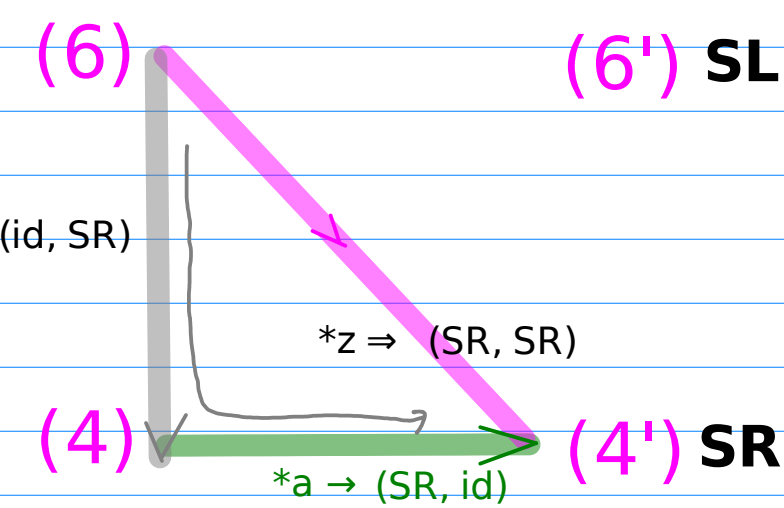
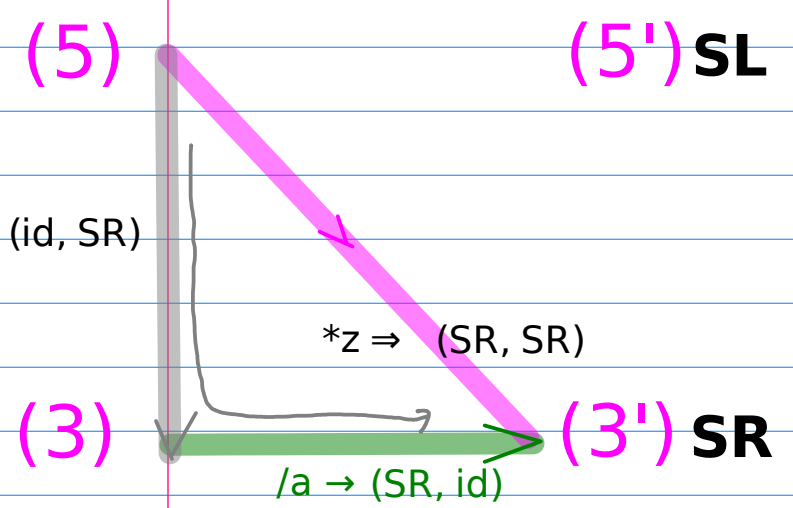
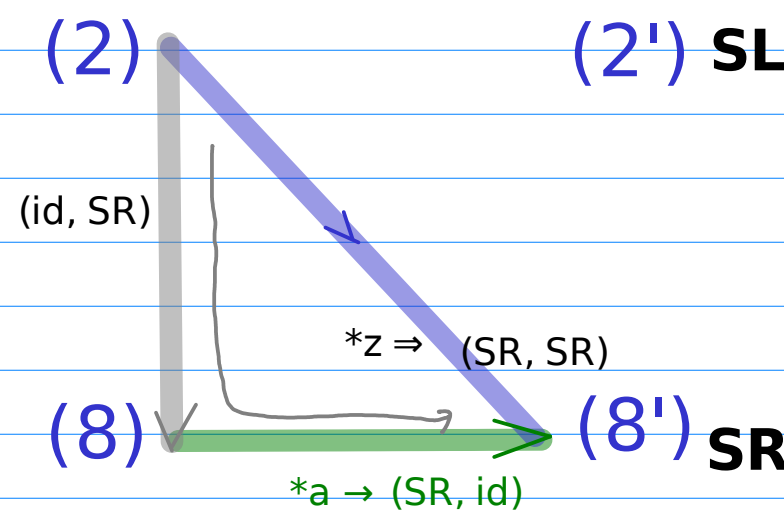
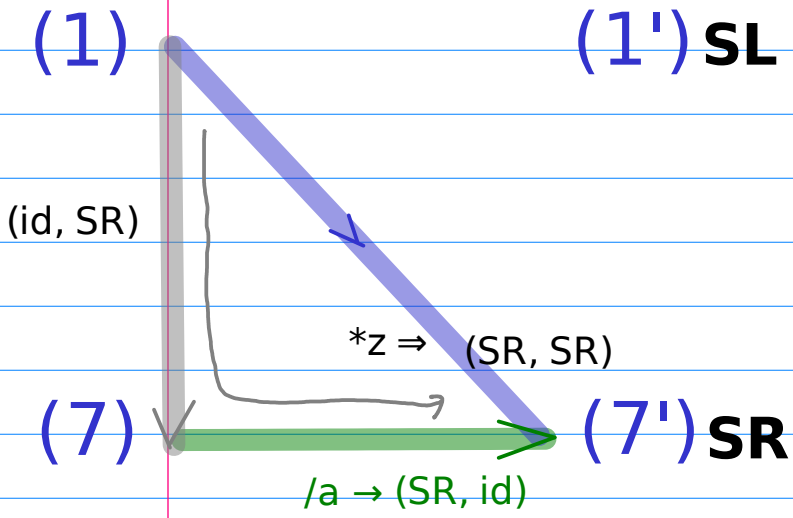
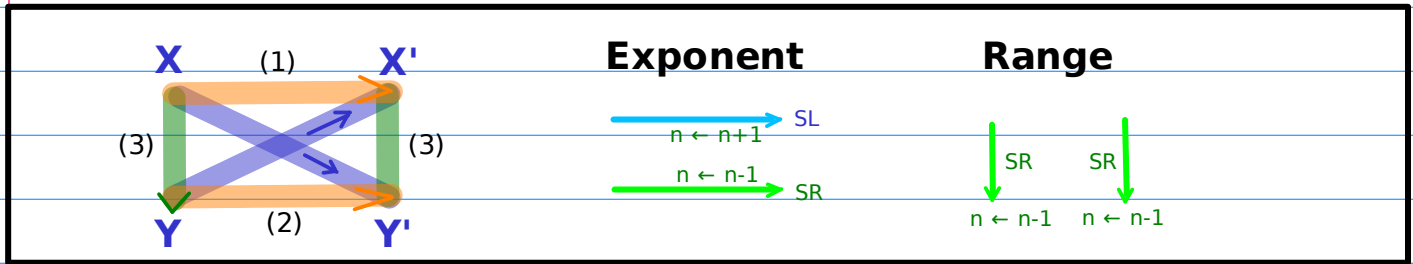


# Decomposition of Exp and Rng Shifts (1)

A Shifting Shifted Sequence 1

Unshifted Sequence  $x$   $\rightarrow$  Shifted Sequence 1  $x'$

$$(id, SR) + (SR, id) = (SR, SR)$$

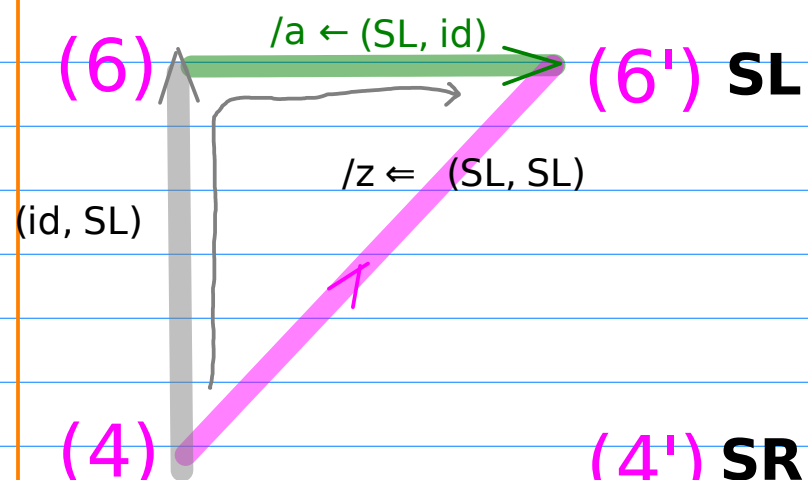
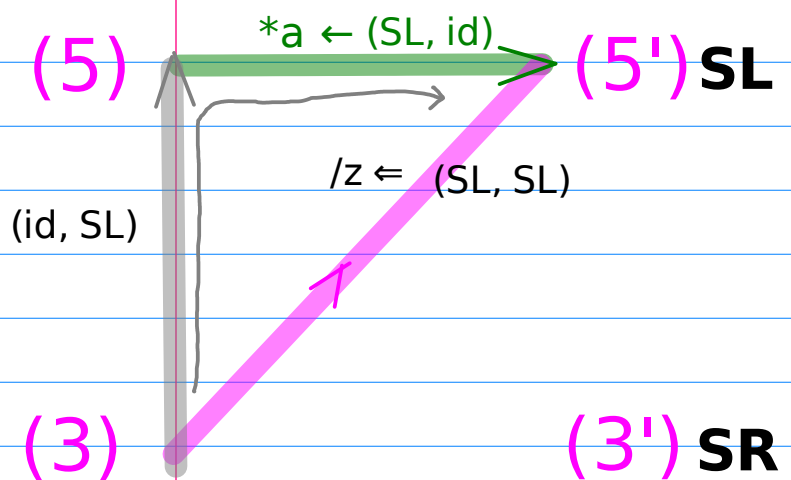
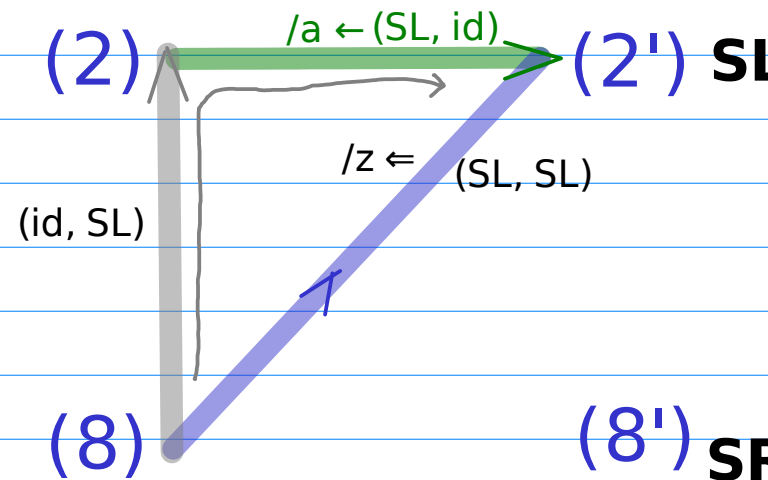
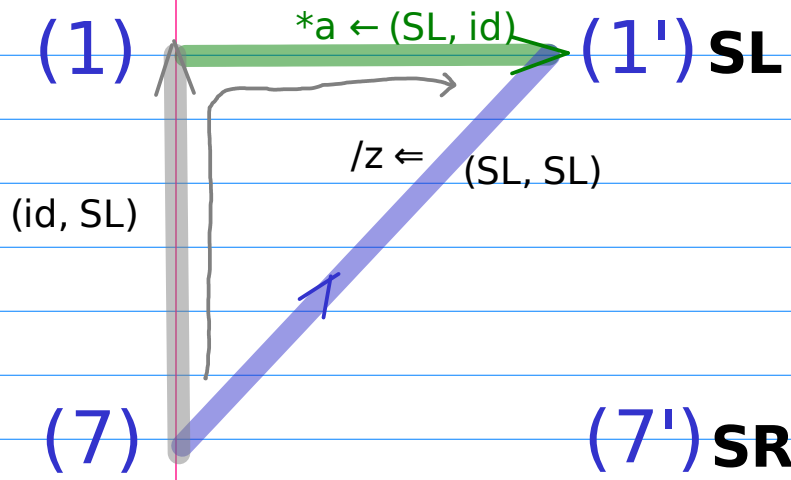
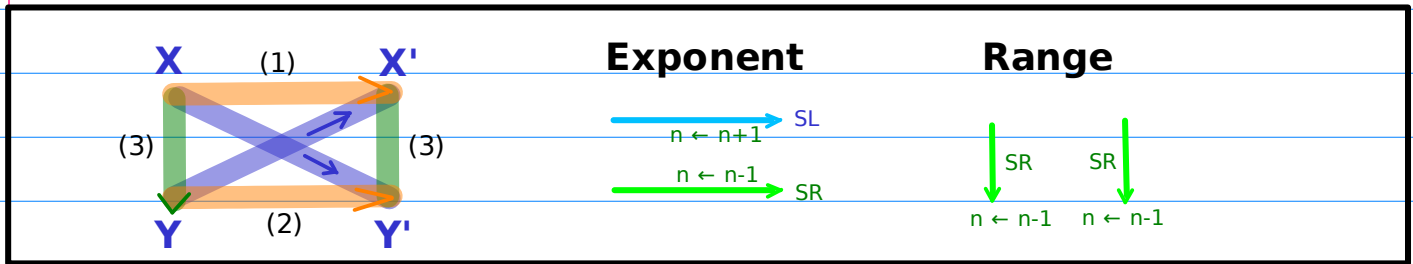


