Random Process Background

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Outline

Space (1)

- A **space** consists of selected mathematical objects that are treated as points, and selected relationships between these points.
  - the *nature* of the points can vary widely: for example, the points can be
    - elements of a set
    - functions on another space
    - subspaces of another space
  - It is the relationships between points that define the *nature* of the **space**.

https://en.wikipedia.org/wiki/Space_(mathematics)

Space (2)

- *modern mathematics* uses many types of **spaces**, such as
  - Euclidean spaces
  - linear spaces
  - topological spaces
  - Hilbert spaces
  - probability spaces
- *modern mathematics* does not define the notion of **space** itself.

https://en.wikipedia.org/wiki/Space_(mathematics)

Space (3)

- a **space** is
  a set (or a universe) with some added features
- it is not always clear
  whether a given mathematical object should be considered
  as a geometric **space**, or an algebraic **structure**
- a general definition of **structure** embraces
  all common types of **space**

https://en.wikipedia.org/wiki/Space_(mathematics)

Mathematical objects (1)

- a **mathematical object** is
  an abstract concept arising in mathematics.
- an **mathematical object** is anything
  that has been (or could be) formally defined,
  and with which one may do
  - deductive reasoning
  - mathematical proofs

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (2)

- typically, a **mathematical object**
  - can be a value that can be assigned to a variable
  - therefore can be involved in formulas

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (3)

- commonly encountered **mathematical objects** include
  - numbers
  - sets
  - functions
  - expressions
  - geometric objects
  - transformations of other mathematical objects
  - spaces

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (4)

- **Mathematical objects** can be very *complex*;
  - for example, the followings are considered as **mathematical objects** in **proof theory**.
    - theorems
    - proofs
    - theories

https://en.wikipedia.org/wiki/Mathematical_object

Structure (1)

- a **structure** is a set
  endowed with some *additional* features on the set
  - an *operation*
  - *relation*
  - *metric*
  - *topology*
- often, the *additional* features are
  attached or related to the set,
  so as to provide it with some
  additional *meaning* or *significance*.

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

Structure (2)

- A partial list of possible **structures** are
  - measures
  - algebraic structures (groups, fields, etc.)
  - topologies
  - metric structures (geometries)
  - orders
  - events
  - equivalence relations
  - differential structures
  - categories.

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

Mathematical space (1)

- A **mathematical space** is, informally,
  a collection of mathematical objects
  under consideration.

- The **universe** of mathematical objects within a **space**
  are *precisely* defined entities
  whose rules of *interaction* come baked
  into the rules of the **space**.

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

Mathematical space (2)

- A **space** differs from a **mathematical set** in several important ways:
  - A **mathematical set** is also a collection of objects
  - but these objects are being pulled from a **space** (or **universe**) of objects where the rules and definitions have already been agreed upon

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

Mathematical space (3)

- A **space** differs from a **mathematical set** in several important ways:
  - a **mathematical set** has no internal structure,
  - a **space** usually has some internal structure.

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

Mathematical space (4)

- having some internal structure
  could mean a variety of things,
  but typically it involves
  - *interactions* and *relationships*
    between elements of the **space**
  - *rules* on how to *create* and *define*
    new elements of the **space**

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

Measurable space (1)

- A **measurable space** is
  any space with a sigma-algebra
  which can then be equipped with a measure
  - collection of subsets of the space
    following certain rules with a way
    to assign sizes to those sets.

https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have

Measurable space (2)

- Intuitively,
  certain sets belonging to a measurable space
  can be given a size in a *consistent way*.

  consistent way means that certain axioms are met:
  - the empty set is given a size of zero
  - if a measurable set is contained inside another one, then its size is less than or equal to the size of the containing set
  - the size of a disjoint union of sets is the sum of the individual sets' sizes

https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have

Probability space

- A **probability space** is simply
  a measurable space equipped with a probability measure.
- A probability measure has the special property
  of giving the entire **space** a size of **1**.
  - this then implies that the size
    of any disjoint union of sets
    (the sum of the **sizes** of the sets)
    in the **probability space**
    is less than or equal to **1**

https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have

Euclidean space definition (1)

- A subset U of the **Euclidean n-space** $\mathbb{R}^n$ is open

  if, for every point x in U,
  there exists a positive real number ε
  (depending on x)

  such that any point in $\mathbb{R}^n$
  whose Euclidean distance from x is smaller than ε

  belongs to U

https://en.wikipedia.org/wiki/Open_set

Euclidean space definition (2)

- Equivalently, a subset U of $\mathbb{R}^n$ is open

  if every point in U is
  the center of an open ball contained in U

- An example of a subset of $\mathbb{R}$ that is not open is

  the closed interval $[0, 1]$,
  since neither $0 - \varepsilon$ nor $1 + \varepsilon$ belongs to $[0, 1]$
  for any $\varepsilon > 0$, no matter how small.

  https://en.wikipedia.org/wiki/Open_set

Metric space definition (1)

- A subset U of a **metric space** (M, d) is called open

  if, for any point x in U, there exists a real number $\varepsilon > 0$
  such that any point $y \in M$ satisfying $d(x, y) < \varepsilon$ belongs to U.

- Equivalently, U is open
  if every point in U
  has a neighborhood contained in U.

