

CLTI Impulse Response (4A)

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- Requirements of an Impulse Response

Solutions of Differential Equations : $h(t)$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{M-1} \frac{dx(t)}{dt} + b_M x(t)$$

requirement at time $t = 0$

All the derivatives of $h(t)$ up to N must match a corresponding derivatives of the impulse up to M at time $t=0$

requirement at time $t \neq 0$

The linear combination of all the derivatives of $h(t)$ must add to zero for any time $t \neq 0$

$y_h(t)u(t)$ is such a function

$y_h(t)$ is the homogeneous solution

Case 1 $N > M$

The derivatives of the $y_h(t)u(t)$ provide all the singularity functions necessary to match the impulse and derivatives of the impulse on the right side and no other terms need to be added

Case 2 $N = M$

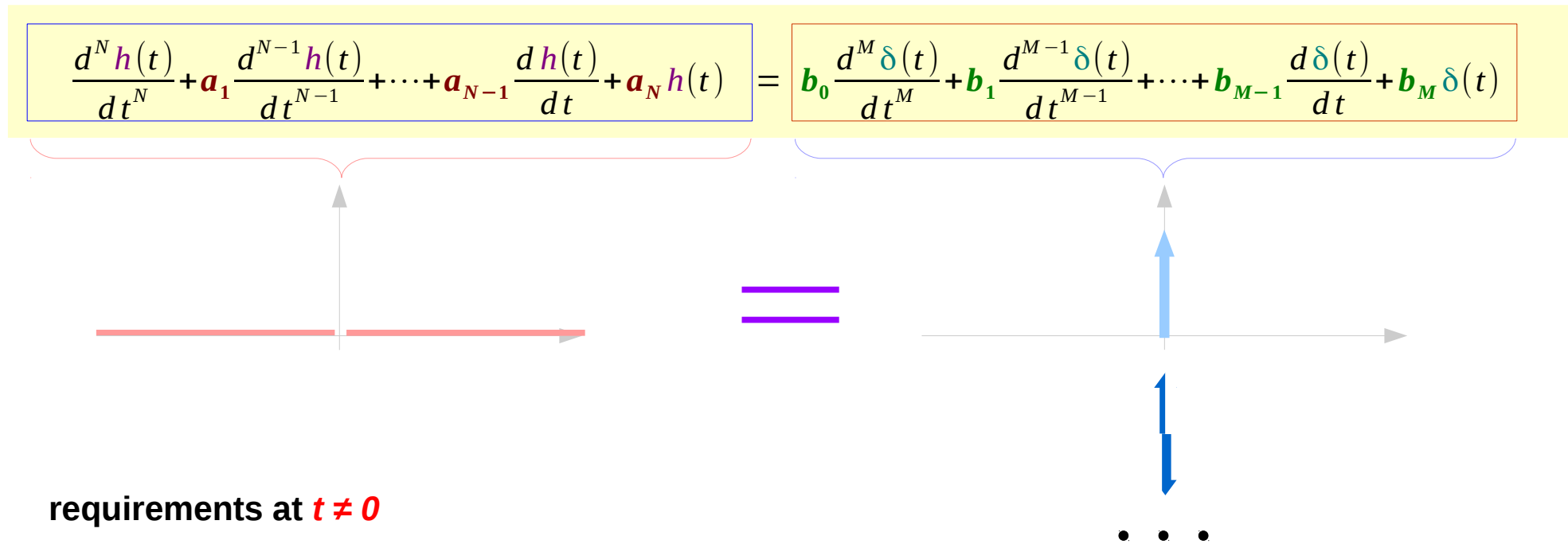
Need to add an impulse term $K_0 \delta(t)$.. and solve for K_0 by matching coefficients of impulses on both sides

Case 3 $N < M$

The n -th derivative of the function we add to $y_h(t)u(t)$ must have a term that matches the M -th derivative of the unit impulse. Must add

$$\begin{aligned} & K_{m-n} u_{m-n}(t) + K_{m-n-1} u_{m-n-1}(t) + \dots + K_0 u_0(t) \\ &= K_{m-n} \delta^{(m-n)}(t) + K_{m-n-1} \delta^{(m-n-1)}(t) + \dots + K_0 \delta(t) \end{aligned}$$

Requirements at $t \neq 0$ (1)



requirements at $t \neq 0$

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) \dots + a_N h(t) = 0 \quad (t \neq 0)$$

The linear combination of all the derivatives of $h(t)$ must add to zero for any time $t \neq 0$

all the derivatives of $\delta(t)$ exists only $t=0$. It is zero for any time $t \neq 0$

Requirements at $t \neq 0$ (2)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \dots + b_{M-1} \frac{d\delta(t)}{dt} + b_M \delta(t)$$

$u(t)=0$
: step function

the *linear combination* of all
the derivatives of $y_p(t)$
results to zero
: *homogeneous solution*

\dots

$$u(t) = 0 \qquad y_h^{(N)} + a_1 y_h^{(N-1)} + \dots + a_N y_h = 0$$

for $t < 0$, $u(t) = 0$
for $t > 0$, $y_h^{(N)} + a_1 y_h^{(N-1)} + \dots + a_N y_h = 0$
derivatives of $\{y_h \cdot u\}$ produce
derivatives of δ when $t = 0$

$y_h(t)u(t)$ when $t \neq 0$
➡ A possible candidate of $h(t)$

Requirements at $t=0$ (1)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \dots + b_{M-1} \frac{d\delta(t)}{dt} + b_M \delta(t)$$

requirements at $t = 0$

All the derivatives of $h(t)$ up to N must **match** the corresponding derivatives of an impulse $\delta(t)$ up to M at time $t=0$

Need to add a $\delta(t)$ and its derivatives in case that $(N \leq M)$

Need to integrate $y_n(t) \cdot u(t)$ several times in case that $(N > M)$

Requirements at t=0 (2)

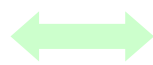
$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \dots + b_{M-1} \frac{d\delta(t)}{dt} + b_M \delta(t)$$

$$\frac{d^N}{dt^N} \{y_h(t)u(t)\} = \frac{d^{N-1}}{dt^{N-1}} \{c_0 \delta(t)\} + \dots$$

$y_h(t) \cdot u(t)$ gives the highest order (N-1)
Need to have the following terms

$$m_{M-N} \frac{d^M}{dt^M} \delta(t) + \dots + m_0 \frac{d^N}{dt^N} \delta(t) \quad (N \leq M)$$

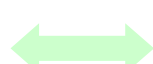
(N ≤ M)



$$b_0 \frac{d^M \delta(t)}{dt^M}$$

the highest order derivatives of $\delta(t)$

(N ≤ M)



$$b_0 \frac{d^M \delta(t)}{dt^M}$$

the highest order derivatives of $\delta(t)$

$$m_0 \frac{d^N}{dt^N} \delta(t) + \dots + m_{M-N} \frac{d^M}{dt^M} \delta(t) \quad \rightarrow$$

Requirements at $t=0$ (3)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \dots + b_{M-1} \frac{d\delta(t)}{dt} + b_M \delta(t)$$

$$m_0 \frac{d^N \delta(t)}{dt^N} + \dots + m_{M-N} \frac{d^M \delta(t)}{dt^M} \rightarrow$$

all the derivatives of $y_h(t) \cdot u(t)$ may not include all the required the derivatives of $\delta(t)$ at the time $t=0$ in case that $N \leq M$.