# Decomposition of Exp and Rng Shifts (2)

A Shifting Shifted Sequence 1

Unshifted Sequence  $x$   $\rightarrow$  Shifted Sequence 1  $x'$

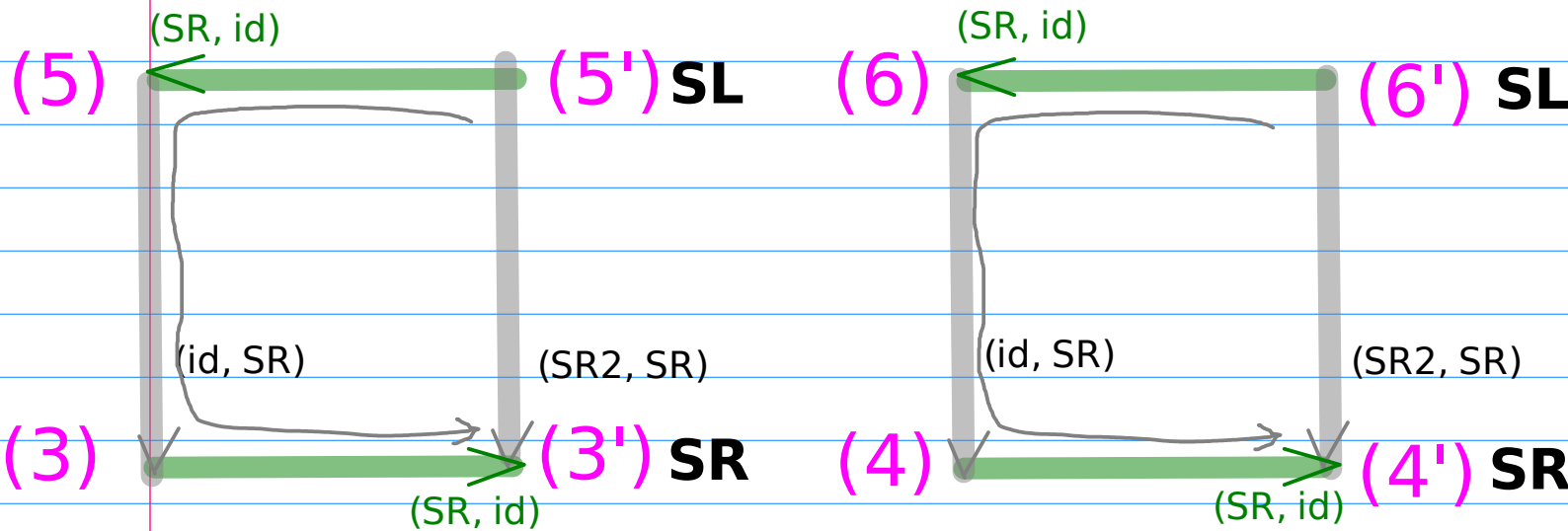
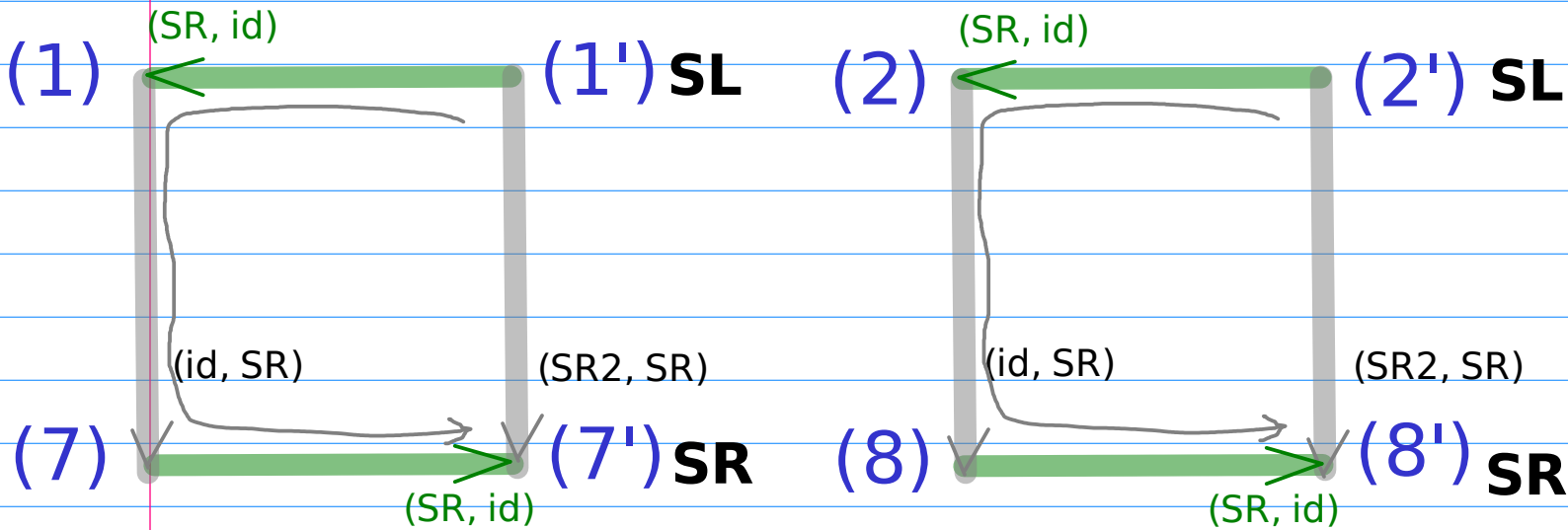
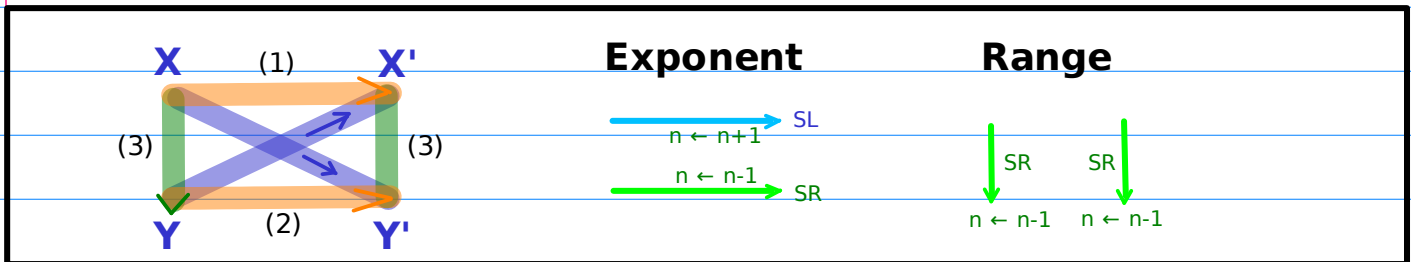
$$(id, SL) + (SL, id) = (SL, SL)$$



