Logical Connectives (2A)

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List of Logical Connectives

Commonly used logical connectives include

- Negation (not): ¬ , N (prefix), ~
- Conjunction (and): ∧ , K (prefix), & , ∙
- Disjunction (or): ∨, A (prefix)
- Material implication (if...then): → , C (prefix), ⇒ , ⊃
- Biconditional (if and only if): ↔ , E (prefix), ≡ , =

Alternative names for biconditional are "iff", "xnor" and "bi-implication".

https://en.wikipedia.org/wiki/Logical_connective

Examples

For example, the meaning of the statements *it is raining* and *I am indoors* is transformed when the two are combined with logical connectives. For statement *P* = *It is raining* and *Q* = *I am indoors*:

- It is **not** raining ($\neg P$)
- It is raining **and** I am indoors ($P \wedge Q$)
- It is raining **or** I am indoors ($P \vee Q$)
- **If** it is raining, **then** I am indoors ($P \rightarrow Q$)
- **If** I am indoors, **then** it is raining ($Q \rightarrow P$)
- I am indoors **if and only if** it is raining ($P \leftrightarrow Q$)

https://en.wikipedia.org/wiki/Logical_connective

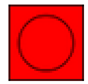
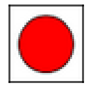
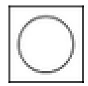
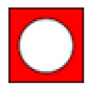
Tautology and Contradiction

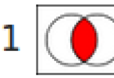
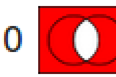
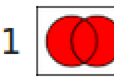
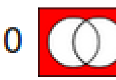
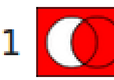
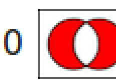
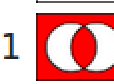
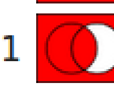
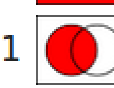
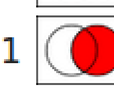
It is also common to consider the *always true* formula and the *always false* formula to be connective:

- True formula (⊤, 1, V [prefix], or T)
- False formula (⊥, 0, O [prefix], or F)

https://en.wikipedia.org/wiki/Logical_connective

Truth Table and Venn Diagram

Name / Symbol		Truth table $P =$ 0	1	Venn diagram
Truth/Tautology	⊤	1	1	
Proposition P		0	1	
False/Contradiction	⊥	0	0	
Negation	¬	1	0	

Binary connectives		$P =$ 0 0 1 1 $Q =$ 0 1 0 1	
Conjunction	∧	0 0 0 1	
Alternative denial	↑	1 1 1 0	
Disjunction	∨	0 1 1 1	
Joint denial	↓	1 0 0 0	
Material conditional	→	1 1 0 1	
Exclusive or	↮	0 1 1 0	
Biconditional	↔	1 0 0 1	
Converse implication	←	1 0 1 1	
Proposition P		0 0 1 1	
Proposition Q		0 1 0 1	

https://en.wikipedia.org/wiki/Logical_connective

Precedence

Order of precedence [edit]

As a way of reducing the number of necessary parentheses, one may introduce precedence rules: ¬ has higher precedence than ∧, ∧ higher than ∨, and ∨ higher than →. So for example, $P \vee Q \wedge \neg R \rightarrow S$ is short for $(P \vee (Q \wedge (\neg R))) \rightarrow S$.

Here is a table that shows a commonly used precedence of logical operators.[15]

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

https://en.wikipedia.org/wiki/Logical_connective

Properties

- **Associativity**: Within an expression containing two or more of the same associative connectives in a row, the order of the operations does not matter as long as the sequence of the operands is not changed.
- **Commutativity**: The operands of the connective may be swapped preserving logical equivalence to the original expression.
- **Distributivity**: A connective denoted by · distributes over another connective denoted by +, if $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all operands a, b, c.
- **Idempotence**: Whenever the operands of the operation are the same, the compound is logically equivalent to the operand.
- **Absorption**: A pair of connectives ∧, ∨ satisfies the absorption law if $a \land (a \lor b) = a$ for all operands a, b.

https://en.wikipedia.org/wiki/Logical_connective

Associativity

Truth functional connectives [edit]

Associativity is a property of some logical connectives of truth-functional propositional logic. The following logical equivalences demonstrate that associativity is a property of particular connectives. The following are truth-functional tautologies.