Need to add a $\delta(t)$ and its derivatives in case that $(N \leq M)$

$$h(t) = y_h(t)u(t) + m_0 \delta(t)$$

$$h(t) = y_h(t)u(t) + m_0 \delta(t) + m_1 \dot{\delta}(t) + \dots + m_{M-N} \delta^{(M-N)}(t)$$

$(N = M)$

$(N < M)$

Requirements at t=0 (4)

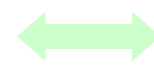
$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \dots + b_{M-1} \frac{d\delta(t)}{dt} + b_M \delta(t)$$

$$\frac{d^N}{dt^N} \{ \text{ } \} = \frac{d^M}{dt^M} \{ c_0 \delta(t) \} + \dots$$

$h(t)$ gives the highest order (M)
 $h(t)$ must have the following terms

$$h(t) = \int_{-\infty}^t \dots \int_{-\infty}^t y_h(t) u(t) dt \dots dt$$

($N > M$)



$$b_0 \frac{d^M \delta(t)}{dt^M}$$

the highest order derivatives of $\delta(t)$

($N \leq M$)



$$b_0 \frac{d^M \delta(t)}{dt^M}$$

the highest order derivatives of $\delta(t)$

$$\frac{d^N}{dt^N} \{ h(t) \} = \frac{d^{M+1}}{dt^{M+1}} \left\{ \frac{d^{N-M-1}}{dt^{N-M-1}} \{ h(t) \} \right\} = \frac{d^{M+1}}{dt^{M+1}} \{ y_h(t) u(t) \}$$

Requirements at $t=0$ (5)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \dots + b_{M-1} \frac{d\delta(t)}{dt} + b_M \delta(t)$$

All the derivatives of $h(t)$ up to N must generate the derivatives of an impulse $\delta(t)$ **only up to M** in case that $N > M$

Need to integrate $y_h(t) \cdot u(t)$ several times in case that $(N > M)$

$$h(t) = \int_{-\infty}^t \dots \int_{-\infty}^t y_h(t) u(t) dt \dots dt$$

$N-M-1$

$(N > M)$

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- Impulse Response in terms of $y_h(t) \cdot u(t)$

Three different $h(t)$'s

$$h(t) = y_h(t)u(t)$$



All the derivatives of $h(t)$ up to N incurs the derivatives of an impulse $\delta(t)$ up to $N-1$

$$h^{(1)}(t) = y_h(t)u(t)$$



All the derivatives of $h(t)$ up to N incurs the derivatives of an impulse $\delta(t)$ up to $N-2$

$$h^{(2)}(t) = y_h(t)u(t)$$



All the derivatives of $h(t)$ up to N incurs the derivatives of an impulse $\delta(t)$ up to $N-3$

Derivatives of three different $h(t)$'s

$$h(t) = y_h(t)u(t)$$

$$h^{(N)}(t) \longrightarrow K_1 \delta^{(N-1)}(t) + K_2 \delta^{(N-2)}(t) + \dots + K_{N-1} \delta^{(1)}(t) + K_N \delta(t)$$

$$h(t) = \int_{-\infty}^t y_h(t)u(t) dt$$

$$h^{(N)}(t) \longrightarrow K_1 \delta^{(N-2)}(t) + K_2 \delta^{(N-3)}(t) + \dots + K_{N-1} \delta(t)$$

$$h(t) = \iint_{-\infty}^t y_h(t)u(t) dt dt$$

$$h^{(N)}(t) \longrightarrow K_1 \delta^{(N-3)}(t) + \dots + K_{N-2} \delta(t)$$

All the derivatives of $h(t)$ up to N

	$h^{(N)}(t)$	$+a_1 h^{(N-1)}(t)$	$+ \dots$	$+a_{N-2} h^{(2)}(t)$	$+a_{N-1} h^{(1)}(t)$	$+a_N h^{(0)}(t)$
(a)	$\frac{d^N}{dt^N} \{y_h u\}$	$\frac{d^{N-1}}{dt^{N-1}} \{y_h u\}$		$\frac{d^2}{dt^2} \{y_h u\}$	$\frac{d}{dt} \{y_h u\}$	$y_h u$
(b)	$\frac{d^{N-1}}{dt^{N-1}} \{y_h u\}$	$\frac{d^{N-2}}{dt^{N-2}} \{y_h u\}$		$\frac{d}{dt} \{y_h u\}$	$y_h u$	$\int_{-\infty}^t y_h u dt$
(c)	$\frac{d^{N-2}}{dt^{N-2}} \{y_h u\}$	$\frac{d^{N-3}}{dt^{N-3}} \{y_h u\}$		$y_h u$	$\int_{-\infty}^t y_h u dt$	$\int_{-\infty}^t \int_{-\infty}^t y_h u dt dt$

$$\begin{array}{llll}
 (N = M+1) & (N-1 = M) & \delta^{(N-1)} = \delta^{(M)} & (M-N+1 = 0) \\
 (N = M+2) & (N-2 = M) & \delta^{(N-2)} = \delta^{(M)} & (M-N+1 = -1) \\
 (N = M+3) & (N-3 = M) & \delta^{(N-3)} = \delta^{(M)} & (M-N+1 = -2)
 \end{array}$$

Negative powers denote integration

(a) $h(t) = y_h(t)u(t) \Rightarrow h(t) = y_h(t)u(t) = g(t)$

(b) $h^{(1)}(t) = y_h(t)u(t) \Rightarrow h(t) = \int_{-\infty}^t y_h(t)u(t) dt = g^{(-1)}(t) \equiv \int_{-\infty}^t g(t) dt$

(c) $h^{(2)}(t) = y_h(t)u(t) \Rightarrow h(t) = \iint_{-\infty}^t y_h(t)u(t) dt dt = g^{(-2)}(t) \equiv \int_{-\infty}^t \int_{-\infty}^t g(t) dt dt$

Derivative of three different $h(t)$: $g(t)$, $g^{(1)}(t)$, $g^{(2)}(t)$

	$h^{(N)}(t)$	$+a_1 h^{(N-1)}(t)$	$+ \dots$	$+a_{N-2} h^{(2)}(t)$	$+a_{N-1} h^{(1)}(t)$	$+a_N h^{(0)}(t)$
(a)	$g^{(N)}(t)$	$g^{(N-1)}(t)$		$g^{(2)}(t)$	$g^{(1)}(t)$	$g(t)$
(b)	$g^{(N-1)}(t)$	$g^{(N-2)}(t)$		$g^{(1)}(t)$	$g(t)$	$g^{(-1)}(t)$
(c)	$g^{(N-2)}(t)$	$g^{(N-3)}(t)$		$g(t)$	$g^{(-1)}(t)$	$g^{(-2)}(t)$

$$g(t) = y_h(t) \cdot u(t)$$

$$h(t) = g^{(M-N+1)}(t) = g^{(0)}(t)$$

$$h(t) = g^{(M-N+1)}(t) = g^{(-1)}(t)$$

$$h(t) = g^{(M-N+1)}(t) = g^{(-2)}(t)$$

$$(N = M+1) \quad (M-N+1 = 0)$$

$$(N = M+2) \quad (M-N+1 = -1)$$

$$(N = M+3) \quad (M-N+1 = -2)$$

Impulse response $h(t)$ in terms of $y_h(t) \cdot u(t)$

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \dots + b_{M-1} \frac{d\delta(t)}{dt} + b_M \delta(t)$$

$(N = M+1)$	$(N-1 = M)$	$\delta^{(N-1)} = \delta^{(M)}$	$(M-N+1 = 0)$	$h(t) = g^{(M-N+1)}(t) = g^{(0)}(t)$
$(N = M+2)$	$(N-2 = M)$	$\delta^{(N-2)} = \delta^{(M)}$	$(M-N+1 = -1)$	$h(t) = g^{(M-N+1)}(t) = g^{(-1)}(t)$
$(N = M+3)$	$(N-3 = M)$	$\delta^{(N-3)} = \delta^{(M)}$	$(M-N+1 = -2)$	$h(t) = g^{(M-N+1)}(t) = g^{(-2)}(t)$

$$g(t) = y_h(t)u(t)$$

$$\left\{ \begin{array}{l} h(t) = g^{(M-N+1)}(t) = \int_{-\infty}^t \dots \int_{-\infty}^t y_h(t)u(t) dt \dots dt \quad (N > M) \\ h(t) = g(t) + m_0 \delta(t) \quad (N = M) \\ h(t) = g(t) + m_0 \delta(t) + m_1 \delta'(t) + \dots + m_{M-N} \delta^{(M-N)}(t) \quad (N < M) \end{array} \right.$$