# Decomposition of Exp and Rng Shifts (3)

A Shifting Shifted Sequence 1  
 Unshifted Sequence  $x$   $\rightarrow$  Shifted Sequence 1  $x'$

$$(SR, id) + (id, SR) + (SR, id) = (SR2, SR)$$



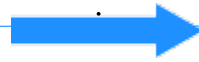




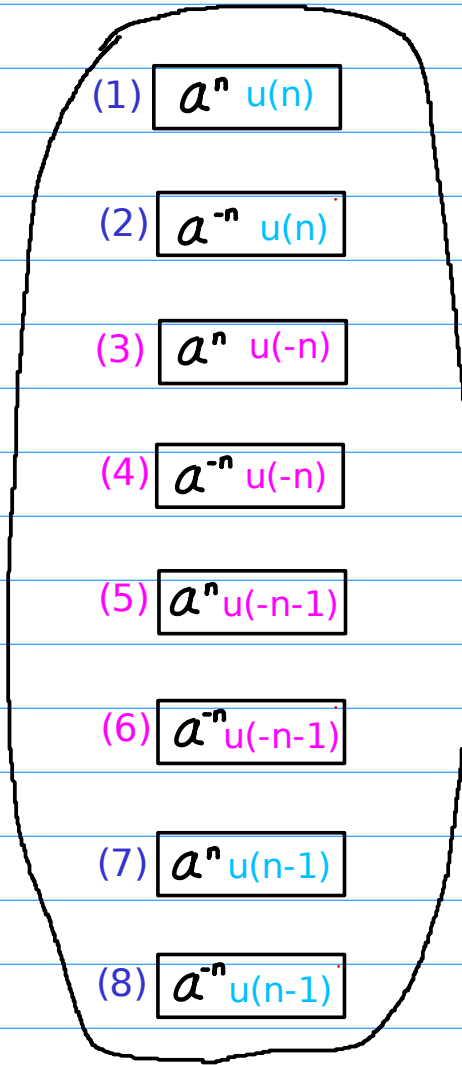
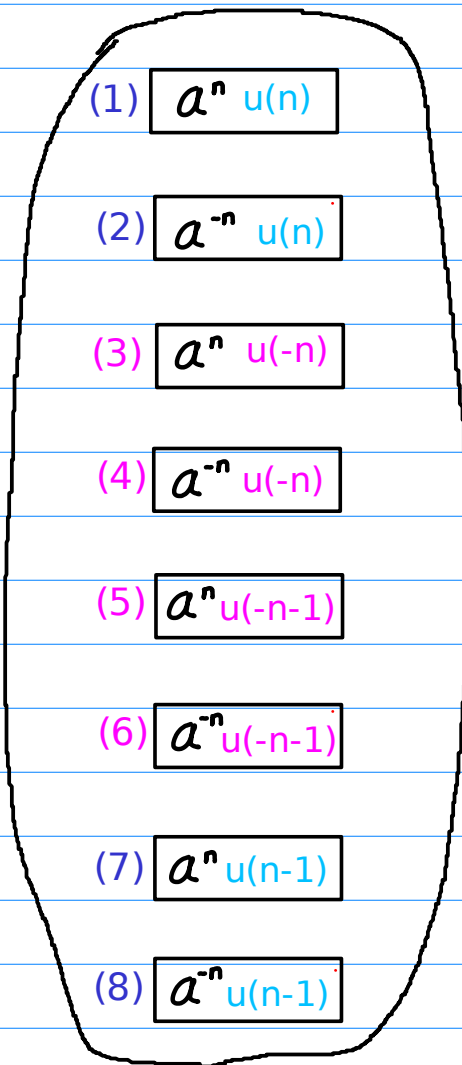


## Intra-Permutations over (1) ~ (8)

Unshifted  
Sequence  $x$



Unshifted  
Sequence  $x$

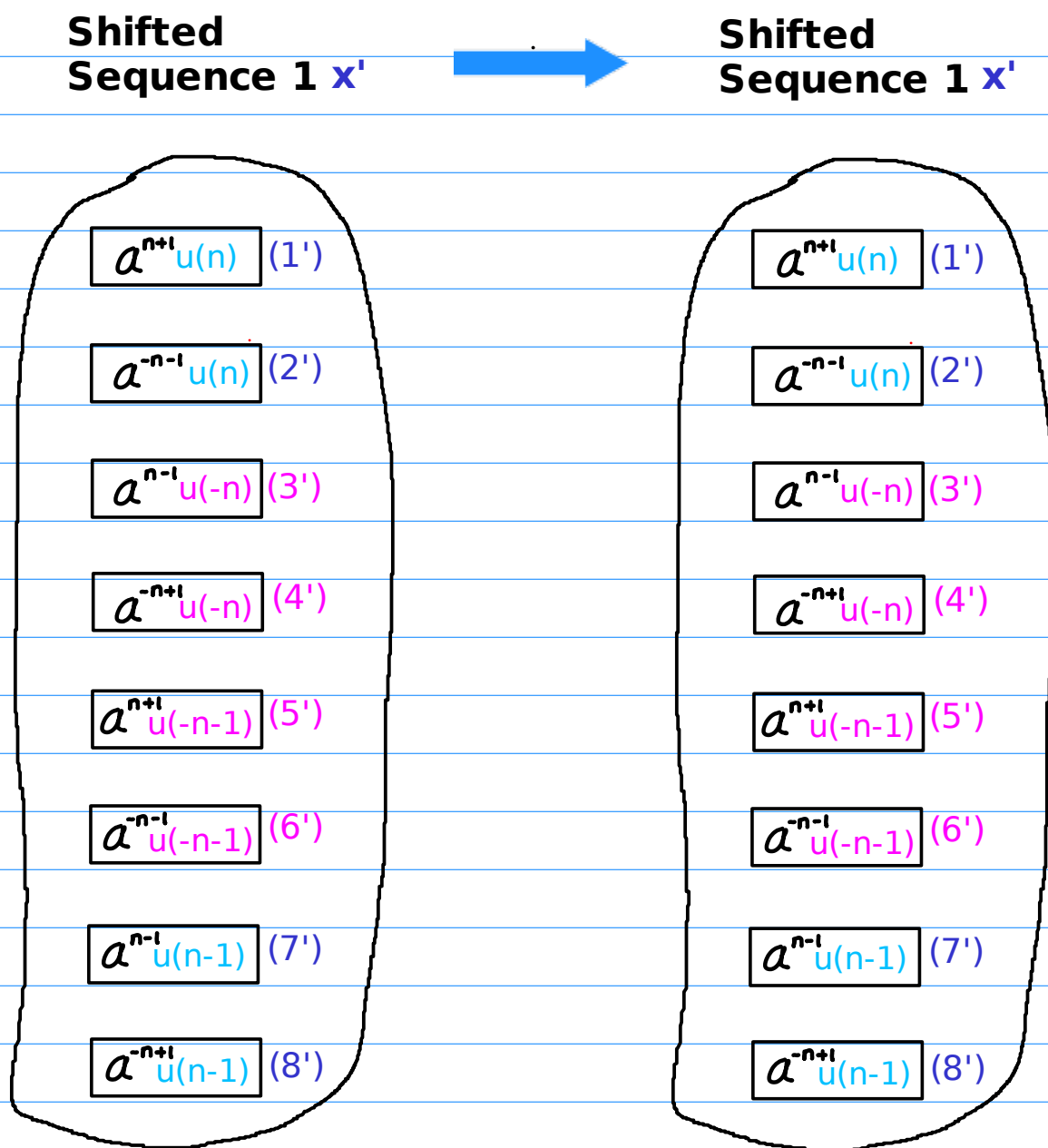


Inter-permutations over unshifted sequence and shifted sequence

Intra-permutations over unshifted sequence

Intra-permutations over shifted sequence

# Intra-Permutations over (1') ~ (8')



Inter-permutations over unshifted sequence and shifted sequence

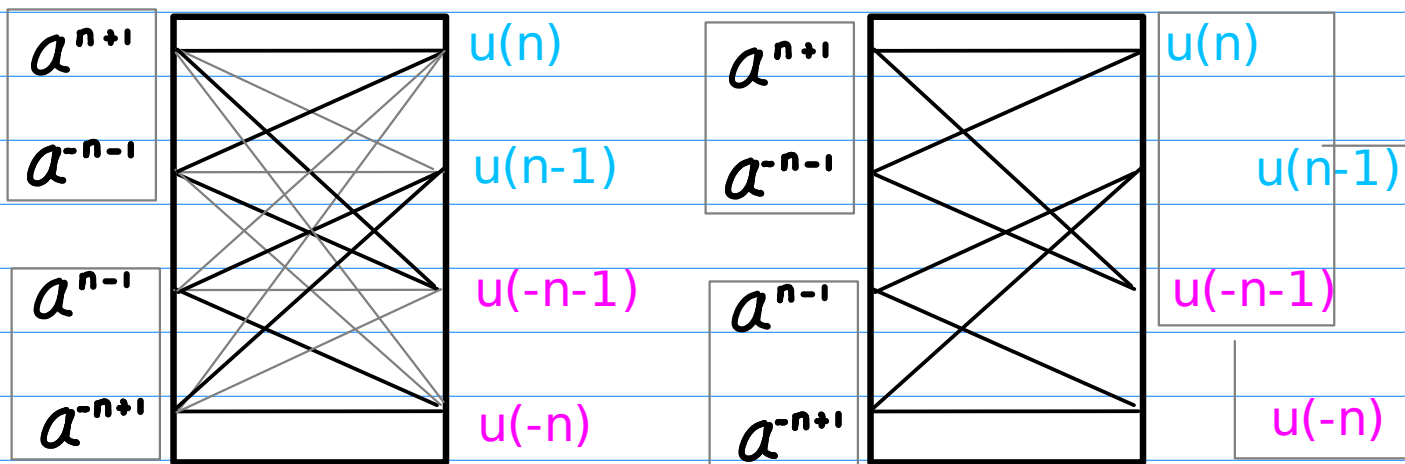
Intra-permutations over unshifted sequence

Intra-permutations over shifted sequence



$$a^n \times R(n)$$

$$\begin{array}{|c|c|} \hline a^{n+1} & a^{-n-1} \\ \hline a^{n-1} & a^{-n+1} \\ \hline \end{array} \times \begin{array}{|c|c|} \hline u(n) & u(-n-1) \\ \hline u(n-1) & u(-n) \\ \hline \end{array}$$




$$\begin{array}{l} n \leftarrow n+2 \text{ or} \\ n \leftarrow n-2 \end{array}$$


$$\begin{array}{|c|c|} \hline (1') a^{n+1}u(n) & a^{-n-1}u(n) \quad (2') \\ \hline (3') a^{n-1}u(-n) & a^{-n+1}u(-n) \quad (4') \\ \hline (5') a^{n+1}u(-n-1) & a^{-n-1}u(-n-1) \quad (6') \\ \hline (7') a^{n-1}u(n-1) & a^{-n+1}u(n-1) \quad (8') \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline (1'') a^{n-1}u(n) & a^{-n+1}u(n) \quad (2'') \\ \hline (3'') a^{n+1}u(-n) & a^{-n-1}u(-n) \quad (4'') \\ \hline (5'') a^{n-1}u(-n-1) & a^{-n+1}u(-n-1) \quad (6'') \\ \hline (7'') a^{n+1}u(n-1) & a^{-n-1}u(n-1) \quad (8'') \\ \hline \end{array}$$



permutation over (1) ~ (8)    (x)        (x)

<b>A. Flipping</b>	(1) - (4)	(5) - (8)
Base Inverting	(2) - (3)	(6) - (7)
Range Flipping	(3) - (2)	(7) - (6)
	(4) - (1)	(8) - (5)
<b>B. Range Shifting</b>	(1) - (7)	(5) - (3)
Range Flipping	(2) - (8)	(6) - (4)
Range Complementing	(3) - (5)	(7) - (1)
	(4) - (6)	(8) - (2)
<b>C. Complementary Inverting</b>	(1) - (6)	(5) - (2)
Base Inverting	(6) - (1)	(6) - (1)
Range Complementing	(2) - (5)	(7) - (4)
	(5) - (2)	(8) - (3)

permutation over (1') ~ (8')    (x')        (x')