Associativity of disjunction:

$$((P \lor Q) \lor R) \leftrightarrow (P \lor (Q \lor R))$$
$$(P \lor (Q \lor R)) \leftrightarrow ((P \lor Q) \lor R)$$

Associativity of conjunction:

$$((P \land Q) \land R) \leftrightarrow (P \land (Q \land R))$$
$$(P \land (Q \land R)) \leftrightarrow ((P \land Q) \land R)$$

Associativity of equivalence:

$$((P \leftrightarrow Q) \leftrightarrow R) \leftrightarrow (P \leftrightarrow (Q \leftrightarrow R))$$
$$(P \leftrightarrow (Q \leftrightarrow R)) \leftrightarrow ((P \leftrightarrow Q) \leftrightarrow R)$$

https://en.wikipedia.org/wiki/Associative_property

Commutativity

Truth functional connectives [edit]

Commutativity is a property of some logical connectives of truth functional propositional logic. The following logical equivalences demonstrate that commutativity is a property of particular connectives. The following are truth-functional tautologies.

Commutativity of conjunction

$$(P \land Q) \leftrightarrow (Q \land P)$$

Commutativity of disjunction

$$(P \lor Q) \leftrightarrow (Q \lor P)$$

Commutativity of implication (also called the law of permutation)

$$(P \to (Q \to R)) \leftrightarrow (Q \to (P \to R))$$

Commutativity of equivalence (also called the complete commutative law of equivalence)

$$(P \leftrightarrow Q) \leftrightarrow (Q \leftrightarrow P)$$

https://en.wikipedia.org/wiki/Commutative_property

Distributivity (1)

Truth functional connectives [edit]

Distributivity is a property of some logical connectives of truth-functional propositional logic. The following logical equivalences demonstrate that distributivity is a property of particular connectives. The following are truth-functional tautologies.

Distribution of conjunction over conjunction

$$(P \land (Q \land R)) \leftrightarrow ((P \land Q) \land (P \land R))$$

Distribution of conjunction over disjunction

$$(P \land (Q \lor R)) \leftrightarrow ((P \land Q) \lor (P \land R))$$

Distribution of disjunction over conjunction

$$(P \lor (Q \land R)) \leftrightarrow ((P \lor Q) \land (P \lor R))$$

Distribution of disjunction over disjunction

$$(P \lor (Q \lor R)) \leftrightarrow ((P \lor Q) \lor (P \lor R))$$

https://en.wikipedia.org/wiki/Distributive_property

Distributivity (2)

Distribution of implication over equivalence

$$(P \to (Q \leftrightarrow R)) \leftrightarrow ((P \to Q) \leftrightarrow (P \to R))$$

Distribution of disjunction over equivalence

$$(P \vee (Q \leftrightarrow R)) \leftrightarrow ((P \vee Q) \leftrightarrow (P \vee R))$$

Double distribution

$$((P \wedge Q) \vee (R \wedge S)) \leftrightarrow (((P \vee R) \wedge (P \vee S)) \wedge ((Q \vee R) \wedge (Q \vee S)))$$

$$((P \vee Q) \wedge (R \vee S)) \leftrightarrow (((P \wedge R) \vee (P \wedge S)) \vee ((Q \wedge R) \vee (Q \wedge S)))$$

https://en.wikipedia.org/wiki/Distributive_property

Logical Conjunction

In logic, mathematics and linguistics, And (∧) is the truth-functional operator of **logical conjunction**; the *and* of a set of operands is true if and only if *all* of its operands are true. The logical connective that represents this operator is typically written as ∧ or · .

"*A* and *B*" is true only if *A* is true and *B* is true.

An operand of a conjunction is a **conjunct**.

Truth table [edit]

The truth table of $A \wedge B$:

INPUT		OUTPUT
A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

https://en.wikipedia.org/wiki/Distributive_property

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Truth table [edit]

The truth table of $A \land B$:

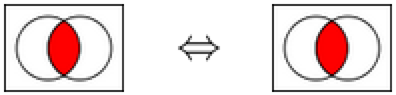
INPUT		OUTPUT
A	B	$A \land B$
T	T	T
T	F	F
F	T	F
F	F	F

https://en.wikipedia.org/wiki/Logical_conjunction

Properties of Conjuction

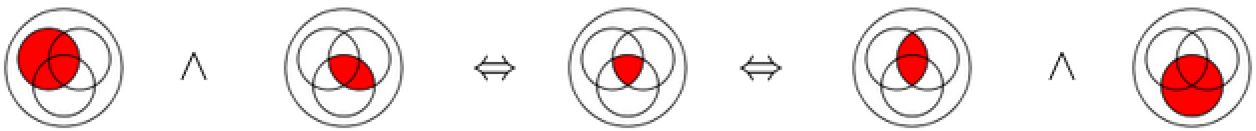
commutativity: yes

$$A \wedge B \Leftrightarrow B \wedge A$$



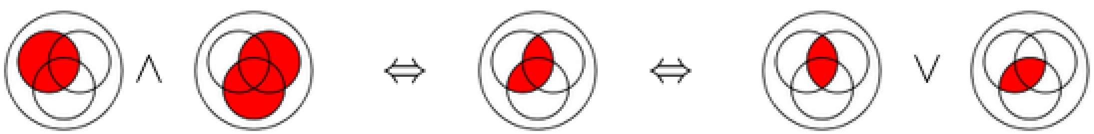
associativity: yes

$$A \wedge (B \wedge C) \Leftrightarrow (A \wedge B) \wedge C$$



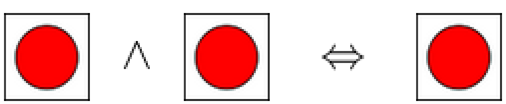
distributivity: with various operations, especially with *or*

$$A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$$



idempotency: yes

$$A \wedge A \Leftrightarrow A$$

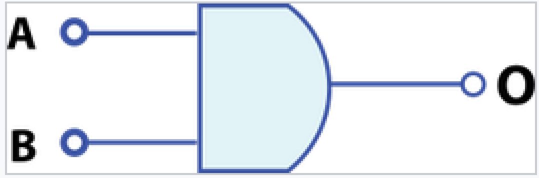


https://en.wikipedia.org/wiki/Logical_conjunction

Conjunction in boolean algebra

Applications in computer engineering [edit]

In high-level computer programming and digital electronics, logical conjunction is commonly represented by an infix operator, usually as a keyword such as "`AND`", an algebraic multiplication, or the ampersand symbol "`&`". Many languages also provide short-circuit control structures corresponding to logical conjunction.



AND logic gate

Logical conjunction is often used for bitwise operations, where `0` corresponds to false and `1` to true:

- `0 AND 0` = `0`,
- `0 AND 1` = `0`,
- `1 AND 0` = `0`,
- `1 AND 1` = `1`.

The operation can also be applied to two binary words viewed as bitstrings of equal length, by taking the bitwise AND of each pair of bits at corresponding positions. For example:

- `11000110 AND 10100011` = `10000010`.

https://en.wikipedia.org/wiki/Logical_conjunction

Disjunction

Logical disjunction is an operation on two logical values, typically the values of two propositions, that has a value of *false* if and only if both of its operands are false. More generally, a disjunction is a logical formula that can have one or more literals separated only by 'or's. A single literal is often considered to be a degenerate disjunction.

Truth table [edit]

The truth table of $A \lor B$:

INPUT		OUTPUT
A	B	$A \lor B$
T	T	T
T	F	T
F	T	T
F	F	F

https://en.wikipedia.org/wiki/Distributive_property

Properties of disjunction

- associativity: $a \lor (b \lor c) \equiv (a \lor b) \lor c$
- commutativity: $a \lor b \equiv b \lor a$
- distributivity: $(a \lor (b \land c)) \equiv ((a \lor b) \land (a \lor c))$

$$(a \lor (b \lor c)) \equiv ((a \lor b) \lor (a \lor c))$$
$$(a \lor (b \equiv c)) \equiv ((a \lor b) \equiv (a \lor c))$$

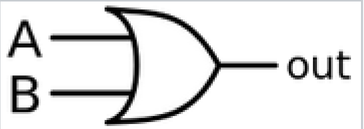
- idempotency: $a \lor a \equiv a$

https://en.wikipedia.org/wiki/Logical_disjunction

Disjunction in boolean algebra

Applications in computer science [edit]

Operators corresponding to logical disjunction exist in most programming languages.