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- Matching an impulse and its derivatives

Impulse Matching ($N > M$)

$$\begin{aligned}
 & \frac{d^N h}{dt^N} + a_1 \frac{d^{N-1} h}{dt^{N-1}} + \dots + \underbrace{a_{N-M-1} \frac{d^{M+1} h}{dt^{M+1}} + a_{N-M} \frac{d^M h}{dt^M} + \dots + a_{N-1} \frac{dh}{dt}}_{M+1} + a_N h = \\
 & \underbrace{K_1 \delta^{(N-1)}(t) + K_2 \delta^{(N-2)}(t) + \dots}_{\text{must have zero coefficients}} + \underbrace{K_{N-M} \delta^{(M)}(t) + K_{N-M+1} \delta^{(M-1)}(t) + \dots + K_N \delta(t)}_{\text{impulse related terms in the left hand side}} + \boxed{y_h(t)u(t)}
 \end{aligned}$$

$b_0 \frac{d^M \delta}{dt^M} + \dots + b_{M-1} \frac{d\delta}{dt} + b_M \delta$

Impulse Matching ($N > M$)

$$\begin{array}{c}
 \overbrace{b_0 \frac{d^M \delta}{dt^M} + \dots + b_{M-1} \frac{d \delta}{dt} + b_M \delta}^{M+1} \\
 \leftarrow N > M \\
 \frac{d^N h}{dt^N} + a_1 \frac{d^{N-1} h}{dt^{N-1}} + \dots + a_M \frac{d^{N-M} h}{dt^{N-M}} + a_{M+1} \frac{d^{N-M-1} h}{dt^{N-M-1}} + \dots + a_{N-1} \frac{dh}{dt} + a_N h = \\
 \underbrace{K_1 \delta^{(M)}(t) + K_2 \delta^{(M-1)}(t) + \dots + K_{M+1} \delta(t)}_{M+1} \quad \underbrace{y_h(t)u(t)}_{\text{do not have an impulse or its derivatives}}
 \end{array}$$

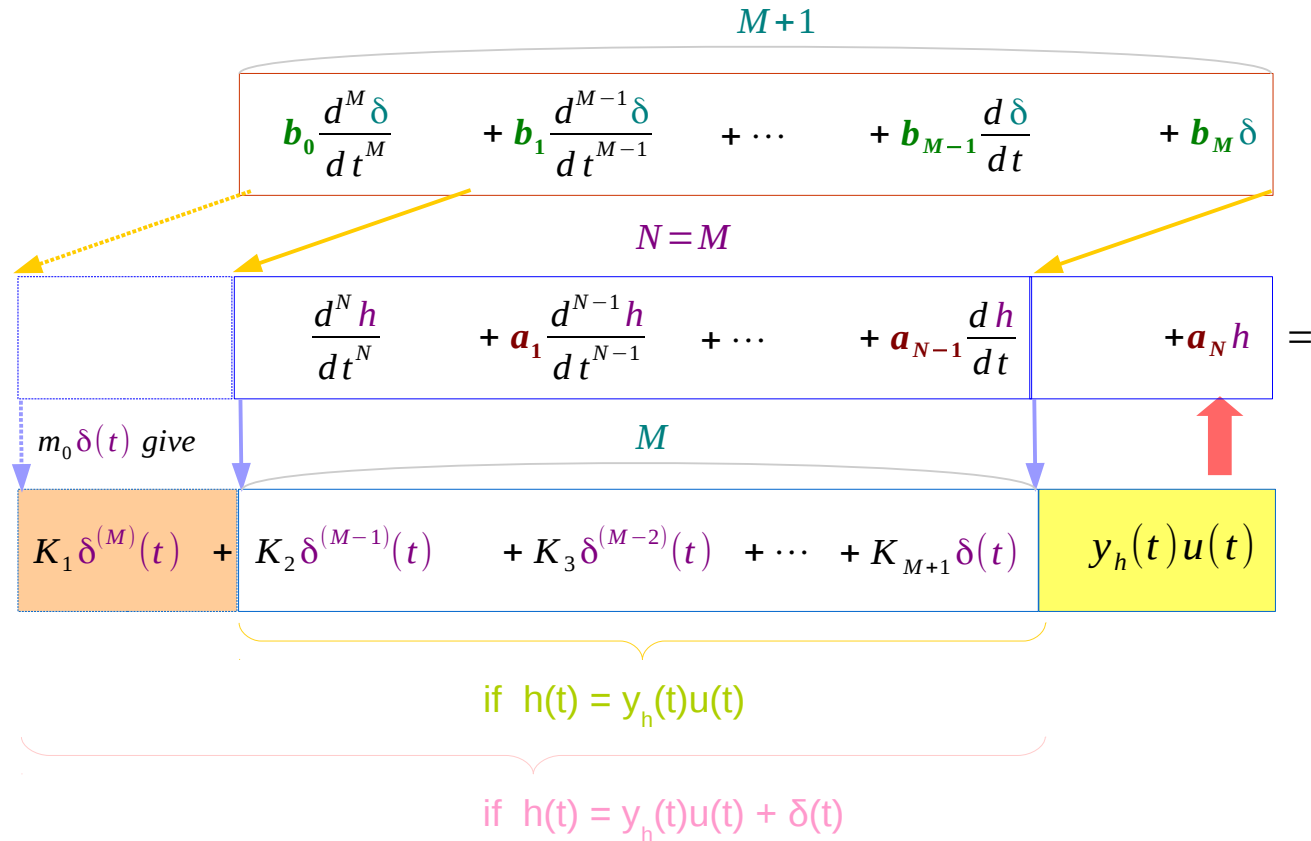
impulse related terms in the left hand side

do not have an impulse or its derivatives

$$g(t) = y_h(t)u(t)$$

$$h(t) = g^{(M-N+1)}(t) = \int_{-\infty}^t \dots \int_{-\infty}^t y_h(t)u(t) dt \dots dt \quad (N > M)$$

Impulse Matching ($N = M$)