<b>D. Flipping2</b>	(1') - (4')	(5') - (8')
Base Inverting	(2') - (3')	(6') - (7')
Shifted Range Flipping	(3') - (2')	(7') - (6')
	(4') - (1')	(8') - (5')
<b>E. Shifting2</b>	(1') - (7')	(5') - (3')
Shifted Range Flipping	(2') - (8')	(6') - (4')
Range Complementing	(3') - (5')	(7') - (1')
	(4') - (6')	(8') - (2')
<b>F. Complementary Inverting</b>	(1') - (6')	(5') - (2')
Base Inverting	(6') - (1')	(6') - (1')
Range Complementing	(2') - (5')	(7') - (4')
	(5') - (2')	(8') - (3')

**Shifted Range Flipping = Exponent Shifting2 + Range Flipping**

**Shifting2 = Shifted Range Flipping + Range Complementing**  
**= Exponent Shifting2 + Range Flipping + Range Complementing**  
**= Exponent Shifting2 + Range (Flipping + Complementing)**  
**= Exponent Shifting2 + Range Shifting**

## Permutation over (1) ~ (8)

Permutations	A	B	C
Base Inverting	X		X
Range Flipping	X	X	
Range Complementing		X	X

## Permutation over (1') ~ (8')

Permutations	D	E	F
Base Inverting	X		X
Shifted Range Flipping	X	X	
Range Complementing		X	X

**Shifted Range Flipping = Exponent Shifting<sub>2</sub> + Range Flipping**

**Shifting<sub>2</sub> = Shifted Range Flipping + Range Complementing**  
**= Exponent Shifting<sub>2</sub> + Range Flipping + Range Complementing**  
**= Exponent Shifting<sub>2</sub> + Range (Flipping + Complementing)**  
**= Exponent Shifting<sub>2</sub> + Range Shifting**

**Over (1) ~ (8)**

**Base Inverting**

$$a^n \longleftrightarrow a^{-n}$$

**Range Flipping**

$$R(n) \longleftrightarrow R(-n)$$

**Range Complementing**

$$R(n) \longleftrightarrow \overline{R(n)}$$

**Over (1') ~ (8')**

**Base Inverting**

$$a^n \longleftrightarrow a^{-n}$$

**Shifted Range Flipping**

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

**Range Complementing**

$$R(n) \longleftrightarrow \overline{R(n)}$$



**A.I Flipping**  
**Base Inverting**  
**Range Flipping**

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-n} R(-n)$$

**D.I Flipping2**  
**Base Inverting**  
**Shifted Range Flipping**

$$a^n \longleftrightarrow a^{-n}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-sh2(n)} R(-n)$$

**B.I Range Shifting**  
**Range Flipping**  
**Range Complementing**

$$R(n) \longleftrightarrow R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

**E.I Shifting2**  
**Shifted Range Flipping**  
**Range Complementing**

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

**C.I Complementary Inverting**  
**Base Inverting**  
**Range Complementing**

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

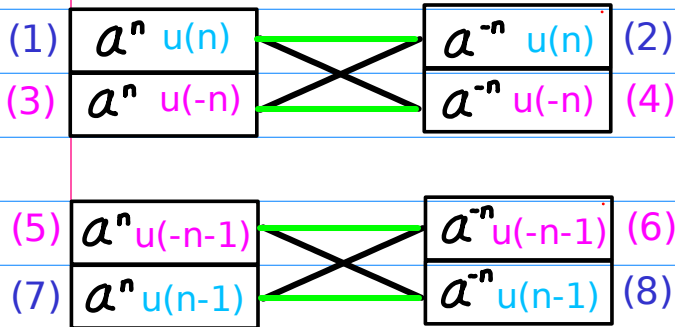
**F.I Complementary Inverting**  
**Base Inverting**  
**Range Complementing**

$$a^n \longleftrightarrow a^{-n}$$

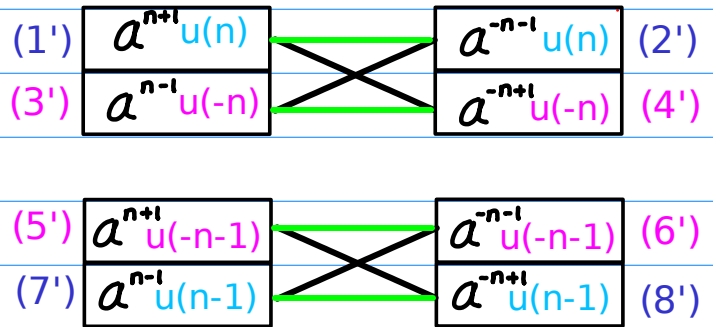
$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

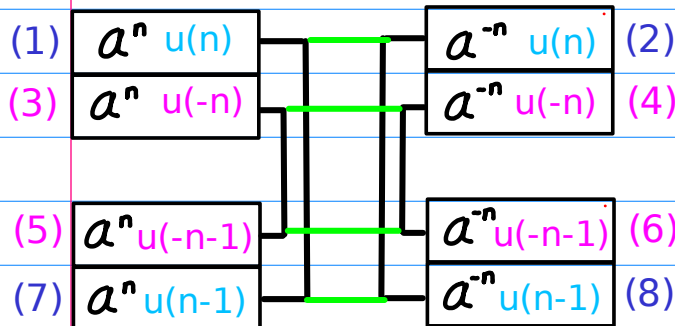
**A.I Flipping**  
**Base Inverting**  
**Range Flipping**



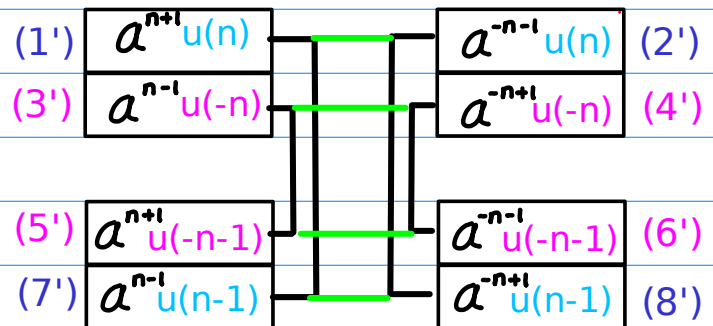
**D.I Flipping2**  
**Base Inverting**  
**Shifted Range Flipping**



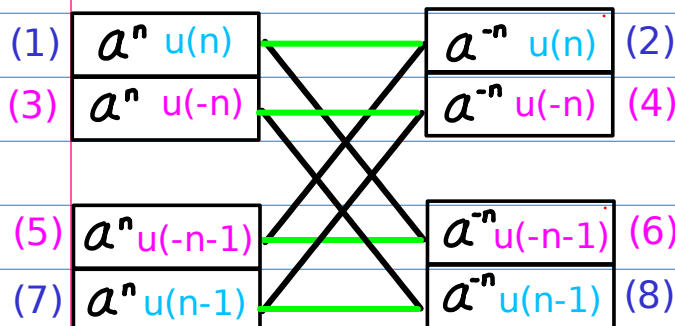
**B.I Range Shifting**  
**Range Flipping**  
**Range Complementing**



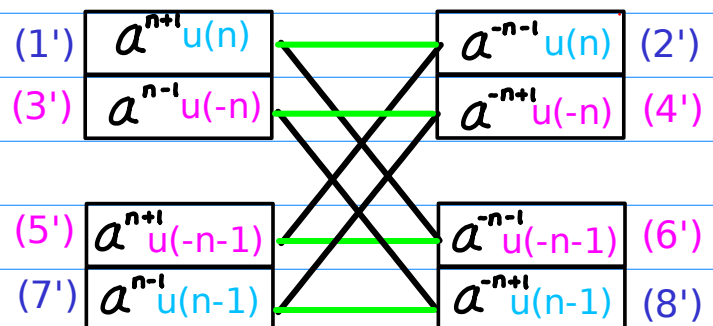
**E.I Shifting2**  
**Shifted Range Flipping**  
**Range Complementing**