- This generalizes the **Euclidean space** example,
  since **Euclidean space** with the **Euclidean distance**
  is a **metric space**.

https://en.wikipedia.org/wiki/Open_set

Metric space definition (2)

- formally, a **metric space** is an ordered pair (M, d)
  where M is a set and
  d is a metric on M,
  i.e., a function

$$d : M \times M \to \mathbb{R}$$

  satisfying the following axioms for all points $x, y, z \in M$:
  - $d(x,x) = 0$.
  - if $x \neq y$, then $d(x,y) > 0$.
  - $d(x,y) = d(y,x)$.
  - $d(x,z) \leq d(x,y) + d(y,z)$.

https://en.wikipedia.org/wiki/Open_set

Metric space definition (3)

- satisfying the following axioms for all points $x, y, z \in M$:
  - the distance from a point *to itself* is zero:
  - (Positivity) the distance between two distinct points is always positive:
  - (Symmetry) the distance from x to y is always the same as the distance from y to x:
  - (Triangle inequality) you can arrive at z from x by taking a detour through y, but this will not make your journey any faster than the shortest path.
- If the metric d is *unambiguous*, one often refers by abuse of notation to "the **metric space** M".

https://en.wikipedia.org/wiki/Open_set

Outline

Topology (1)

- **topology**
  from the Greek words
  τόπος, 'place, location',
  and λόγος, 'study'

https://en.wikipedia.org/wiki/Topology

Topology (2)

- **topology** is concerned with the *properties* of a geometric object that are *preserved*
  - under continuous deformations such as
    - stretching
    - twisting
    - crumpling
    - bending
  - that is, *without*
    - closing holes
    - opening holes
    - tearing
    - gluing
    - passing through itself

https://en.wikipedia.org/wiki/Topology

Topological space (1)

- a **topological space** is, roughly speaking,

  a geometrical space
  in which closeness is defined

  but cannot necessarily be measured
  by a numeric distance.

https://en.wikipedia.org/wiki/Borel_set

Topological space (2)

- More specifically, a **topological space** is
  - a set whose elements are called points,
  - along with an additional structure called a topology,
- which can be defined as
  - a set of neighbourhoods for each point
  - that satisfy some axioms
    formalizing the concept of closeness.

https://en.wikipedia.org/wiki/Borel_set

Topological space (3)

- There are several *equivalent* definitions of a **topology**,

  the most commonly used of which is
  the definition through open sets,

  which is easier than the others to manipulate.

https://en.wikipedia.org/wiki/Borel_set

Topological space (4)

- A **topological space** is
  the most general type of a mathematical space
  that allows for the definition of
  - limits
  - continuity
  - connectedness
- Although very general,
  the concept of **topological spaces** is fundamental,
  and used in virtually every branch of modern mathematics.
- The study of **topological spaces** in their own right
  is called point-set topology or general topology.

https://en.wikipedia.org/wiki/Topological_space

Topological space (5)

- Common types of **topological spaces** include
  - Euclidean spaces : a set of points satisfying certain relationships, expressible in terms of distance and angles.
  - metric spaces : a set together with a notion of distance between points. The distance is measured by a function called a metric or distance function.
  - manifolds : a topological space that *locally* resembles Euclidean space near each point. More precisely, an n-manifold is a topological space with the property that each point has a neighborhood that is homeomorphic to an open subset of n-dimensional Euclidean space.

https://en.wikipedia.org/wiki/Topological_space

Topological space definition (1-1)

- A topology τ on a set X is

  a set of subsets of X with the *properties* below.

  a topology τ on a set X : a set of subsets of X
  members of τ : subsets of X
- each member of τ is called an open set.
- X together with τ is called a **topological space**

https://en.wikipedia.org/wiki/Open_set

Topological space definition (1-2)

- a topology τ on a set X is
  a set of subsets of X with the *properties* below.
  Each member of τ is called an open set.[
  - $X \in \tau$ and $\varnothing \in \tau$
  - any union of sets in τ belong to τ :
    any union of subsets of X belong to τ :
    if $\{U_i : i \in I\} \subseteq \tau$ then
    $$\bigcup_{i \in I} U_i \in \tau$$
  - any finite intersection of sets in τ belong to τ
    any finite intersection of subsets of X belong to τ :
    if $U_1, \ldots, U_n \in \tau$ then
    $$U_1 \cap \cdots \cap U_n \in \tau$$

Topological space definition (2)

- Infinite intersections of open sets need not be open.
- For example, the intersection of all intervals of the form $(-1/n, 1/n)$, where n is a positive integer, is the set $\{0\}$ which is not open in the real line.
- A metric space is a topological space, whose topology consists of the collection of all subsets that are unions of open balls.
- There are, however, topological spaces that are not metric spaces.

https://en.wikipedia.org/wiki/Open_set

Topological space via open sets (1)

- A **topology** on a set X may be defined as a collection τ of subsets of X, called open sets and satisfying the following axioms:
  - The empty set and X itself belong to τ .
  - Any arbitrary (finite or infinite) union of members of τ belongs to τ .
  - The intersection of any finite number of members of τ belongs to τ .

https://en.wikipedia.org/wiki/Topological_space

Topological space via open sets (2)

- As this definition of a topology is the most commonly used, the set τ of the open sets is commonly called a **topology** on X.
- A subset $C \subseteq X$ is said to be closed in (X, τ) if its complement $X \setminus C$ is an open set.

https://en.wikipedia.org/wiki/Topological_space

Examples of topoloy (1)

- Given $X = \{1, 2, 3, 4\}$,
  the trivial or indiscrete topology on X is
  the family $\tau = \{\{\}, \{1, 2, 3, 4\}\} = \{\varnothing, X\}$
  consisting of only the two subsets of X
  required by the axioms
  forms a topology of X.
- Given $X = \{1, 2, 3, 4\}$,
  the family $\tau = \{\{\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$
  $= \{\varnothing, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, X\}$
  of six subsets of X forms another topology of X.

https://en.wikipedia.org/wiki/Topological_space

Examples of topoloy (2)

- Given $X = \{1, 2, 3, 4\}$,
  the discrete topology on X is
  the power set of X, which is the family $\tau = \wp(X)$
  consisting of all possible subsets of X.
  In this case the topological space (X, τ)
  is called a discrete space.
- Given $X = \mathbb{Z}$, the set of integers,
  the family τ of all finite subsets
  of the integers plus $\mathbb{Z}$ itself
  is not a topology,
  because (for example) the union of all finite sets
  not containing zero is not finite
  but is also not all of $\mathbb{Z}$, and so it cannot be in τ .