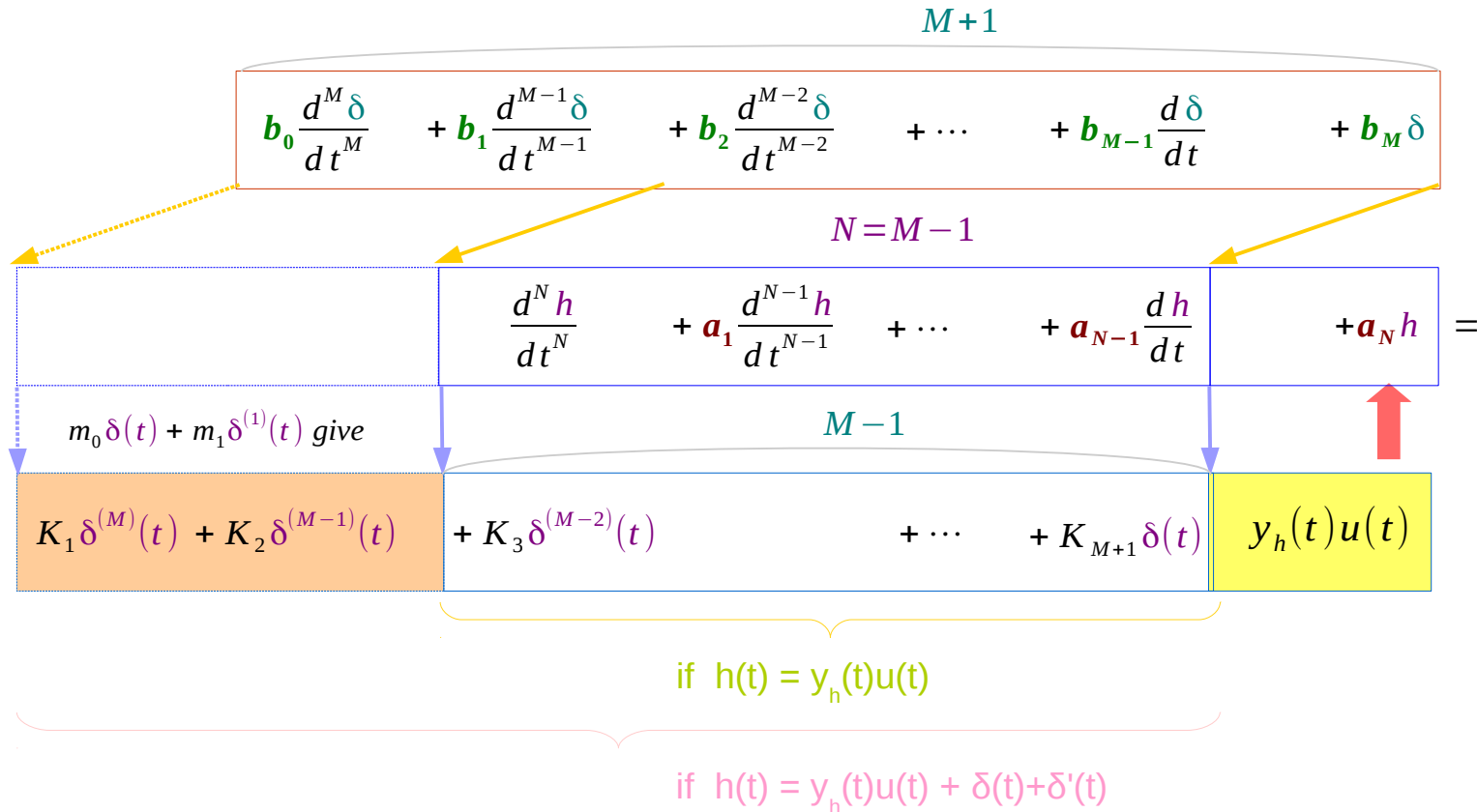


$$g(t) = y_h(t)u(t)$$

$$h(t) = g(t) + m_0 \delta(t)$$

$(N=M)$

Impulse Matching ($N < M$)



$$g(t) = y_h(t)u(t)$$

$$h(t) = g(t) + m_0\delta(t) + m_1\delta^{(1)}(t) + \dots + m_{M-N}\delta^{(M-N)}(t) \quad (N < M)$$

Integrals of $y_h(t) \cdot u(t)$: has the same form as $y_h(t) \cdot u(t)$

$$\frac{d^2 h(t)}{dt^2} + a_1 \frac{dh(t)}{dt} + a_2 h(t)$$

linear equation with constant coefficients

$$y_h(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} \quad m_1, m_2 = (-a_1 \pm \sqrt{a_1^2 - 4a_2})/2$$

$$y_h(t)u(t) = (C_1 e^{m_1 t} + C_2 e^{m_2 t})u(t)$$

$$\int_{-\infty}^t y_h(t)u(t) dt$$

$$\int_{-\infty}^t \int_{-\infty}^t y_h(t)u(t) dt dt$$

$$y_h(t) = c_1 e^{m_1 t} + c_2 t e^{m_1 t} \quad m_1, m_2 = -a_1/2$$

$$y_h(t)u(t) = (C_1 e^{m_1 t} + C_2 t e^{m_1 t})u(t)$$

$$\int_{-\infty}^t y_h(t)u(t) dt$$

$$\int_{-\infty}^t \int_{-\infty}^t y_h(t)u(t) dt dt$$

$$y_h(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} \quad s_1, s_2 = (-a_1 \pm i\sqrt{4a_2 - a_1^2})/2$$

$$y_h(t)u(t) = (C_1 e^{s_1 t} + C_2 e^{s_2 t})u(t)$$

$$\int_{-\infty}^t y_h(t)u(t) dt$$

$$\int_{-\infty}^t \int_{-\infty}^t y_h(t)u(t) dt dt$$

Integrals of $y_h(t) \cdot u(t)$

$$\int_{-\infty}^t \cdots \int_{-\infty}^t y_h(t) u(t) dt \cdots dt$$



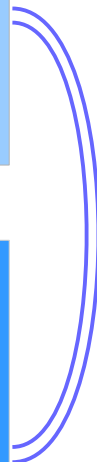
$$y_h(t) u(t)$$

linear equation with constant coefficients

$$g(t) = y_h(t) u(t)$$

$$\left\{ \begin{array}{l} h(t) = g^{(M-N+1)}(t) = \int_{-\infty}^t \cdots \int_{-\infty}^t y_h(t) u(t) dt \cdots dt \quad (N > M) \\ h(t) = g(t) + m_0 \delta(t) \quad (N = M) \\ h(t) = g(t) + m_0 \delta(t) + m_1 \delta'(t) + \cdots + m_{M-N} \delta^{(M-N)}(t) \quad (N < M) \end{array} \right.$$

$$\left\{ \begin{array}{l} h(t) = y_h(t) u(t) \quad (N > M) \\ h(t) = y_h(t) u(t) + m_0 \delta(t) \quad (N = M) \\ h(t) = y_h(t) u(t) + m_0 \delta(t) + m_1 \delta'(t) + \cdots + m_{M-N} \delta^{(M-N)}(t) \quad (N < M) \end{array} \right.$$



Case Examples

($N > M$)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) =$$

$$\xrightarrow{\quad} b_0 \frac{d^{N-1} \delta(t)}{dt^{N-1}} + b_1 \frac{d^{N-2} \delta(t)}{dt^{N-2}} + \dots + b_{N-2} \frac{d\delta(t)}{dt} + b_{N-1} \delta(t)$$

$M = N - 1$

($N = M$)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) =$$

$$b_0 \frac{d^N \delta(t)}{dt^N} + b_1 \frac{d^{N-1} \delta(t)}{dt^{N-1}} + \dots + b_{N-1} \frac{d\delta(t)}{dt} + b_N \delta(t)$$

$M = N$

($N < M$)

$$\xleftarrow{\quad} \frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = \delta(t)$$

$$b_0 \frac{d^{N+1} \delta(t)}{dt^{N+1}} + b_1 \frac{d^N \delta(t)}{dt^N} + \dots + b_N \frac{d\delta(t)}{dt} + b_{N+1} \delta(t)$$

$M = N + 1$

in most systems $N \geq M$

$h(t) =$

$$y_h(t)u(t)$$

$h(t) =$

$$y_h(t)u(t) + m_0 \delta(t)$$

seldom used $N < M$

$h(t) =$

$$y_h(t)u(t) + m_0 \delta(t) + m_1 \delta(t)$$

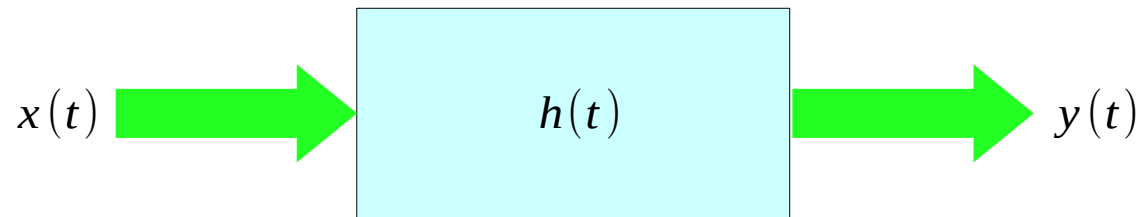
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- An Impulse Response of a Causal LTI System