**C.I Complementary Inverting**  
**Base Inverting**  
**Range Complementing**

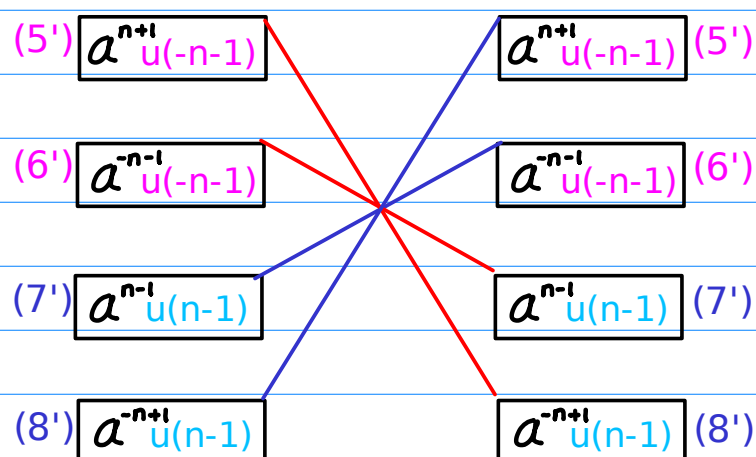
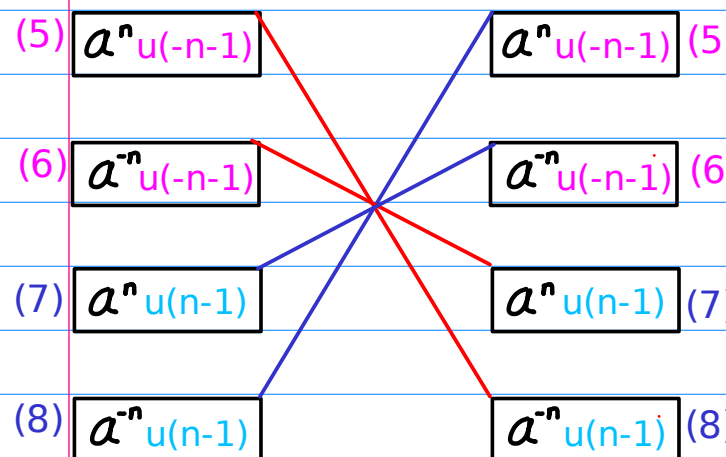
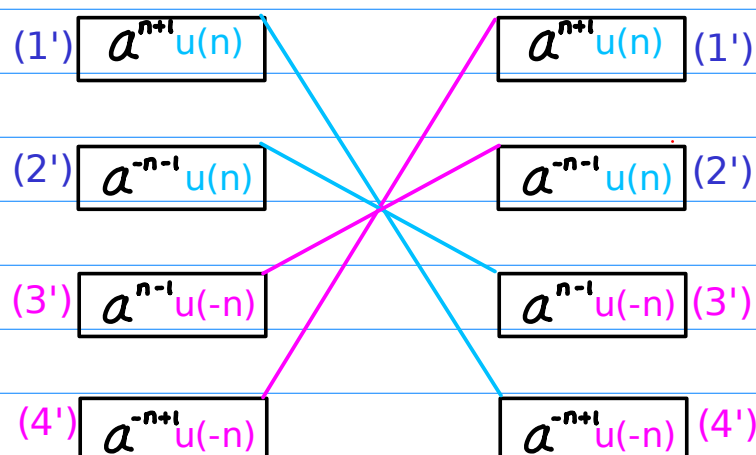
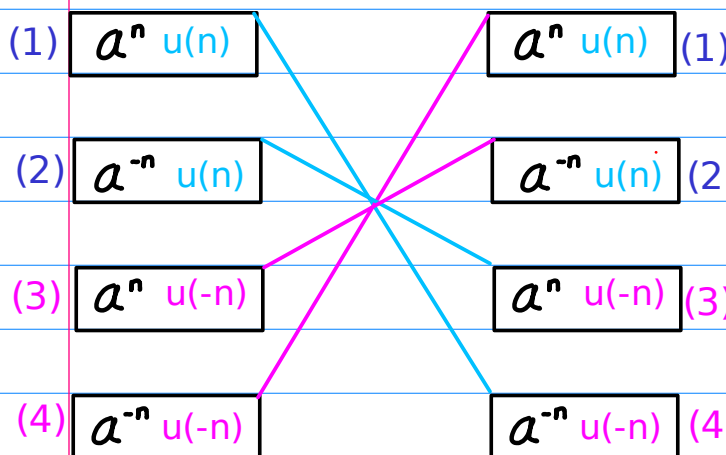
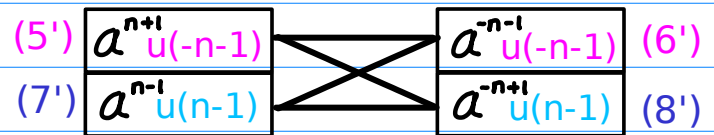
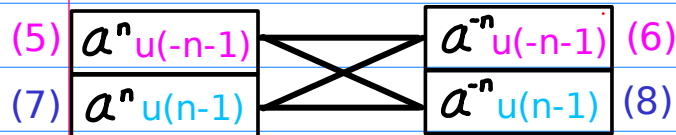
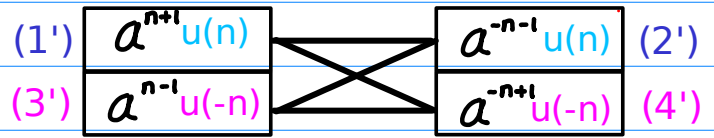
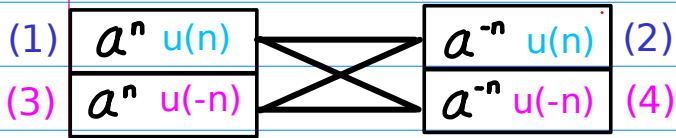


**F.I Complementary Inverting**  
**Base Inverting**  
**Range Complementing**



**A.I Flipping**  
**Base Inverting**  
**Range Flipping**

**D.I Flipping2**  
**Base Inverting**  
**Shifted Range Flipping**

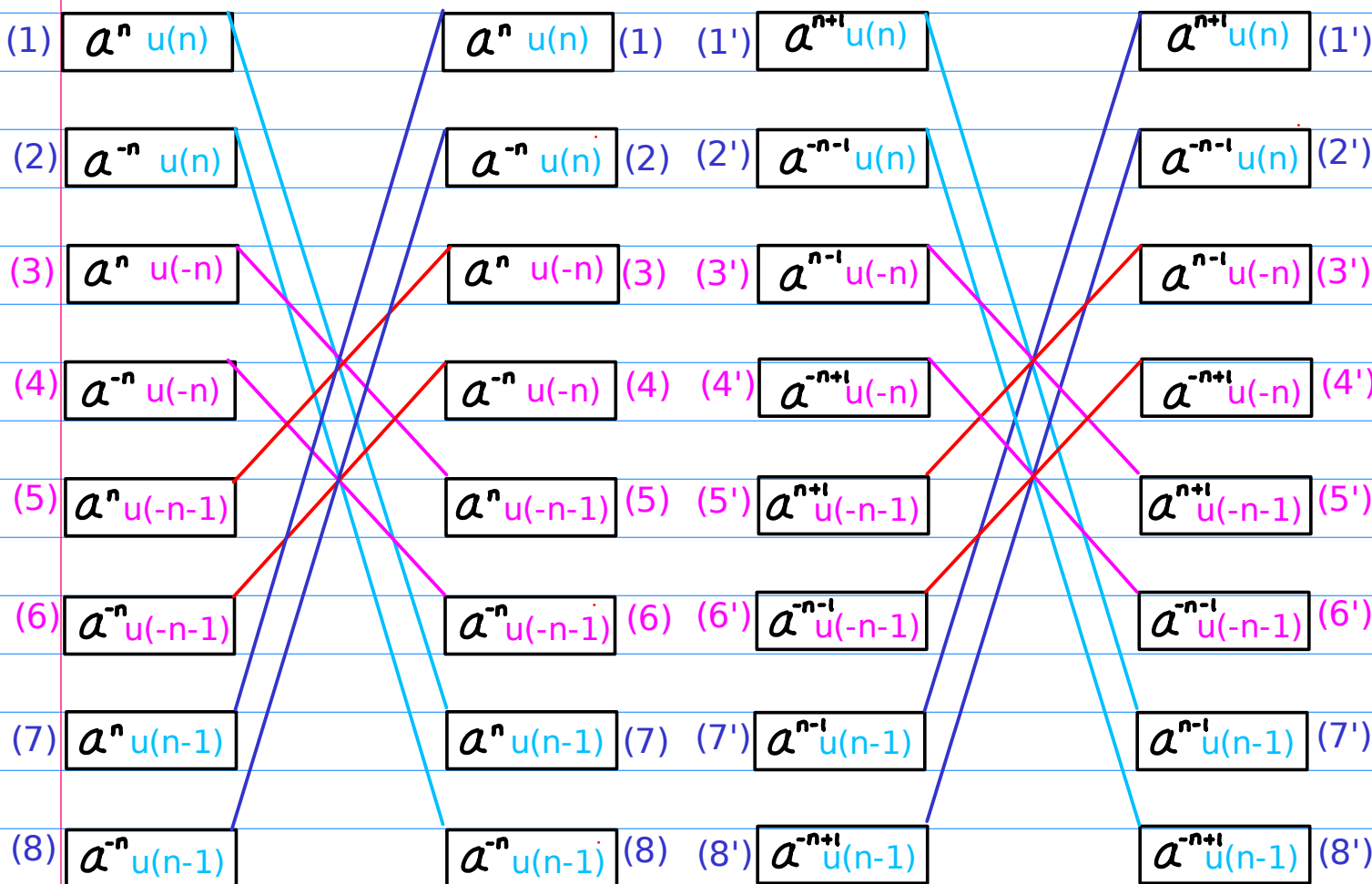
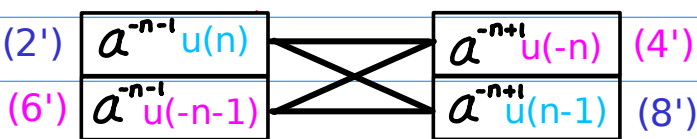
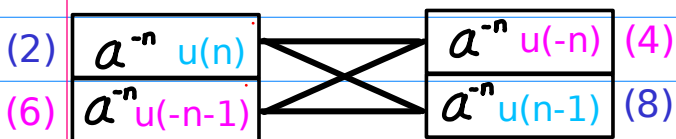
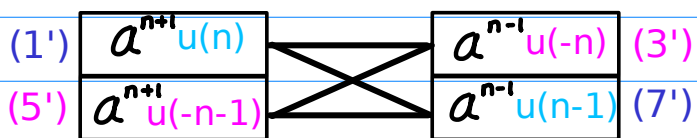
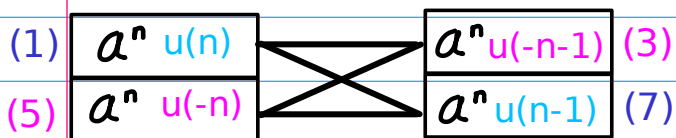


- (1) - (4)
- (2) - (3)
- (3) - (2)
- (4) - (1)
- (5) - (8)
- (6) - (7)
- (7) - (6)
- (8) - (5)

- (1') - (4')
- (2') - (3')
- (3') - (2')
- (4') - (1')
- (5') - (8')
- (6') - (7')
- (7') - (6')
- (8') - (5')