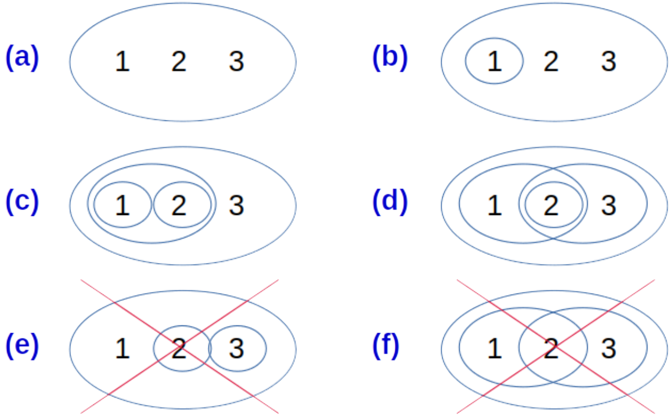
https://en.wikipedia.org/wiki/Topological_space

Examples of topoloy (3)

- Let τ be denoted with the circles, here are four examples **(a)**, **(b)**, **(c)**, **(d)**, and two non-examples **(e)**, **(f)** of topologies on the three-point set $\{1, 2, 3\}$.
- **(e)** is not a topology because the union of $\{2\}$ and $\{3\}$ [i.e. $\{2, 3\}$] is missing;
- **(f)** is not a topology because the intersection of $\{1, 2\}$ and $\{2, 3\}$ [i.e. $\{2\}$], is missing.

https://en.wikipedia.org/wiki/Topological_space

Examples of topoloy (4)



Topological space via neighborhoods (1)

- This axiomatization is due to Felix Hausdorff.
- Let X be a set;
- the elements of X are usually called points, though they can be any mathematical object.
- We allow X to be empty.

https://en.wikipedia.org/wiki/Topological_space

Topological space via neighborhoods (2)

- Let $\mathcal{N}$ be a function assigning to each x (point) in X a non-empty collection $\mathcal{N}(x)$ of subsets of X.
- The elements of $\mathcal{N}(x)$ will be called neighbourhoods of x with respect to $\mathcal{N}$ (or, simply, neighbourhoods of x).
- The function $\mathcal{N}$ is called a neighbourhood topology if *the axioms* below are satisfied; and
- then X with $\mathcal{N}$ is called a topological space.

https://en.wikipedia.org/wiki/Topological_space

Topological space via neighborhoods (3)

- If N is a neighbourhood of x (i.e., $N \in \mathcal{N}(x)$), then $x \in N$. In other words, each point belongs to every one of its neighbourhoods.
- If N is a subset of X and includes a neighbourhood of x, then N is a neighbourhood of x. I.e., every superset of a neighbourhood of a point $x \in X$ is again a neighbourhood of x.
- The intersection of two neighbourhoods of x x is a neighbourhood of x.
- Any neighbourhood $\mathcal{N}$ of x includes a neighbourhood $\mathcal{M}$ of x such that $\mathcal{N}$ is a neighbourhood of each point of M.

https://en.wikipedia.org/wiki/Topological_space

Topological space via neighborhoods (4)

- The first three axioms for neighbourhoods have a clear meaning.
- The fourth axiom has a very important use in the structure of the theory, that of linking together the neighbourhoods of different points of X.
- A standard example of such a system of neighbourhoods is for the real line $\mathbb{R}$, where a subset N of $\mathbb{R}$ is defined to be a neighbourhood of a real number x if it includes an open interval containing x.

https://en.wikipedia.org/wiki/Topological_space

Topological space via neighborhoods (3)

- Given such a structure, a subset U of X is defined to be **open** if U is a neighbourhood of all points in U.
- The **open sets** then satisfy the axioms given below.
- Conversely, when given the **open sets** of a topological space, the neighbourhoods satisfying the above axioms can be recovered by defining N to be a neighbourhood of x if N includes an open set U such that $x \in U$.

https://en.wikipedia.org/wiki/Topological_space

Definitions via closed sets

- Using **de Morgan's laws**,
  the above axioms defining **open sets**
  become axioms defining **closed sets**:
- The empty set and X are **closed**.
  - The intersection of any collection of **closed sets**
    s also **closed**.
  - The union of any finite number of **closed sets**
    is also **closed**.
- Using these axioms, another way to define a **topological space**
  is as a set X together with a collection τ of **closed subsets** of X.
  Thus the **sets** in the topology τ are the **closed sets**,
  and their complements in X are the **open sets**.

https://en.wikipedia.org/wiki/Open_set

Homeomorphism (1)

- a **homeomorphism**
  (from Greek ὅμοιος (homoios) 'similar, same',
  and μορφή (morphē) 'shape, form',
  named by Henri Poincaré), **topological isomorphism**,
  or **bicontinuous function** is
  a bijective and continuous function
  between topological spaces
  that has a continuous inverse function.