ODE's and Causal LTI Systems

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \dots + b_{M-1} \frac{d\delta(t)}{dt} + b_M \delta(t)$$

$N > M$: (N-M) Integrator

$N < M$: (M-N) differentiator – magnify high frequency components of noise (seldom used)

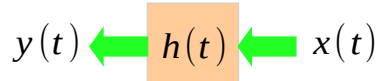


$N \geq M$ in most systems

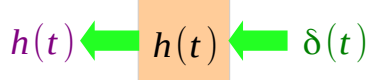
$$h(t) = y_h(t)u(t) \quad (N > M)$$

$$h(t) = y_h(t)u(t) + m_0 \delta(t) \quad (N = M)$$

$h(t)$ can have at most a $\delta(t)$ for most systems



$$y^{(N)} + a_1 y^{(N-1)} + \dots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \dots + b_{N-1} x^{(1)} + b_N x$$



$$h^{(N)} + a_1 h^{(N-1)} + \dots + a_{N-1} h^{(1)} + a_N h = b_0 \delta^{(N)} + b_1 \delta^{(N-1)} + \dots + b_{N-1} \delta^{(1)} + b_N \delta$$



if h contain δ

$$C \delta^{(N)} \dots \dots \dots = b_0 \delta^{(N)}$$

the highest order derivatives of $\delta(t)$



if h contain $\delta^{(1)}$

$$C \delta^{(N+1)} \dots \dots \dots \neq b_0 \delta^{(N)}$$

the highest order derivatives of $\delta(t)$

cannot be matched



if h contain $\delta^{(2)}$

$$C \delta^{(N+2)} \dots \dots \dots \neq b_0 \delta^{(N)}$$

the highest order derivatives of $\delta(t)$

cannot be matched

$t=0$

$h(t)$ can have at most an impulse $b_0 \delta(t)$
no derivatives of $\delta(t)$ possible at all

in most systems

$$N \geq M$$

$h(t)$ can have at most a $\delta(t)$ ($N \geq M$)

$$N = M \quad Q(D)y(t) = P(D)x(t)$$

$$\begin{array}{c} N \\ (D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) h(t) \end{array} = \begin{array}{c} M \quad (N \geq M) \\ (b_0 D^N + b_1 D^{N-1} + \dots + b_{N-1} D + b_N) \delta(t) \end{array}$$



If $\delta'(t)$ is included in $h(t)$
then the highest order term
in the LHS



the highest order term
in the RHS

$$\frac{d^N h}{dt^N} \rightarrow \delta^{(N+1)}(t) \quad \boxed{\delta^{(N+1)}(t)} \neq \boxed{\delta^{(N)}(t)} \quad \rightarrow \text{contradiction}$$

$h(t)$ cannot contain $\delta^{(0)}(t)$ at all \rightarrow $h(t)$ can contain at most $\delta(t)$
only when $N = M$

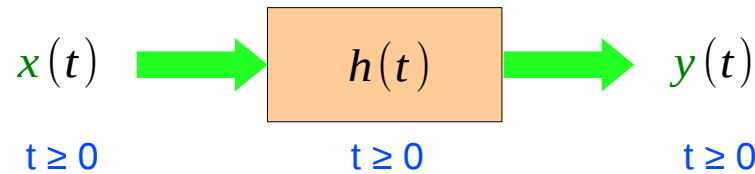
-
- An Impulse Response and System Responses

Causality

Causal Signals:
the input signals
starts at time $t=0$

Causal System:
the response $h(t)$ cannot
begin before the input

Causal Signals:
the output signals
starts at time $t=0$



$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau$$

$$x(\tau) = 0 \quad (\tau < 0)$$

$$h(t-\tau) = 0 \quad (t-\tau < 0)$$

causal signal: $x(t) = 0 \quad (t < 0)$

causal signal: $h(t) = 0 \quad (t < 0)$

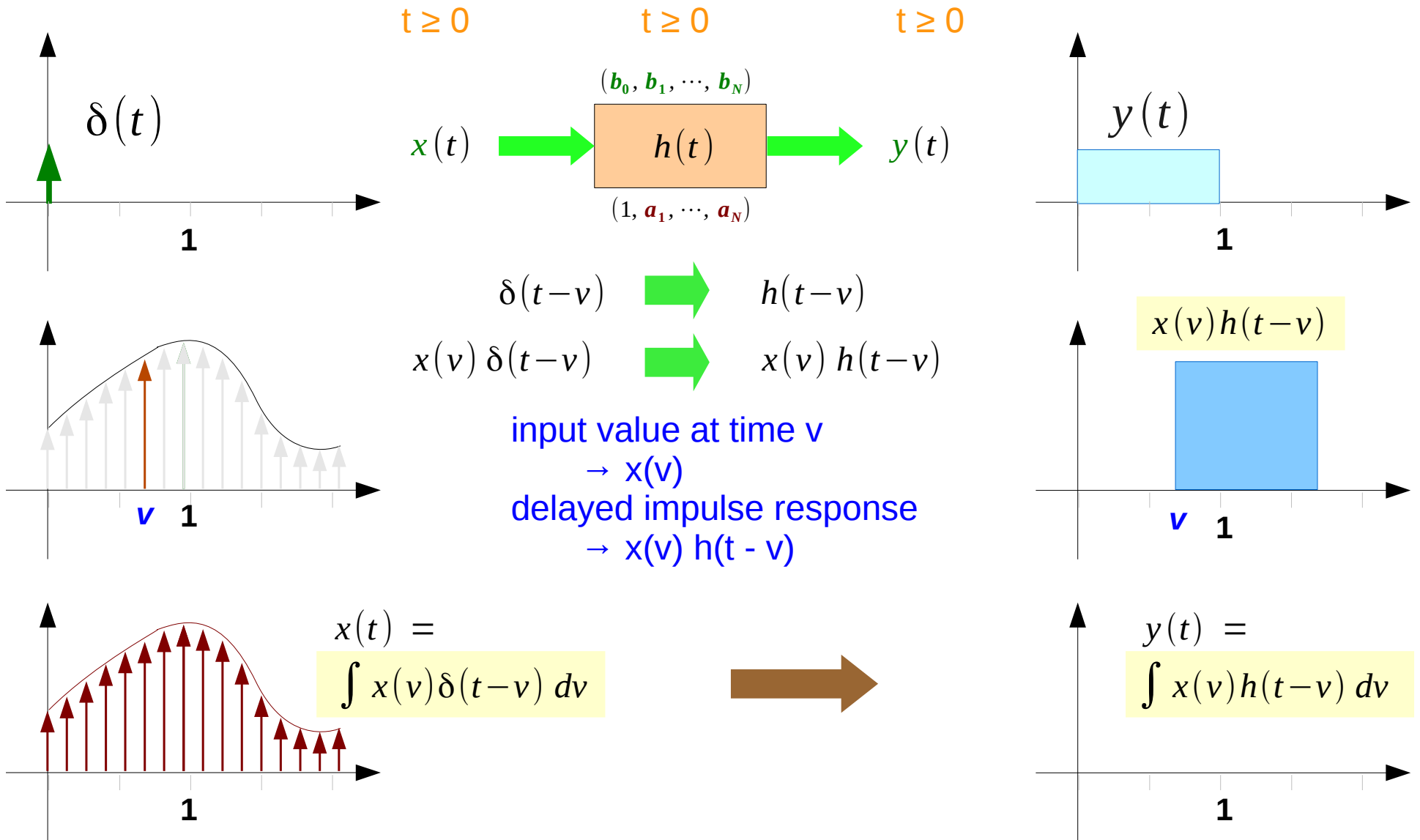


causal signal: $y(t) = 0 \quad (t < 0)$

$$\begin{aligned}
 y(t) &= x(t) * h(t) = \int_{0^-}^t x(\tau) h(t-\tau) d\tau \\
 &= \int_{0^-}^t x(t-\tau) h(\tau) d\tau
 \end{aligned}$$