OR logic gate

Bitwise operation [edit]

Disjunction is often used for bitwise operations. Examples:

- 0 or 0 = 0
- 0 or 1 = 1
- 1 or 0 = 1
- 1 or 1 = 1
- 1010 or 1100 = 1110

The `or` operator can be used to set bits in a bit field to 1, by `or`-ing the field with a constant field with the relevant bits set to 1. For example, `x = x | 0b00000001` will force the final bit to 1 while leaving other bits unchanged.

https://en.wikipedia.org/wiki/Logical_disjunction

Material conditional

The **material conditional** (also known as *material implication*, *material consequence*, or simply *implication*, *implies*, or *conditional*) is a logical connective (or a binary operator) that is often symbolized by a forward arrow "→". The material conditional is used to form statements of the form $p \to q$ (termed a *conditional statement*) which is read as "if *p* then *q*". Unlike the English construction "if...then...", the material conditional statement $p \to q$ does not specify a causal relationship between *p* and *q*. It is merely to be understood to mean "if *p* is true, then *q* is also true" such that the statement $p \to q$ is false only when *p* is true and *q* is false.[1] The material conditional only states that *q* is true when (but not necessarily only when) *p* is true, and makes no claim that *p* causes *q*.

https://en.wikipedia.org/wiki/Material_conditional

Disjunction in boolean algebra

As a truth function [edit]

In classical logic, the compound $p \to q$ is logically equivalent to the negative compound: not both p and not q. Thus the compound $p \to q$ is *false* if and only if both p is true and q is false. By the same stroke, $p \to q$ is *true* if and only if either p is false or q is true (or both). Thus $\to$ is a function from pairs of truth values of the components p, q to truth values of the compound $p \to q$, whose truth value is entirely a function of the truth values of the components. Hence, this interpretation is called *truth-functional*. The compound $p \to q$ is logically equivalent also to $\neg p \lor q$ (either not p, or q (or both)), and to $\neg q \to \neg p$ (if not q then not p). But it is not equivalent to $\neg p \to \neg q$, which is equivalent to $q \to p$.

https://en.wikipedia.org/wiki/Material_conditional

Disjunction in boolean algebra

Truth table [edit]

The truth table associated with the material conditional $p \to q$ is identical to that of $\neg p \vee q$. It is as follows:

p	q	$p \to q$
T	**T**	T
T	**F**	F
F	**T**	T
F	**F**	T

https://en.wikipedia.org/wiki/Material_conditional

Logical Equivalence

In logic, statements p and q are **logically equivalent** if they have the same logical content. This is a semantic concept; two statements are equivalent if they have the same truth value in every model (Mendelson 1979:56). The logical equivalence of p and q is sometimes expressed as $p \equiv q$, Epq, or $p \iff q$. However, these symbols are also used for material equivalence; the proper interpretation depends on the context. Logical equivalence is different from material equivalence, although the two concepts are closely related.

https://en.wikipedia.org/wiki/Logical_equivalence

Laws of logical equivalence (1)

Equivalence	***Name***
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws

https://en.wikipedia.org/wiki/Logical_equivalence

Laws of logical equivalence (2)

$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

https://en.wikipedia.org/wiki/Logical_equivalence

Logical equivalence and conditionals

Logical equivalences involving conditional statements:

1. $p \implies q \equiv \neg p \vee q$
2. $p \implies q \equiv \neg q \implies \neg p$
3. $p \vee q \equiv \neg p \implies q$
4. $p \wedge q \equiv \neg(p \implies \neg q)$
5. $\neg(p \implies q) \equiv p \wedge \neg q$
6. $(p \implies q) \wedge (p \implies r) \equiv p \implies (q \wedge r)$
7. $(p \implies q) \vee (p \implies r) \equiv p \implies (q \vee r)$
8. $(p \implies r) \wedge (q \implies r) \equiv (p \vee q) \implies r$
9. $(p \implies r) \vee (q \implies r) \equiv (p \wedge q) \implies r$

https://en.wikipedia.org/wiki/Logical_equivalence

Logical equivalence and bi-conditionals

Logical equivalences involving biconditionals:

1. $p \iff q \equiv (p \implies q) \wedge (q \implies p)$
2. $p \iff q \equiv \neg p \iff \neg q$
3. $p \iff q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
4. $\neg(p \iff q) \equiv p \iff \neg q$

https://en.wikipedia.org/wiki/Logical_equivalence

Logical equivalence and bi-conditionals

https://en.wikipedia.org/wiki/Logical_disjunction

References

[1] http://en.wikipedia.org/
[2]