https://en.wikipedia.org/wiki/Homeomorphism

Homeomorphism (2)

- **Homeomorphisms** are the isomorphisms in the category of topological spaces – the mappings that preserve all the topological properties of a given space.
- Two spaces with a **homeomorphism** between them are called homeomorphic, and from a topological viewpoint they are the same.

https://en.wikipedia.org/wiki/Homeomorphism

Homeomorphism (3)

- Very roughly speaking,
  a **topological space** is a geometric object,
  and the **homeomorphism** is
  a *continuous* stretching and bending
  of the object into a *new* shape.

https://en.wikipedia.org/wiki/Homeomorphism

Homeomorphism (4)

- Thus, a *square* and a *circle* are homeomorphic to each other, but a *sphere* and a *torus* are not.
- However, this description can be misleading.
- Some *continuous deformations* are not homeomorphisms, such as the *deformation* of a *line* into a *point*.
- Some homeomorphisms are not *continuous deformations*, such as the homeomorphism between a *trefoil knot* and a *circle*.

https://en.wikipedia.org/wiki/Homeomorphism

Outline

Sigma algebra (1)

- We term the **structures** which allow us to use **measure** to be **sigma algebras**
- the only requirements for **sigma algebras** (on a **set** X) are:
    - the $\{\}$ and X are in the **set**.
    - if A is in the **set**, $complement(A)$ is in the **set**.
    - for any **sets** E_i in the set, $\bigcup_i E_i$ is in the **set** (for countable i).

https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7

Sigma algebra (2)

- The most intuitive way to think about a **sigma algebra** is that it is the kind of **structure** we can do probability on.
  - for example, we can assign ratios of areas and length, so the measure on such a set X tells something about the probability of its subsets.
  - we can find the probability of subsets A and B because we know their ratios with respect to a set X ;
  - we also know that
    - (the measure of) their complements are defined, and
    - their unions and intersections are defined,
    - so we know how to find the probability of things in this set X.

https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7

Sigma algebra (3)

- The **sigma algebra** which contains the standard topology on R (that is, *all* open sets on R) is called the **Borel Sigma Algebra**, and the elements of this set are called **Borel sets**.
- What this gives us, is the set of sets on which outer measure gives our list of dreams. That is, if we take a **Borel set** and we check that length follows translation, additivity, and interval length, it will always hold.

https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5ee0ee2e7

Sigma algebra (4)

- The set of Lebesgue measurable sets is
  the set of **Borel sets**,
  along with (union) all the sets
  which differ from a Borel set
  by a set of measure 0.
- More intuitively, it is all the sets
  we can normally measure,
  plus a bunch of stuff
  that doesn't affect our ideas of area or volume
  (think about the border of the circle above).

https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7

Borel Sets (1-1)

- a **Borel set** is any set in a topological space
  that can be formed from open sets
  (or, equivalently, from closed sets)
  through the operations of
  - countable union,
  - countable intersection, and
  - relative complement.

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (1-2)

- For a **topological space X**,
the collection of all Borel sets on X forms a σ-algebra,
known as the **Borel algebra** or **Borel σ-algebra**.
- The B**orel algebra** on **X** is the smallest **σ-algebra**
containing all open sets (or, equivalently, all closed sets).

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (1-3)

- **Borel sets** are important in measure theory,
  since any measure defined on the open sets of a space,
  or on the closed sets of a space,
  must also be defined on all Borel sets of that space.
- Any measure defined on the Borel sets
  is called a **Borel measure**.
- **Borel sets** and the associated **Borel hierarchy** also play a fundamental role in descriptive set theory.

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (2)

- **Borel sets** are those obtained from intervals by means of the operations allowed in a **σ-algebra**. So we may construct them in a (transfinite) "sequence" of steps:
- . . . And again and again.

https://math.stackexchange.com/questions/220248/understanding-borel-sets

Borel Sets (3-1)

1. Start with finite unions of closed-open intervals.
   These sets are completely elementary, and they form an algebra.
2. Adjoin countable unions and intersections of elementary sets.
   What you get already includes open sets and closed sets, intersections of an open set and a closed set, and so on.
   Thus you obtain an algebra, that is still not a σ-algebra.

https://math.stackexchange.com/questions/220248/understanding-borel-sets

Borel Sets (3)

3. Again, adjoin countable unions and intersections to 2.
   Observe that you get a strictly larger class, since a countable intersection of countable unions of intervals is not necessarily included in 2.
   Explicit examples of sets in 3 but not in 2 include F_σ sets, like, say, the set of *rational numbers*.
4. And do the same again.

https://math.stackexchange.com/questions/220248/understanding-borel-sets

Borel Sets (4-1)

- And even after a sequence of steps we are not yet finished. Take, say, a countable union of a set constructed at step 1, a set constructed at step 2, and so on. This union may very well not have been constructed at any step yet. By axioms of σ-algebra, you should include it as well - if you want, as step ∞

https://math.stackexchange.com/questions/220248/understanding-borel-sets

Borel Sets (4-2)

- (or, technically, the first infinite ordinal, if you know what that means).
- And then continue in the same way until you reach the first uncountable ordinal. And only then will you finally obtain the generated σ -algebra.

https://math.stackexchange.com/questions/220248/understanding-borel-sets

Outline

Open set (1)