0^- to include an impulse function $\delta(t)$ in $h(t)$

$h(t)$ as a ZSR ($t \geq 0$)



h(t) as a ZIR (t > 0)

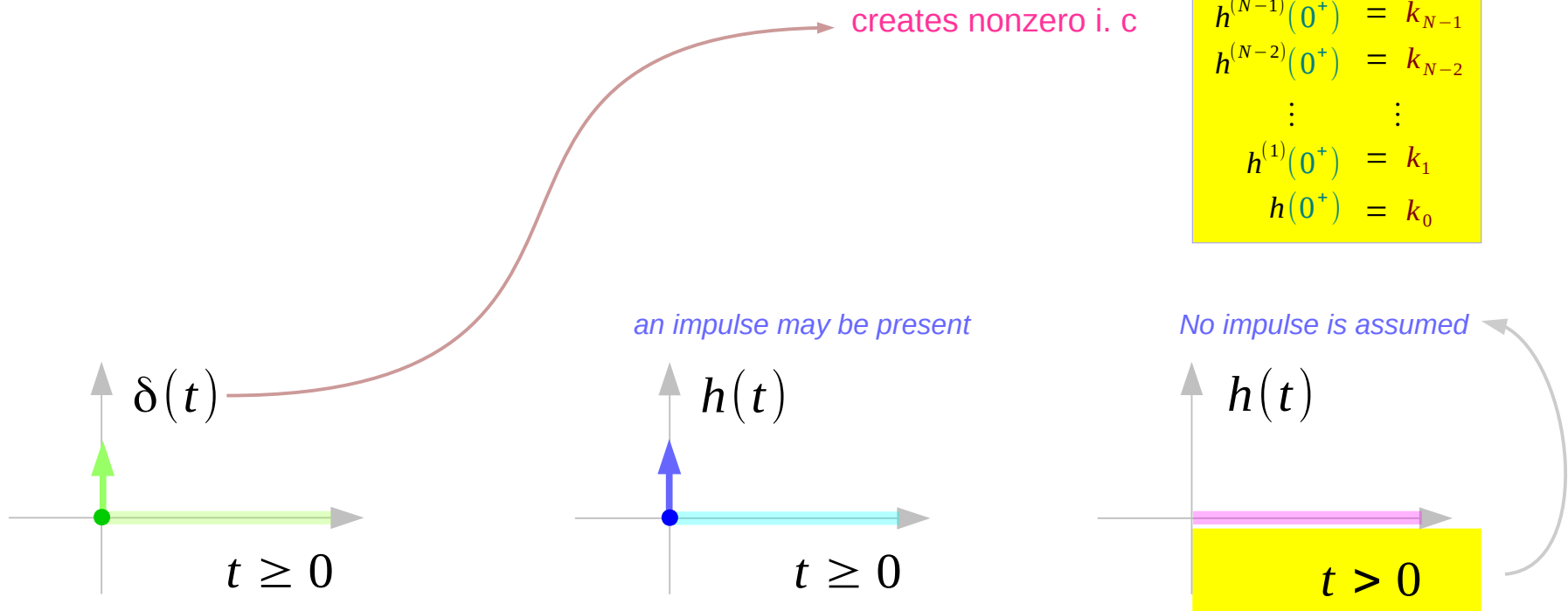
$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \dots + b_{M-1} \frac{d\delta(t)}{dt} + b_M \delta(t)$$

Interval of Validity (t > 0)

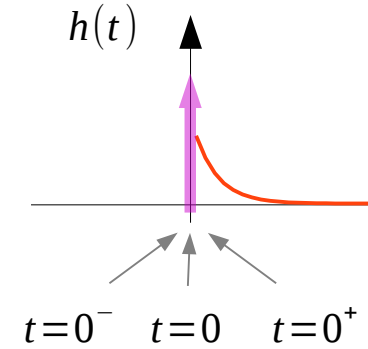
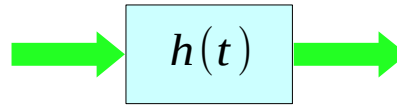
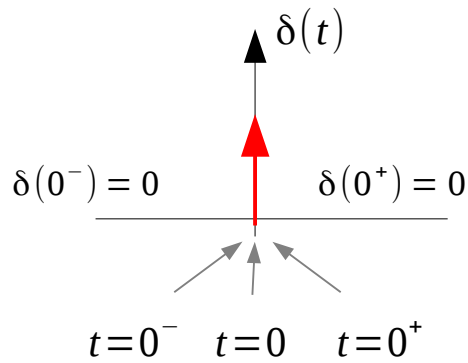
$$h(t) = y_h(t)u(t)$$

The solution of the IVP with the following I.C.

$$\begin{aligned} h^{(N-1)}(0^+) &= k_{N-1} \\ h^{(N-2)}(0^+) &= k_{N-2} \\ &\vdots \\ h^{(1)}(0^+) &= k_1 \\ h(0^+) &= k_0 \end{aligned}$$



Impulse Response $h(t)$



General $y(t)$ cannot be a **ZIR** for a general input $x(t)$ (generally $x(t) \neq 0$ for $t > 0$)

$(N \geq M)$

$h(t)$: **ZIR** with the created I.C. (impulse input vanishes $x(t) = 0$ for $t > 0$)

$$\begin{aligned} h^{(N-1)}(0^+) &= k_{N-1} \\ h^{(N-2)}(0^+) &= k_{N-2} \\ &\vdots \\ h^{(1)}(0^+) &= k_1 \\ h(0^+) &= k_0 \end{aligned}$$

non-zero i. c.

$$\exists i, k_i \neq 0$$

$$\begin{aligned} t &\geq 0^+ \\ (t &\neq 0) \end{aligned}$$

$$h(t) = \text{char mode terms}$$

$$t = 0$$

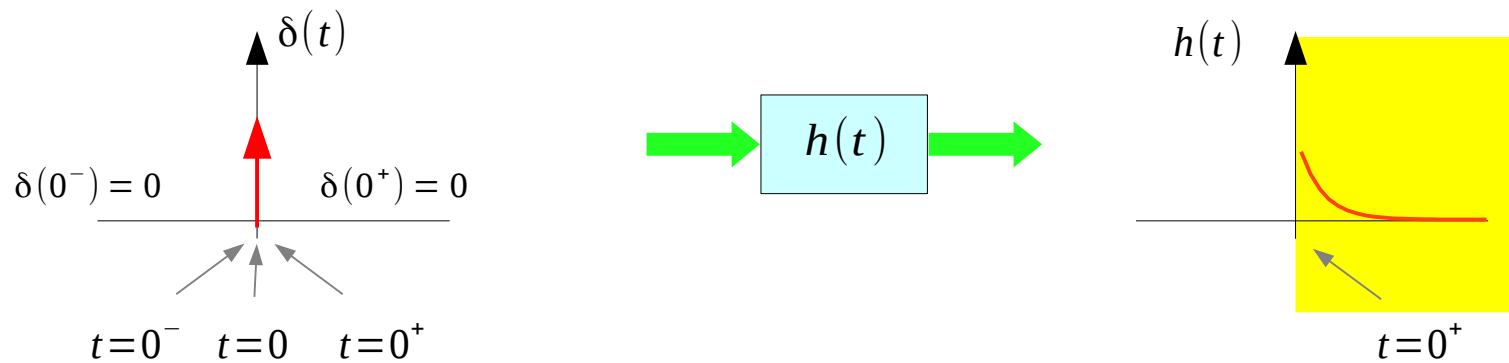
$$h(t) = b_0 \delta(t) \text{ at most an impulse}$$