**B.I Range Shifting  
Range Flipping  
Range Complementing**

**E.I Shifting2  
Shifted Range Flipping  
Range Complementing**

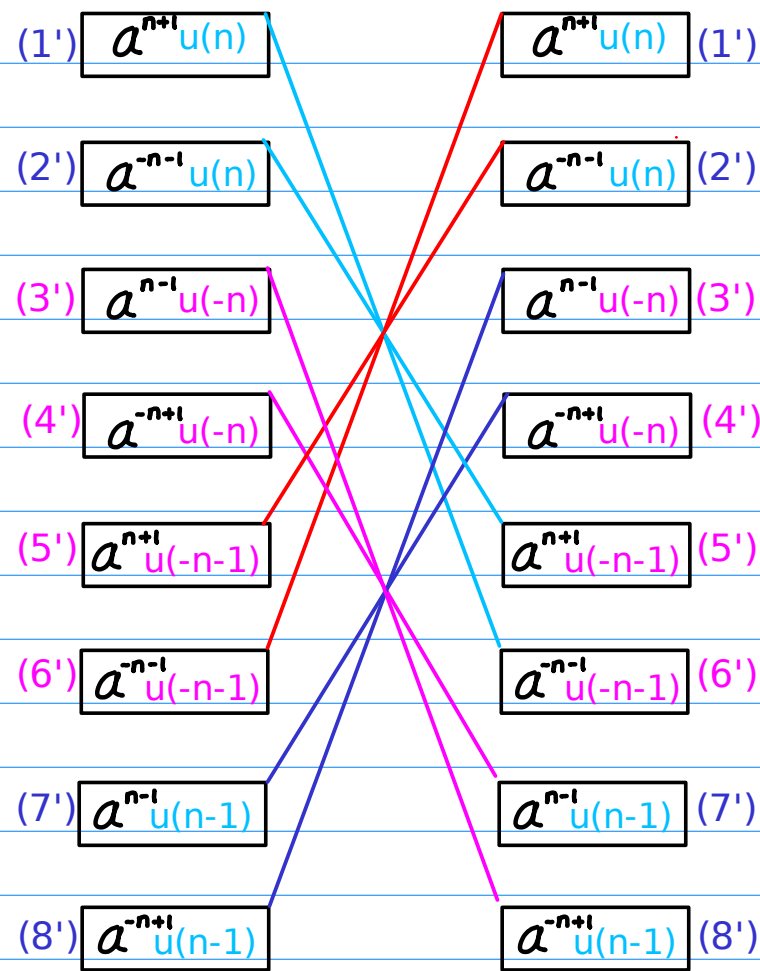
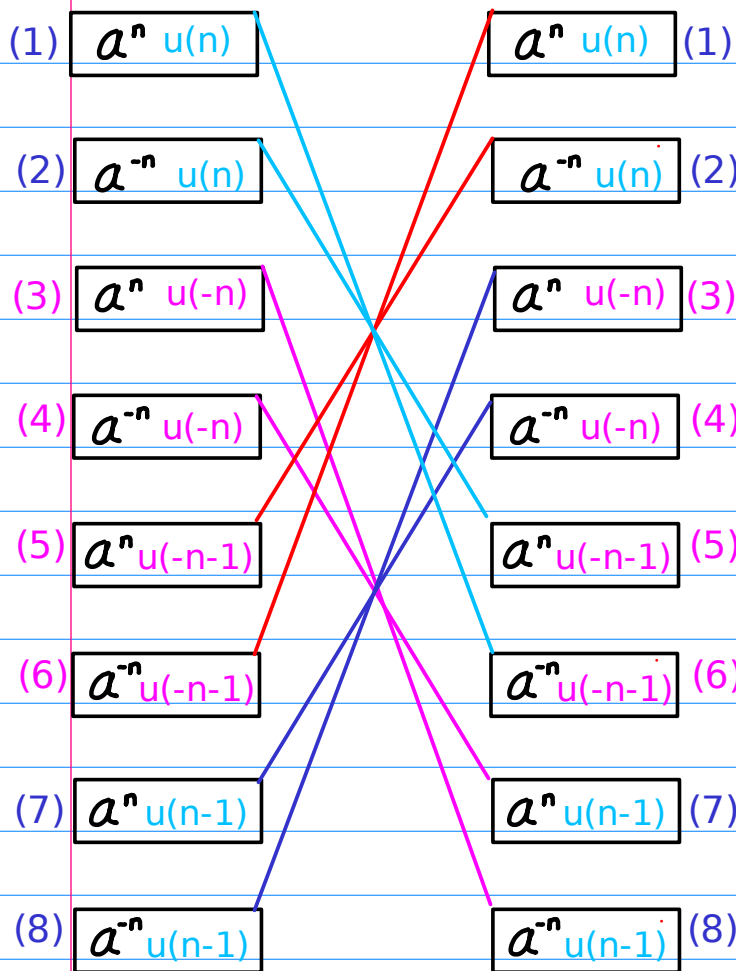
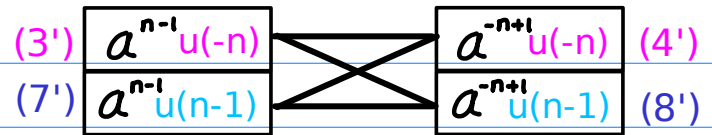
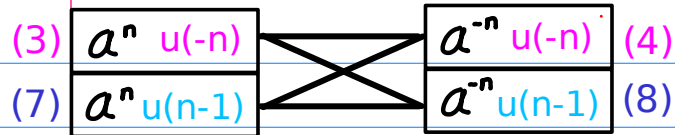
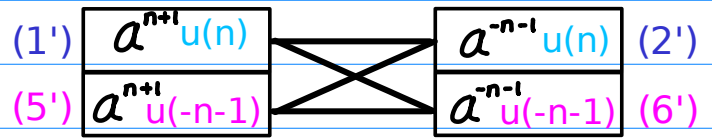
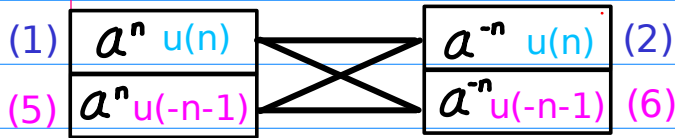


(1) - (7)      (5) - (3)  
 (2) - (8)      (6) - (4)  
 (3) - (5)      (7) - (1)  
 (4) - (6)      (8) - (2)

(1') - (7')      (5') - (3')  
 (2') - (8')      (6') - (4')  
 (3') - (5')      (7') - (1')  
 (4') - (6')      (8') - (2')

### C.I Complementary Inverting Base Inverting Range Complementing

### F.I Complementary Inverting Base Inverting Range Complementing



(1) - (6)      (5) - (2)  
 (6) - (1)      (6) - (1)  
 (2) - (5)      (7) - (4)  
 (5) - (2)      (8) - (3)

(1') - (6')      (5') - (2')  
 (6') - (1')      (6') - (1')  
 (2') - (5')      (7') - (4')  
 (5') - (2')      (8') - (3')

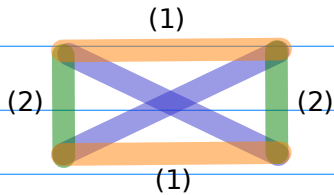


### A.I Flipping

- (1) Base Inverting
- (2) Range Flipping

$$\begin{array}{|c|} \hline (1) \ a^n u(n) \\ \hline (3) \ a^n u(-n) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(n) \quad (2) \\ \hline a^{-n} u(-n) \quad (4) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (5) \ a^n u(-n-1) \\ \hline (7) \ a^n u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(-n-1) \quad (6) \\ \hline a^{-n} u(n-1) \quad (8) \\ \hline \end{array}$$

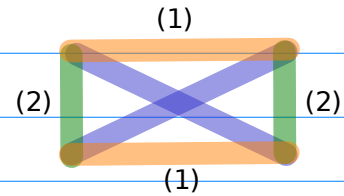


### D.I Flipping2

- (1) Base Inverting
- (2) Shifted Range Flipping

$$\begin{array}{|c|} \hline (1') \ a^{n+1} u(n) \\ \hline (3') \ a^{n+1} u(-n) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n-1} u(n) \quad (2') \\ \hline a^{-n+1} u(-n) \quad (4') \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (5') \ a^{n+1} u(-n-1) \\ \hline (7') \ a^{n-1} u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n-1} u(-n-1) \quad (6') \\ \hline a^{-n+1} u(n-1) \quad (8') \\ \hline \end{array}$$



$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & R(-n) \\ a^n R(n) & \longleftrightarrow & a^{-n} R(-n) \end{array}$$

$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ a^n R(n) & \xleftrightarrow{(2)} & a^{sh2(n)} R(-n) \\ a^n R(n) & \longleftrightarrow & a^{-sh2(n)} R(-n) \end{array}$$

$b^n$	$b^{-n}$
$a^n$	$a^{-n}$
$a^{-n}$	$a^n$

$b^n$	$b^{-sh2(n)}$
$a^{(n+1)}$	$a^{-(n-1)}$
$a^{-(n+1)}$	$a^{(n-1)}$
$a^{(n-1)}$	$a^{-(n+1)}$
$a^{-(n-1)}$	$a^{(n+1)}$

$R(n)$	$R(-n)$
$u(n)$	$u(-n)$
$u(n-1)$	$u(-n-1)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

$R(n)$	$R(-n)$
$u(n)$	$u(-n)$
$u(n-1)$	$u(-n-1)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

## B.I Range Shifting

(1) Range Complementing  
(2) Range Flipping

## E.I Shifting2

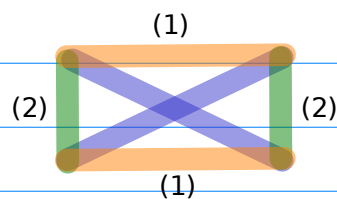
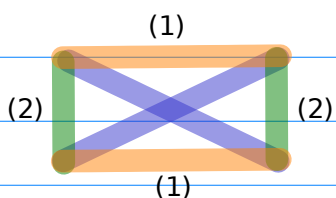
(1) Shifted Range Flipping  
(2) Range Complementing

$$\begin{array}{l} (1) \\ (5) \end{array} \begin{array}{|c|} \hline a^n u(n) \\ \hline a^n u(-n) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \quad \diagup \\ \hline \end{array} \begin{array}{|c|} \hline a^n u(-n-1) \\ \hline a^n u(n-1) \\ \hline \end{array} \begin{array}{l} (3) \\ (7) \end{array}$$

$$\begin{array}{l} (1') \\ (5') \end{array} \begin{array}{|c|} \hline a^{n+1} u(n) \\ \hline a^{n+1} u(-n-1) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \quad \diagup \\ \hline \end{array} \begin{array}{|c|} \hline a^{n+1} u(-n) \\ \hline a^{n+1} u(n-1) \\ \hline \end{array} \begin{array}{l} (3') \\ (7') \end{array}$$

$$\begin{array}{l} (2) \\ (6) \end{array} \begin{array}{|c|} \hline a^{-n} u(n) \\ \hline a^{-n} u(-n-1) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \quad \diagup \\ \hline \end{array} \begin{array}{|c|} \hline a^{-n} u(-n) \\ \hline a^{-n} u(n-1) \\ \hline \end{array} \begin{array}{l} (4) \\ (8) \end{array}$$

$$\begin{array}{l} (2') \\ (6') \end{array} \begin{array}{|c|} \hline a^{-n-1} u(n) \\ \hline a^{-n-1} u(-n-1) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \quad \diagup \\ \hline \end{array} \begin{array}{|c|} \hline a^{-n-1} u(-n) \\ \hline a^{-n-1} u(n-1) \\ \hline \end{array} \begin{array}{l} (4') \\ (8') \end{array}$$



$$\begin{array}{l} R(n) \xleftrightarrow{(1)} \overline{R(-n)} \\ R(n) \xleftrightarrow{(2)} \overline{R(n)} \\ R(n) \xleftrightarrow{\quad} \overline{R(-n)} \end{array}$$

$$\begin{array}{l} a^n R(n) \xleftrightarrow{(1)} \overline{a^{sh2(n)} R(-n)} \\ R(n) \xleftrightarrow{(2)} \overline{R(n)} \\ a^n R(n) \xleftrightarrow{\quad} \overline{a^{sh2(n)} R(-n)} \end{array}$$

$b^n$	$b^{sh2(n)}$
$a^{(n+1)}$	$a^{(n-1)}$
$a^{-(n+1)}$	$a^{-(n-1)}$
$a^{(n-1)}$	$a^{(n+1)}$
$a^{-(n-1)}$	$a^{-(n+1)}$