- an **open set** is a generalization of an **open interval** in the real line.
- a **metric space** is a set along with a distance defined between any two points
- in a **metric space**,
  an **open set** is a set that, along with every point P, contains all points that are sufficiently near to P
  - all points whose distance to P is less than some value depending on P

https://en.wikipedia.org/wiki/Open_set

Open set (2)

- More generally, an **open set** is
  a member of a given collection of subsets of a given set,
  a collection that has the property of containing
  - every union of its members
  - every finite intersection of its members
  - the empty set
  - the whole set itself

https://en.wikipedia.org/wiki/Open_set

Open set (2)

- A set in which such a collection is given is called a **topological space**, and the collection is called a **topology**.
- These conditions are very loose, and allow enormous flexibility in the choice of open sets.
- For example,
  - every subset can be open (the discrete topology), or
  - no subset can be open (the indiscrete topology) except
    - the space itself and
    - the empty set .

https://en.wikipedia.org/wiki/Open_set

Open set (3)

Example:

- The *circle* represents the set of points (x, y) satisfying $x^2 + y^2 = r^2$.
- The *disk* represents the set of points (x, y) satisfying $x^2 + y^2 < r^2$.
- The *circle* set is an **open set**,
- the *disk* set is its **boundary set**, and
- the union of the *circle* and *disk* sets is a **closed set**.

https://en.wikipedia.org/wiki/Open_set

Open set (4)

- A **set** is a collection of distinct objects.
- Given a **set** A, we say that a is an **element** of A if a is one of the distinct objects in A, and we write $a \in A$ to denote this
- Given two **sets** A and B, we say that A is a **subset** of B if every element of A is also an element of B write $A \subseteq B$ to denote this.

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Open set (5) Open Balls

- We give these definitions in general,
  for when one is working in $\mathbb{R}^n$
  since they are really not all that different
  to define in $\mathbb{R}^n$ than in $\mathbb{R}^2$
- An **open ball** $B_r(\boldsymbol{a})$ in $\mathbb{R}^n$
  centered at $\boldsymbol{a} = (a_1, \ldots a_n) \in \mathbb{R}^n$ with radius r
  is the set of all points $\boldsymbol{x} = (x_1, \ldots x_n) \in \mathbb{R}^n$
  such that the distance between x and $\boldsymbol{a}$ is less than r
- In $\mathbb{R}^2$ an **open ball** is often called an **open disk**

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Open set (6) Interior points

- Suppose that $S \subseteq \mathbb{R}^n$.
- A **point** $\boldsymbol{p} \in S$ is an **interior point** of S if there exists an open ball $B_r(\boldsymbol{p}) \subseteq S$.
- Intuitively, $\boldsymbol{p}$ is an **interior point** of S if we can *squeeze* an entire open ball centered at $\boldsymbol{p}$ within S

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Open set (7) Boundary points

- A point $\boldsymbol{p} \in \mathbb{R}^n$ is a **boundary point** of S if all open balls centered at $\boldsymbol{p}$ contain both points in S and points not in S .
- The **boundary** of S is the set ∂S that consists of all of the **boundary points** of S.

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Open set (8) Open and Closed Sets

- A set $O \subseteq \mathbb{R}^n$ is **open**
  if every point in O is an interior point.
- A set $C \subseteq \mathbb{R}^n$is **closed**
  if it contains all of its boundary points.

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Open set (8) Bounded and Unbounded

- A set S is **bounded** if there is an open ball $B_M(0)$ such that

$$S \subseteq B.$$

  - intuitively, this means that we can enclose all of the set S within a large enough ball centered at the origin.
- A set that is not **bounded** is called **unbounded**

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd(

Topologically distinguishable points

- Intuitively, an open set provides a *method* to *distinguish* two points.
- two points in a topological space,
  there exists an open set
  - containing one point but
  - not containing the other (distinct) point
  - the two points are **topologically distinguishable**.

https://en.wikipedia.org/wiki/Open_set

Metric spaces

- In this manner, one may speak of whether two points, or more generally two subsets, of a topological space are "**near**" without concretely defining a distance.
- Therefore, **topological spaces** may be seen as a generalization of **spaces** equipped with a notion of distance, which are called **metric spaces**.

https://en.wikipedia.org/wiki/Open_set

The set of all real numbers

- In the set of all real numbers,
  one has the natural Euclidean metric; that is,
  a function which *measures* the distance
  between two real numbers: $d(x, y) = |x - y|$.

https://en.wikipedia.org/wiki/Open_set

All points close to a real number x

- Therefore, given a real number x, one can speak of the set of all points **close** to that real number x; that is, **within** ε of x.
- In essence, points **within** ε of x ***approximate*** x to an accuracy of degree ε.
- Note that $\varepsilon > 0$ always,
  but as ε becomes *smaller* and *smaller*,
  one obtains points that approximate x
  to a *higher* and *higher* degree of accuracy.

https://en.wikipedia.org/wiki/Open_set

The points within ε of x

- For example, if $x = 0$ and $\varepsilon = 1$,
  the points within ε of x are precisely
  the points of the interval $(-1, 1)$;
- However, with $\varepsilon = 0.5$,
  the points within ε of x are precisely
  the points of $(-0.5, 0.5)$.
- Clearly, these points approximate x
  to a *greater* degree of accuracy than when $\varepsilon = 1$.

https://en.wikipedia.org/wiki/Open_set

without a concrete Euclidean metric

- The previous examples shows,
  for the case $x = 0$, that one may **approximate** x
  to *higher* and *higher* degrees of accuracy
  by defining ε to be *smaller* and *smaller*.
- In particular, sets of the form $(-\varepsilon, \varepsilon)$
  give us a lot of information
  about points **close** to $x = 0$.
- Thus, rather than speaking of a concrete Euclidean metric,
  one may *use* **sets** to *describe* points **close** to x.