$$t \geq 0$$

$$h(t) = b_0 \delta(t) + \text{char mode terms}$$

$h(t)$: **ZSR** to an impulse input

Input $\delta(t)$ creates new initial conditions



- generates energy storage
- creates nonzero i. c. at $t=0^+$

$$\begin{aligned} h^{(N-1)}(0^-) &= 0 \\ h^{(N-2)}(0^-) &= 0 \\ &\vdots \\ h^{(1)}(0^-) &= 0 \\ h(0^-) &= 0 \end{aligned}$$

initially at rest

$$\begin{aligned} h^{(N)}(0) &= f_{N-1}(k_i, \delta^{(i)}) \\ h^{(N-1)}(0) &= f_{N-2}(k_i, \delta^{(i)}) \\ &\vdots \\ h^{(2)}(0) &= f_1(k_i, \delta^{(i)}) \\ h^{(1)}(0) &= f_0(k_i, \delta^{(i)}) \end{aligned}$$

assumed finite jumps

Impulse matching

$$\begin{aligned} h^{(N-1)}(0^+) &= k_{N-1} \\ h^{(N-2)}(0^+) &= k_{N-2} \\ &\vdots \\ h^{(1)}(0^+) &= k_1 \\ h(0^+) &= k_0 \end{aligned}$$

non-zero initial conditions

$\exists i, k_i \neq 0$

Total Response

zero input response
+
zero state response

natural response
+
forced response

$[-\infty, 0^-]$

$$y(t) = y_{zi}(t) \leftarrow t \leq 0^-$$

because the input
has not started yet

$$y(0^-) = y_{zi}(0^-) = y_{zi}(0^+)$$

$$\dot{y}(0^-) = \dot{y}_{zi}(0^-) = \dot{y}_{zi}(0^+)$$

$[0^+, +\infty]$

the total response

$$y(0^+) \neq y(0^-)$$

possible discontinuity
at $t = 0$

$$y(0^+) = y_{zi}(0^+) + y_{zs}(0^+)$$

$$\dot{y}(0^+) = \dot{y}_{zi}(0^+) + \dot{y}_{zs}(0^+)$$

$[0^+, +\infty]$

$$y(0^+) = y_{zi}(0^+) + y_{zs}(0^+)$$

$$= y_h(0^+) + y_p(0^+)$$

$$\dot{y}(0^+) = \dot{y}_{zi}(0^+) + \dot{y}_{zs}(0^+)$$

$$= \dot{y}_h(0^+) + \dot{y}_p(0^+)$$

Interval of validity $t > 0$

$$\begin{cases} y_h(0^-) \neq y_{zi}(0^-) \\ \dot{y}_h(0^-) \neq \dot{y}_{zi}(0^-) \end{cases}$$

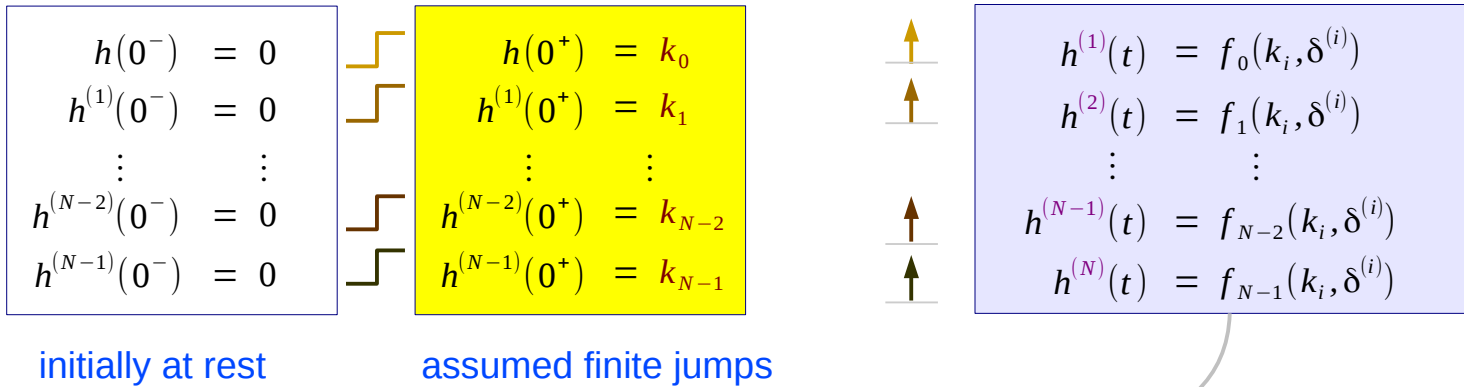
$$\begin{cases} y_p(0^-) \neq y_{zi}(0^-) \\ \dot{y}_p(0^-) \neq \dot{y}_{zi}(0^-) \end{cases}$$

$$\begin{cases} y(0^-) \neq y_{zi}(0^-) \\ \dot{y}(0^-) \neq \dot{y}_{zi}(0^-) \end{cases}$$

-
- Determining the Coefficients of an Impulse Response

Impulse Matching Method Summary

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \dots + b_{M-1} \frac{d\delta(t)}{dt} + b_M \delta(t)$$

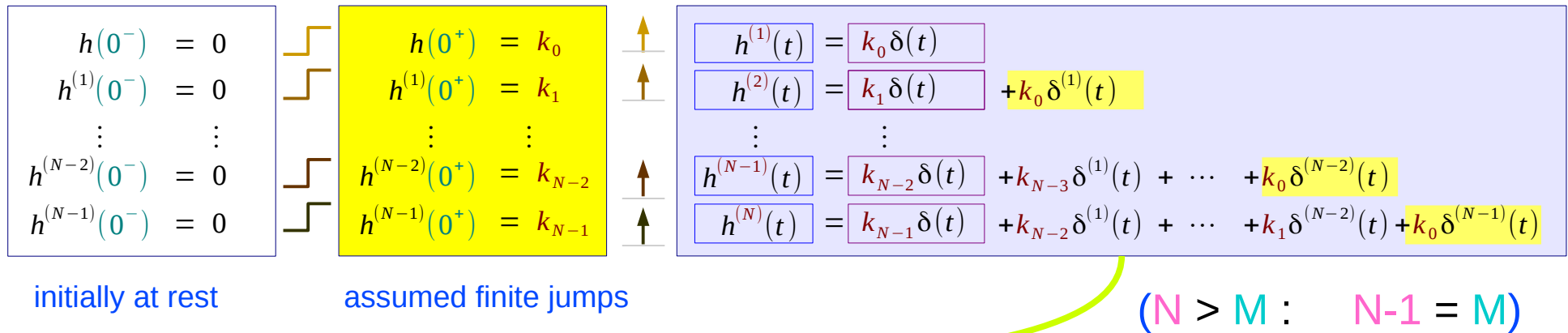


$$h^{(N)}(0) + a_1 h^{(N-1)}(0) + \dots + a_{N-1} h^{(1)}(0) + a_N h(0) = b_0 \delta^{(N)}(t) + b_1 \delta^{(N-1)}(t) + \dots + b_{N-1} \delta^{(1)}(t) + b_N \delta(t)$$

Impulse matching

$$h(t) = (c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \dots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N-1} t}) \cdot u(t)$$