$R(n)$	$\overline{R(-n)}$
$u(n)$	$u(n-1)$
$u(n-1)$	$u(n)$
$u(-n)$	$u(-n-1)$
$u(-n-1)$	$u(-n)$

$R(n)$	$\overline{R(-n)}$
$u(n)$	$u(n-1)$
$u(n-1)$	$u(n)$
$u(-n)$	$u(-n-1)$
$u(-n-1)$	$u(-n)$



### C.I Complementary Inverting

(1) Base Inverting  
(2) Range Complementing

### F.I Complementary Inverting

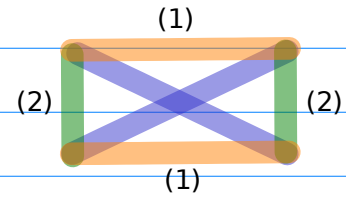
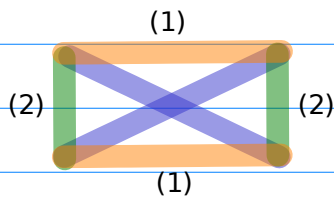
(1) Base Inverting  
(2) Range Complementing

$$\begin{array}{|c|} \hline (1) \quad a^n u(n) \\ \hline (5) \quad a^n u(-n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(n) \quad (2) \\ \hline a^{-n} u(-n-1) \quad (6) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (1') \quad a^{n+1} u(n) \\ \hline (5') \quad a^{n+1} u(-n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n-1} u(n) \quad (2') \\ \hline a^{-n-1} u(-n-1) \quad (6') \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (3) \quad a^n u(-n) \\ \hline (7) \quad a^n u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(-n) \quad (4) \\ \hline a^{-n} u(n-1) \quad (8) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (3') \quad a^{n-1} u(-n) \\ \hline (7') \quad a^{n-1} u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n+1} u(-n) \quad (4') \\ \hline a^{-n+1} u(n-1) \quad (8') \\ \hline \end{array}$$



$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\ a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)} \end{array}$$

$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\ a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)} \end{array}$$

$b^n$	$b^{-n}$
$a^n$	$a^{-n}$
$a^{-n}$	$a^n$

$b^n$	$b^{-n}$
$a^{(n+1)}$	$a^{-(n+1)}$
$a^{-(n+1)}$	$a^{(n+1)}$
$a^{(n-1)}$	$a^{-(n-1)}$
$a^{-(n-1)}$	$a^{(n-1)}$

$R(n)$	$\overline{R(n)}$
$u(n)$	$u(-n-1)$
$u(n-1)$	$u(-n)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

$R(n)$	$\overline{R(n)}$
$u(n)$	$u(-n-1)$
$u(n-1)$	$u(-n)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

## A.I Flipping

- (1) Base Inverting
- (2) Range Flipping

$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & R(-n) \\
 a^n R(n) & \longleftrightarrow & a^{-n} R(-n)
 \end{array}$$

## D.I Flipping2

- (1) Base Inverting
- (2) Shifted Range Flipping

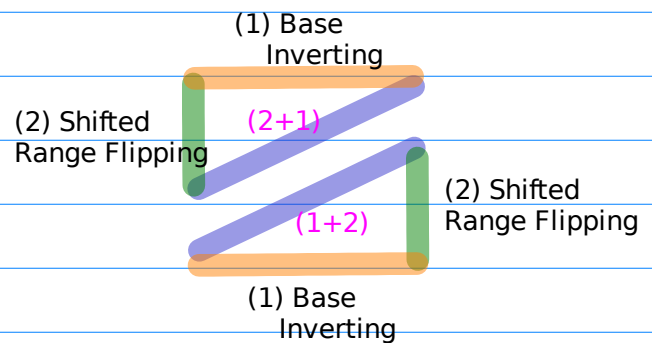
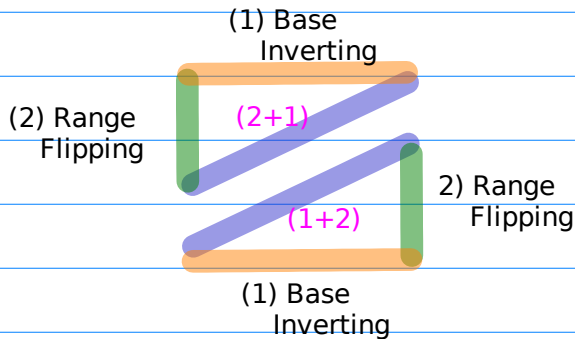
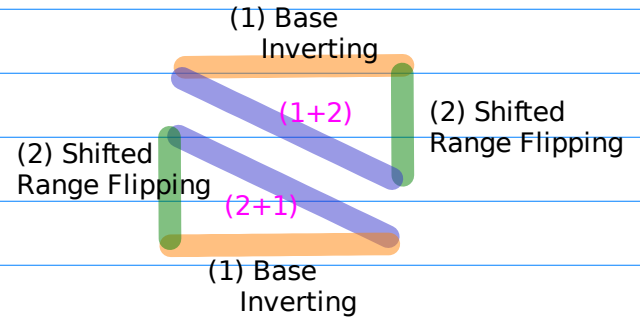
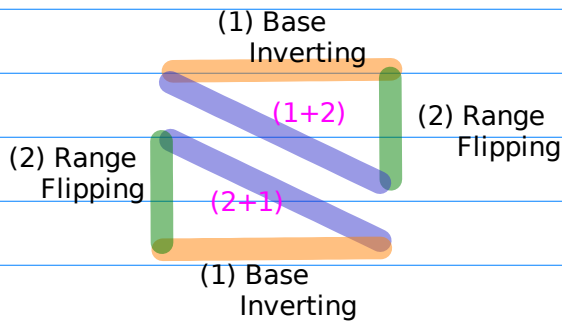
$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 a^n R(n) & \xleftrightarrow{(2)} & a^{sh2(n)} R(-n) \\
 a^n R(n) & \longleftrightarrow & a^{-sh2(n)} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-sh2(n)} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n R(-n) \\
 & \xrightarrow{(1)} & a^{-n} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^{sh2(n)} R(-n) \\
 & \xrightarrow{(1)} & a^{-sh2(n)} R(-n)
 \end{array}$$



**B.I Range Shifting**  
**(1) Range Complementing**  
**(2) Range Flipping**

**E.I Shifting2**  
**(1) Shifted Range Flipping**  
**(2) Range Complementing**

$$R(n) \xleftrightarrow{(1)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

$$a^n R(n) \xleftrightarrow{(1)} a^{sh2(n)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$

$$\xrightarrow{(2)} a^{-n} \overline{R(n)}$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$

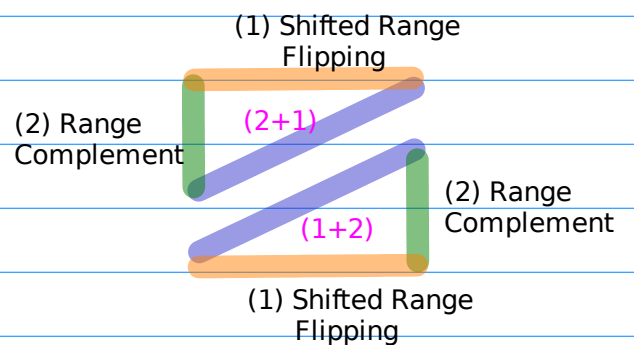
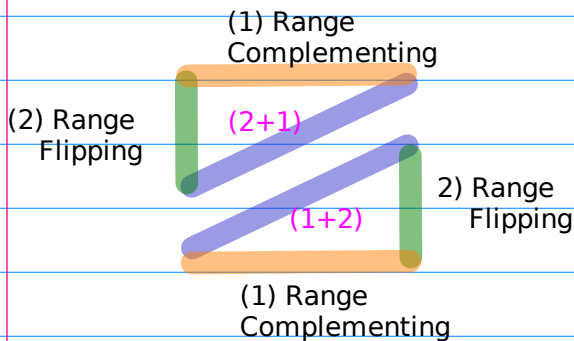
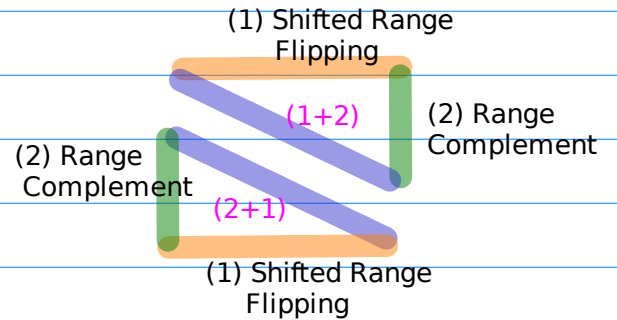
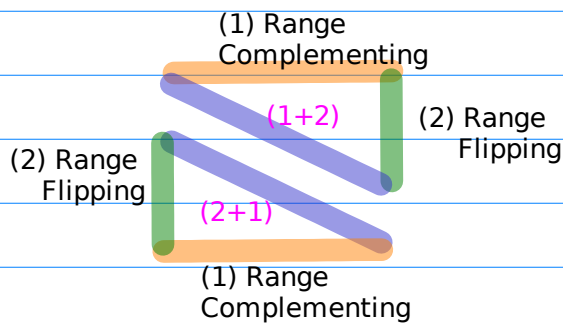
$$\xrightarrow{(2)} a^{-sh2(n)} R(-n)$$

$$a^n \text{rng}(n) \xrightarrow{(2)} a^n \overline{R(n)}$$

$$\xrightarrow{(1)} a^{-n} \overline{R(n)}$$

$$a^n R(n) \xrightarrow{(2)} a^{sh2(n)} R(-n)$$

$$\xrightarrow{(1)} a^{-sh2(n)} R(-n)$$



## C.I Complementary Inverting

(1) Base Inverting  
(2) Range Complementing

## F.I Complementary Inverting

(1) Base Inverting  
(2) Range Complementing

$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\
 a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)}
 \end{array}$$