https://en.wikipedia.org/wiki/Open_set

Different collections of sets containing 0

- This innovative idea has far-reaching consequences; in particular, by defining

  different collections of sets containing 0
  (distinct from the sets $(-\varepsilon, \varepsilon)$),
  one may find different results
  regarding the distance
  between 0 and other real numbers.

https://en.wikipedia.org/wiki/Open_set

A set for measuring distance

- For example, if we were to define $\boldsymbol{R}$ as the *only* such set for "*measuring distance*", all points are close to 0
- since there is only one possible degree of accuracy one may achieve in **approximating** 0: being a member of $\boldsymbol{R}$.

https://en.wikipedia.org/wiki/Open_set

The measure as a binary condition

- Thus, we find that in some sense, every real number is distance 0 away from 0.
- It may help in this case to think of the measure as being a binary condition:
  - all things in $\boldsymbol{R}$ are equally close to 0,
  - while any item that is not in R is not close to 0.

https://en.wikipedia.org/wiki/Open_set

Family of sets (1)

- a **collection** F of subsets of a given set S is called
  a **family** of **subsets** of S, or
  a **family** of **sets** over S.
- More generally,
  a **collection** of any sets whatsoever is called
  a **family** of **sets**,
  set family, or
  a **set system**

https://en.wikipedia.org/wiki/Family_of_sets

Family of sets (2)

- The term "**collection**" is used here because,
  - in some contexts,
    a **family** of **sets** may be allowed
    to contain repeated copies of any given member, and
  - in other contexts
    it may form a proper class rather than a set.

https://en.wikipedia.org/wiki/Family_of_sets

Family of sets – examples

- The set of all subsets of a given set S is called the **power set** of S and is denoted by $\wp(S)$.
  The **power set** $\wp(S)$ of a given set S is a **family** of **sets** over S.
- A subset of S having k elements is called a k-subset of S.
  The k-subset $S^{(k)}$ of a set S form a **family** of **sets**.
- Let $S = \{a, b, c, 1, 2\}$. An example of a **family** of **sets** over S (in the multiset sense) is given by $F = \{A_1, A_2, A_3, A_4\}$, where $A_1 = \{a, b, c\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2\}$, and $A_4 = \{a, b, 1\}$.

https://en.wikipedia.org/wiki/Family_of_sets

Filter

- a **filter** on a set X is a family $\mathscr{B}$ of subsets such that:
- $X \in \mathscr{B}$ and $\emptyset \notin \mathscr{B}$ if $A \in \mathscr{B}$ and $B \in \mathscr{B}$,
  then $A \cap B \in \mathscr{B}$
  If $A, B \subset X, A \in \mathscr{B}$, and $A \subset B$,
  then $B \in \mathscr{B}$
- A **filter** on a set may be thought of
  as representing a "collection of large subsets",
  one intuitive example being the **neighborhood filter**.
- **Filters** appear in order theory, model theory, and set theory,
  but can also be found in topology, from which they originate.
  The dual notion of a **filter** is an ideal.

https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

Neighbourhood basis (1)

- A **neighbourhood basis** or **local basis** (or **neighbourhood base** or **local base**) for a point x is a filter base of the neighbourhood filter;
- this means that it is a subset $\mathscr{B} \subseteq \mathscr{N}(x)$ such that for all $V \in \mathscr{N}(x)$, there exists some $B \in \mathscr{B}$ such that $B \subseteq V$. That is, for any **neighbourhood** V we can find a **neighbourhood** B in the **neighbourhood basis** that is contained in V.

https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis

Neighbourhood basis (2)

- Equivalently, $\mathscr{B}$ is a local basis at x if and only if the neighbourhood filter $\mathscr{N}$ can be recovered from $\mathscr{B}$ in the sense that the following equality holds:

$$\mathscr{N}(x) = \{V \subseteq X \ : \ B \subseteq V \text{ for some } B \in \mathscr{B}\}$$

- A family $\mathscr{B} \subseteq \mathscr{N}(x)$ is a neighbourhood basis for x if and only if $\mathscr{B}$ is a cofinal subset of $(\mathscr{N}(x), \supseteq)$ with respect to the partial order $\supseteq$ (importantly, this partial order is the superset relation and not the subset relation).

https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis

A collection of sets around x

- In general, one refers to the family of sets containing 0, used to **approximate** 0, as a neighborhood basis;
- a member of this neighborhood basis is referred to as an **open set**.
- In fact, one may generalize these notions to an arbitrary set (X); rather than just the real numbers.
- In this case, given a point (x) of that set (X), one may define a **collection** of sets "around" (that is, containing) x, used to **approximate** x.

https://en.wikipedia.org/wiki/Open_set

Smaller sets containing x

- Of course, this collection would have to *satisfy* certain properties (known as axioms) for otherwise we may not have a *well-defined method* to measure distance.
- For example, every point in X should **approximate** x to some degree of accuracy.
- Thus X should be in this family.
- Once we begin to define "smaller" sets containing x, we tend to **approximate** x to a greater degree of accuracy.
- Bearing this in mind, one may define the remaining axioms that the family of sets about x is required to satisfy.

https://en.wikipedia.org/wiki/Open_set

Open ball (1)