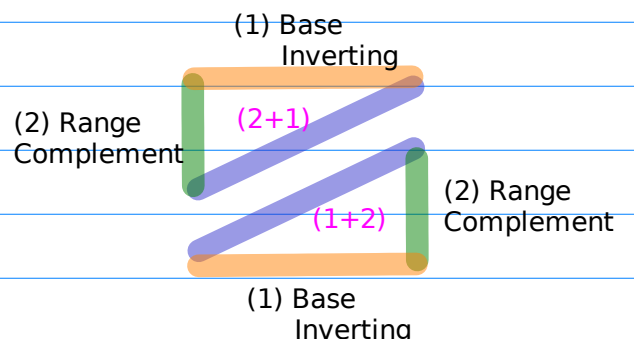
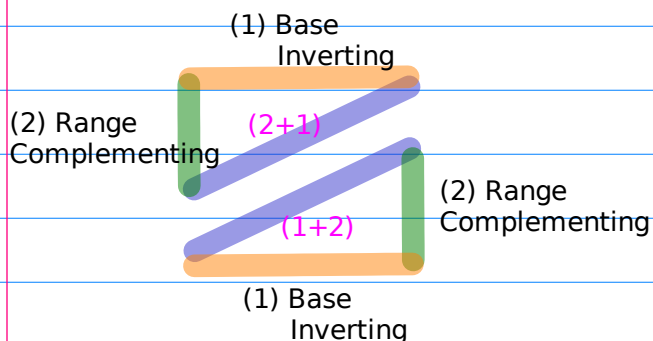
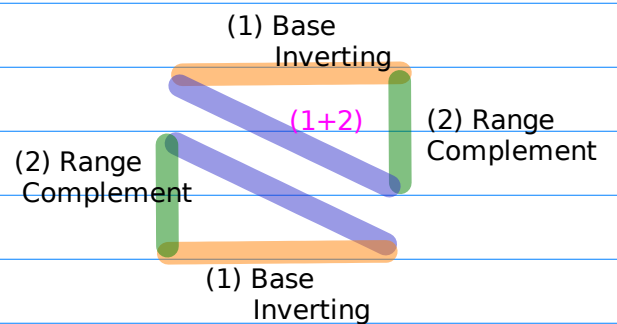
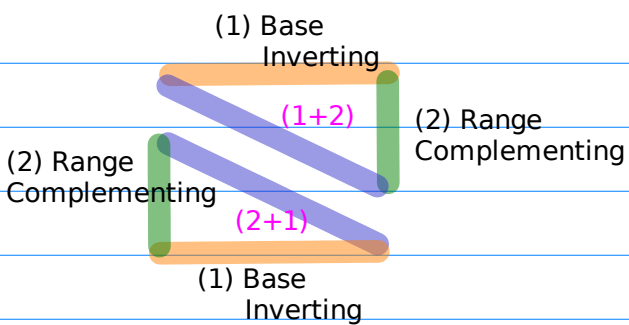
$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\
 a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n \overline{R(n)} \\
 & \xrightarrow{(1)} & a^{-n} \overline{R(n)}
 \end{array}$$

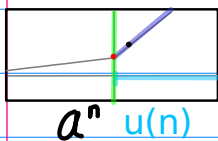
$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n \overline{R(n)} \\
 & \xrightarrow{(1)} & a^{-n} \overline{R(n)}
 \end{array}$$



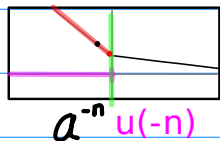


## A.II Flipping Base Inverting Range Flipping

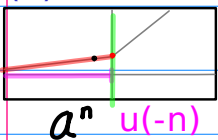
(1) 0000



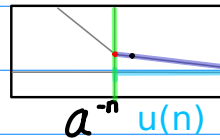
(4) 0011



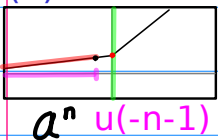
(3) 0010



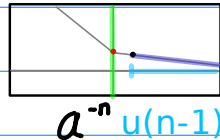
(2) 0001



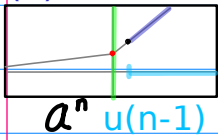
(5) 0100



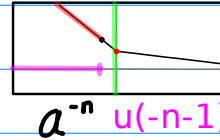
(8) 0111



(7) 0110

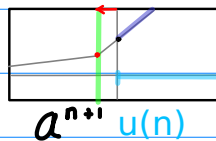


(6) 0101

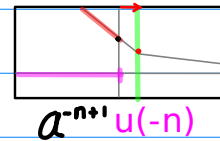


## D.II Flipping2 Base Inverting Shifted Range Flipping

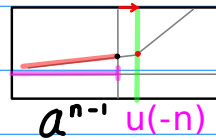
(1') 1000



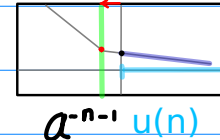
(4') 1011



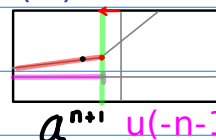
(3') 1010



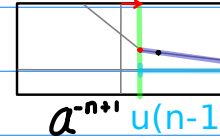
(2') 1001



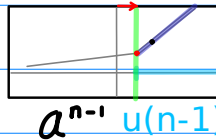
(5') 1100



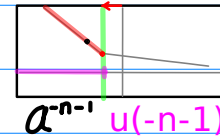
(8') 1111



(7') 1110



(6') 1101



Shifted Range Flipping  
= Exponent Shifting2  
+ Range Flipping

$$a^n R(n) \longleftrightarrow a^{-n} R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-sh2(n)} R(-n)$$

$$a^n \longleftrightarrow a^{-n}$$

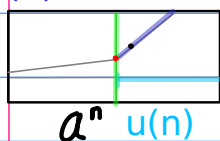
$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow R(-n)$$

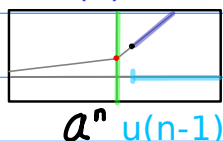
$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

## B.II Range Shifting Range Flipping Range Complementing

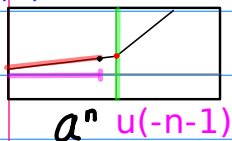
(1) 0000



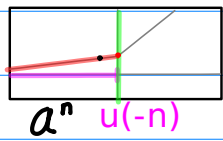
(7) 0110



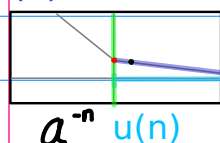
(5) 0100



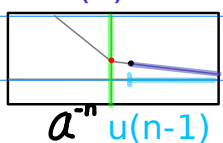
(3) 0010



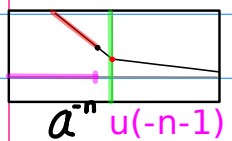
(2) 0001



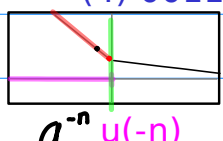
(8) 0111



(6) 0101



(4) 0011



Range Shifting  
= Range Flipping  
+ Range Complementing

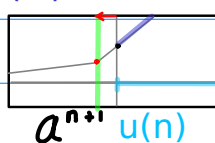
$$R(n) \longleftrightarrow \overline{R(-n)}$$

$$R(n) \longleftrightarrow R(-n)$$

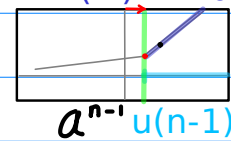
$$R(n) \longleftrightarrow \overline{R(n)}$$

## E.II Shifting2 Shifted Range Flipping Range Complementing

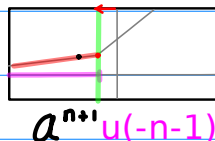
(1') 1000



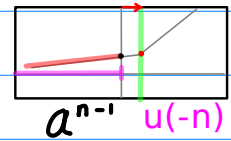
(7') 1110



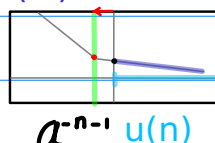
(5') 1100



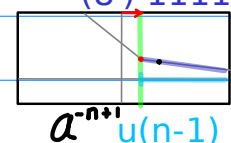
(3') 1010



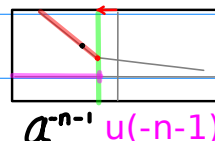
(2') 1001



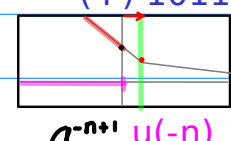
(8') 1111



(6') 1101



(4') 1011



Shifted Range Flipping  
= Exponent Shifting2  
+ Range Flipping

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

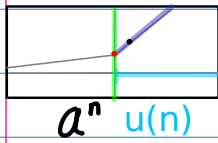
$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

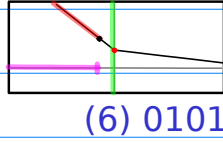
## C.II Complementary Inverting Base Inverting Range Complementing

## F.II Complementary Inverting Base Inverting Range Complementing

(1) 0000

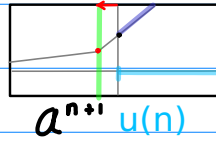


$a^{-n} u(-n-1)$



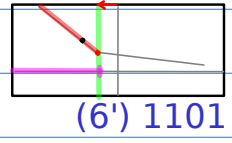
(6) 0101

(1') 1000



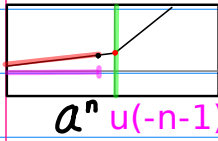
$a^{n+1} u(n)$

$a^{-n-1} u(-n-1)$



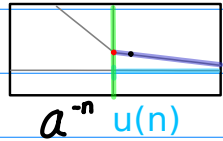
(6') 1101

(5) 0100



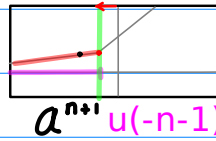
$a^n u(-n-1)$

(2) 0001



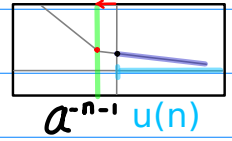
$a^{-n} u(n)$

(5') 1100



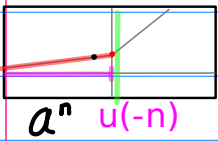
$a^{n+1} u(-n-1)$

(2') 1001



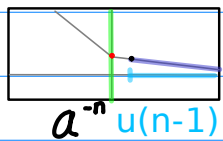
$a^{-n-1} u(n)$

(3) 0010



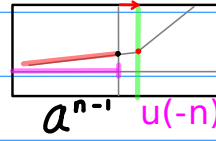
$a^n u(-n)$

(8) 0111



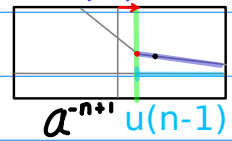
$a^{-n} u(n-1)$

(3') 1010



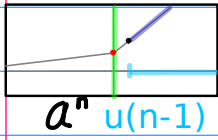
$a^{n-1} u(-n)$

(8') 1111



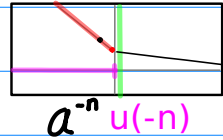
$a^{-n+1} u(n-1)$

(7) 0110



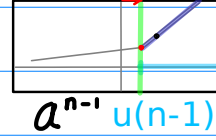
$a^n u(n-1)$

(4) 0011



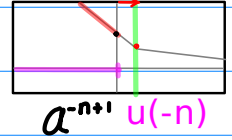
$a^{-n} u(-n)$

(7') 1110



$a^{n-1} u(n-1)$

(4') 1011



$a^{-n+1} u(-n)$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$