- a **ball** is the solid figure bounded by a **sphere**; it is also called a **solid sphere**.
  - a **closed ball** includes the *boundary points* that constitute the sphere
  - an **open ball** excludes them

https://en.wikipedia.org/wiki/Ball_(mathematics)

Open ball (2)

- A **ball** in n dimensions is called a **hyperball** or **n-ball** and is bounded by a **hypersphere** or $(\boldsymbol{n}-1)$**-sphere**
- One may talk about **balls** in any topological space X, not necessarily induced by a metric.
- An n-dimensional **topological ball** of X is any subset of X which is homeomorphic to an **Euclidean n-ball**.

https://en.wikipedia.org/wiki/Ball_(mathematics)

Outline

Class (1)

- a **class** is a collection of sets
  (or sometimes other mathematical objects)
  that can be unambiguously defined
  by a property that all its members share.
- **Classes** act as a way to have set-like collections
  while differing from sets so as to avoid **Russell's paradox**

https://en.wikipedia.org/wiki/Class_(set_theory)

Class (2)

- A **class** *that is not a set* is called a **proper class**, and
- a **class** *that is a set* is sometimes called a **small class**.
- the followings are **proper classes** in many formal systems
  - the **class** of all sets
  - the **class** of all ordinal numbers
  - the **class** of all cardinal numbers

https://en.wikipedia.org/wiki/Class_(set_theory)

Class (3)

- consider "the set of all sets with property X."
- especially when dealing with categories, since the objects of a concrete category are all sets with certain additional structure.
- However, **if** we're not *careful* about this we can get into serious *trouble* –

https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects

Class (4)

- let X be the set of all sets which do not contain *themselves*
- Since X is a set, we can ask whether X is an element of *itself*.
- But then we run into a paradox – **if** X contains *itself* as an element, **then** it does not, and vice versa.

https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects

Class (5)

- In order to avoid this paradox,
  we cannot consider the collection of all sets
  to be itself a set.
- This means we have to *throw out*
  the whole "the set of all sets with property X" construction.
  But we wanted that.
- So the way we get around it is to say that
  there's something called a **class**,
  which is like a set but not a set.

https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects

Class (6)

- Then we can talk about
  "the **class** X of all sets with property Y."
- Since X is not a set,
  it can't be an element of itself, and we're fine.
- Of course, if we need to talk about the collection of all classes, we need to create another name that goes another step back, and so forth.

https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects

Class Examples (1)

- The **collection** of all algebraic structures of a given type will usually be a **proper class**.
  (a **class** *that is* <u>*not*</u> *a set* is called a **proper class**)
  - the **class** of all groups
  - the **class** of all vector spaces
  - and many others.
- Within set theory, many **collections** of sets turn out to be proper classes.

https://en.wikipedia.org/wiki/Class_(set_theory)

Class Examples (2)

- One way to *prove* that a **class** is **proper** is to place it in bijection with the class of all ordinal numbers.
  - **Cardinal numbers** indicate an amount how many of something we have: one, two, three, four, five.
  - **Ordinal numbers** indicate position in a series: first, second, third, fourth, fifth.

https://en.wikipedia.org/wiki/Class_(set_theory)
https://editarians.com/cardinals-ordinals/

Class Paradoxes (1)

- The **paradoxes** of naive set theory can be explained in terms of the *inconsistent tacit assumption* that "all classes are sets".
- These paradoxes do not arise with classes because there is no notion of classes containing classes.
- Otherwise, one could, for example, define a class of all classes that do not contain themselves, which would lead to a Russell paradox for classes.

https://en.wikipedia.org/wiki/Class_(set_theory)

Class Paradoxes (2)

- With a rigorous foundation,
  these **paradoxes** instead *suggest proofs*
  that certain classes are proper (i.e., that they are not sets).
  - **Russell's paradox** *suggests* a *proof*
    that the class of all sets
    which do not contain themselves is proper
  - the **Burali-Forti paradox** *suggests*
    that the class of all ordinal numbers is proper.

https://en.wikipedia.org/wiki/Class_(set_theory)

Russell's Paradox (1)

- According to the unrestricted comprehension principle,
  for any sufficiently well-defined property,
  there is the set of all
  and only the objects that have that property.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (2)

- Let R be the set of all sets ($R = \{x \mid x \notin x\}$) that are not members of themselves ($R \notin R$).
  - *if* R is not a member of itself ($R \notin R$), *then* its definition (the set of all sets) entails that it is a member of itself ($R \in R$);
  - yet, *if* it is a member of itself ($R \in R$), *then* it is not a member of itself ($R \notin R$), since it is the set of all sets that are not members of themselves ($R \notin R$)
- the resulting contradiction is **Russell's paradox**.
- Let $R = \{x \mid x \notin x\}$, then $R \in R \Longleftrightarrow R \notin R$

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (3)

- *Most* *sets* commonly encountered are not members of themselves.
- For example, consider the set of all squares in a plane.
- This set is not itself a square in the plane, thus it is not a member of itself.
- Let us call a set "**normal**" if it is not a member of itself, and "**abnormal**" if it is a member of itself.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (4)

- Clearly *every* set must be either normal or abnormal.
- The set of squares in the plane is normal.
- In contrast, the complementary set
that contains everything which is not a square in the plane
is itself not a square in the plane,
and so it is one of its own members
and is therefore abnormal.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (5)

- Now we consider the set of all normal sets, R, and try to determine whether R is normal or abnormal.
  - ***If*** R were normal, it would be contained in the set of all normal sets (itself), and therefore be abnormal;
  - on the other hand ***if*** R were abnormal, it would not be contained in the set of all normal sets (itself), and therefore be normal.
- This leads to the conclusion that R is neither normal nor abnormal: **Russell's paradox.**

https://en.wikipedia.org/wiki/Russell%27s_paradox

Stochastic Process (1)

In probability theory and related fields,
a **stochastic** (/stoʊˈkæstɪk/) or **random** process is
a mathematical object usually defined as
a family of **random variables**.

The word stochastic in English was originally used as an adjective with the definition "pertaining to **conjecturing**", and stemming from a Greek word meaning "to aim at a mark, guess", and the Oxford English Dictionary gives the year 1662 as its earliest occurrence.

From Ancient Greek στοχαστικός (stokhastikós), from στοχάζομαι (stokházomai, "aim at a target, guess"), from στόχος (stókhos, "an aim, a guess").

https://en.wikipedia.org/wiki/Stochastic
https://en.wiktionary.org/wiki/stochastic

Stochastic Process (2)

The definition of a **stochastic process** varies,
but a **stochastic process** is *traditionally* defined as
a collection of **random variables** indexed by some set.

The terms **random process** and **stochastic process**
are considered synonyms and are used interchangeably,
without the **index set** being precisely specified.

Both "**collection**", or "**family**" are used
while instead of "**index set**", sometimes
the terms "**parameter set**" or "**parameter space**" are used.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process (3)

The term **random function** is also used to refer to
a **stochastic** or **random process**,
though sometimes it is only used
when the stochastic process takes real values.

This term is also used when the **index sets** are
mathematical spaces other than the **real line**,

while the terms **stochastic process** and **random process**
are usually used when the **index set** is interpreted as time,

and other terms are used such as **random field**
when the **index set** is n-dimensional **Euclidean space** $\mathbb{R}^n$
or a manifold

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process (4)

A **stochastic process** can be denoted,
by $\{X(t)\}_{t \in T}$, $\{X_t\}_{t \in T}$, $\{X(t)\}$, $\{X_t\}$ or simply as X or $X(t)$,
although $X(t)$ is regarded as an abuse of function notation.

For example, $X(t)$ or X_t are used to refer to
the **random variable** with the **index** t,
and not the entire **stochastic process**.

If the **index set** is $T = [0, \infty)$, then one can write,
for example, $(X_t, t \geq 0)$ to denote the **stochastic process**.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (1)

A **stochastic process** is defined as
a collection of **random variables**
defined on a common **probability space** $(\Omega, \mathscr{F}, P)$,

- Ω is a **sample space**,
- $\mathscr{F}$ is a σ **-algebra**,
- P is a **probability measure**;
- the **random variables**, indexed by some set T,
- all take values in the same **mathematical space** S, which must be **measurable** with respect to some σ -algebra Σ

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (2)

In other words, for a given **probability space** $(\Omega, \mathscr{F}, P)$
and a **measurable space** (S, Σ),
a **stochastic process** is a
collection of S-valued **random variables**,
which can be written as:

$$\{X(t) : t \in T\}.$$

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (3)

Historically, in many problems from the natural sciences
a point $t \in T$ had the meaning of time,
so $X(t)$ is a **random variable** representing
a value observed at time t.

A **stochastic process** can also be written as $\{X(t,\omega) : t \in T\}$
to reflect that it is actually a function of two variables, $t \in T$ and $\omega \in \Omega$.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (4)

There are other ways to consider a stochastic process,
with the above definition being considered the traditional one.

For example, a stochastic process
can be interpreted or defined as
a S^T-valued **random variable**,
where S^T is the space of all the possible functions
from the set T into the space S.

However this alternative definition as
a "**function-valued random variable**"
in general requires additional regularity assumptions
to be **well-defined**.

https://en.wikipedia.org/wiki/Stochastic_process

Index set (1)

The set T is called the **index set** or **parameter set** of the **stochastic process**.

Often this set is some subset of the real line, such as the natural numbers or an interval, giving the set T the interpretation of time.

https://en.wikipedia.org/wiki/Stochastic_process

Index set (2)

In addition to these sets, the index set T can be another set with a **total order** or a more general set, such as the Cartesian plane R^2 or n-dimensional **Euclidean space**, where an element $t \in T$ can represent a point in space.

That said, many results and theorems are only possible for **stochastic processes** with a **totally ordered index set**.

https://en.wikipedia.org/wiki/Stochastic_process

State space

The **mathematical space** S of a **stochastic process** is called its **state space**.

This mathematical space can be defined using integers, real lines, n-dimensional Euclidean spaces, complex planes, or more abstract mathematical spaces.

The **state space** is defined using elements that reflect the different values that the **stochastic process** can take.

https://en.wikipedia.org/wiki/Stochastic_process

Sample function (1)

A **sample function** is a single **outcome** of a **stochastic process**, so it is formed by taking a single possible value of each **random variable** of the **stochastic process**.

More precisely, if $\{X(t,\omega) : t \in T\}$ is a **stochastic process**, then for any point $\omega \in \Omega$, the mapping $X(\cdot,\omega) : T \to S$, is called a **sample function**, a **realization**, or, particularly when T is interpreted as time, a **sample path** of the **stochastic process** $\{X(t,\omega) : t \in T\}$.

https://en.wikipedia.org/wiki/Stochastic_process

Sample function (2)

This means that for a fixed $\omega \in \Omega$,
there exists a **sample function**
that maps the **index set** T to the **state space** S.

Other names for a **sample function** of a **stochastic process**
include **trajectory**, **path function** or **path**

https://en.wikipedia.org/wiki/Stochastic_process