Differentiate each initial condition ($N-1 = M$)



substitute $h^{(m)}(t)$

$$\boxed{h^{(N)}(t)} + a_1 \boxed{h^{(N-1)}(t)} + \dots + a_{N-1} \boxed{h^{(1)}(t)} + a_N \boxed{h(t)} = b_0 \delta^{(M)}(t) + b_1 \delta^{(M-1)}(t) + \dots + b_{M-1} \delta^{(1)}(t) + b_M \delta(t)$$

N variables / N equations

Find the finite jumps

$$\begin{aligned}
 h^{(1)}(t) &= k_0 \delta(t) \\
 h^{(2)}(t) &= k_1 \delta(t) + k_0 \delta^{(1)}(t) \\
 &\vdots \\
 h^{(N-1)}(t) &= k_{N-2} \delta(t) + k_{N-3} \delta^{(1)}(t) + \dots + k_0 \delta^{(N-2)}(t) \\
 h^{(N)}(t) &= k_{N-1} \delta(t) + k_{N-2} \delta^{(1)}(t) + \dots + k_1 \delta^{(N-2)}(t) + k_0 \delta^{(N-1)}(t)
 \end{aligned}$$

substitute

$h^{(m)}(t)$

Impulse matching

($N > M$: $N-1 = M$)

$$h^{(N)}(t) + a_1 h^{(N-1)}(t) + \dots + a_{N-1} h^{(1)}(t) + a_N h(t) \iff b_0 \delta^{(M)}(t) + b_1 \delta^{(M-1)}(t) + \dots + b_{M-1} \delta^{(1)}(t) + b_M \delta(t)$$

Impulse matching

Find coefficients

Find the finite jumps

$$\begin{aligned}
 d_0 \delta^{(N-1)}(t) &\iff b_0 \delta^{(M)}(t) \\
 d_1 \delta^{(N-2)}(t) &\iff b_1 \delta^{(M-1)}(t) \\
 &\vdots \\
 d_{N-2} \delta^{(1)}(t) &\iff b_1 \delta^{(1)}(t) \\
 d_{N-1} \delta^{(0)}(t) &\iff b_M \delta^{(0)}(t)
 \end{aligned}$$







$$\begin{aligned}
 &d_0 \\
 &d_1 \\
 &\vdots \\
 &d_{N-2} \\
 &d_{N-1}
 \end{aligned}$$



$$\begin{aligned}
 &k_0 \\
 &k_1 \\
 &\vdots \\
 &k_{N-2} \\
 &k_{N-1}
 \end{aligned}$$

N variables / N equations

Find the homogeneous solution

$h(0^-) = 0$		$h(0^+) = k_0$
$h^{(1)}(0^-) = 0$		$h^{(1)}(0^+) = k_1$
\vdots	\vdots	\vdots
$h^{(N-2)}(0^-) = 0$		$h^{(N-2)}(0^+) = k_{N-2}$
$h^{(N-1)}(0^-) = 0$		$h^{(N-1)}(0^+) = k_{N-1}$

initially at rest

found finite jumps

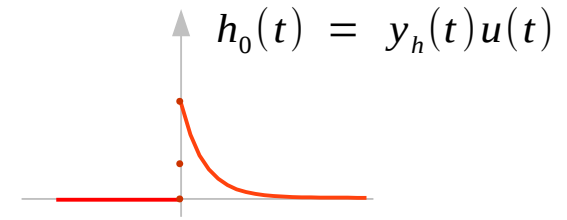
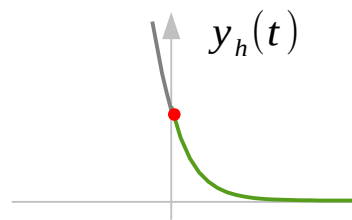
substitute

$$y_h(t) = (c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \dots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N-1} t})$$

($N > M$: $N-1 = M$)

$$\begin{matrix} c_0 \\ c_1 \\ \vdots \\ c_{N-2} \\ c_{N-1} \end{matrix}$$

found homogeneous response



found impulse response

$$h(t) = y_h(t)u(t)$$

A $\delta(t)$ needs to be added to characteristic modes ($N = M$)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \dots + b_{M-1} \frac{d\delta(t)}{dt} + b_M \delta(t)$$

$h^{(1)}(0) =$		$k_0 \delta(t)$	$+b_0 \delta^{(1)}(t)$
$h^{(2)}(0) =$	$k_1 \delta(t)$	$+k_0 \delta^{(1)}(t)$	$+b_0 \delta^{(2)}(t)$
\vdots	\vdots	\vdots	
$h^{(N-1)}(0) =$	$k_{N-2} \delta(t)$	$+ \dots + k_1 \delta^{(N-1)}(t) + k_0 \delta^{(N-2)}(t)$	$+b_0 \delta^{(N-1)}(t)$
$h^{(N)}(0) =$	$k_{N-1} \delta(t)$	$+k_{N-2} \delta^{(1)}(t) + \dots + k_1 \delta^{(N-2)}(t) + k_0 \delta^{(N-1)}(t)$	$+b_0 \delta^{(N)}(t)$

$$h(t) = b_0 \delta(t) + y_h(t)u(t)$$

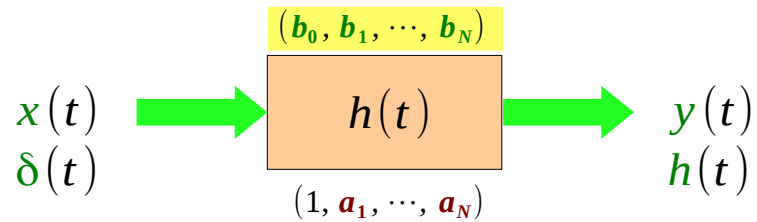
$\delta^{(N)}(t)$	$\delta^{(N-1)}(t)$	$\delta^{(1)}(t)$	
\uparrow	\uparrow	\uparrow	
$h^{(N)}(0) + a_1 h^{(N-1)}(0) + \dots + a_{N-1} h^{(1)}(0) + a_N h(0)$	$=$	$b_0 \delta^{(N)}(t) + b_1 \delta^{(N-1)}(t) + \dots + b_{N-1} \delta^{(1)}(t) + b_N \delta(t)$	

Impulse matching

-
- Superposition of an Impulse Response and its Derivatives

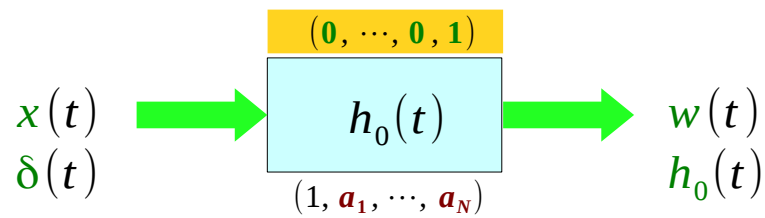
A General System S and a Base System S_0

General System S



$h(t)$: the impulse response of S

Base System S_0



$h_0(t)$: the impulse response of S_0

ZIR of a Base System S_0

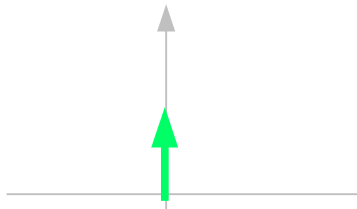
$N \geq M$: $h_0(t)$ include no impulse $\delta(t)$ ($t > 0$)

→ $h_0(t)$ include characteristic modes only

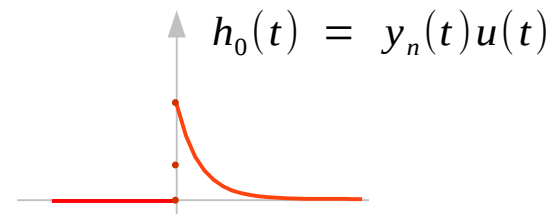
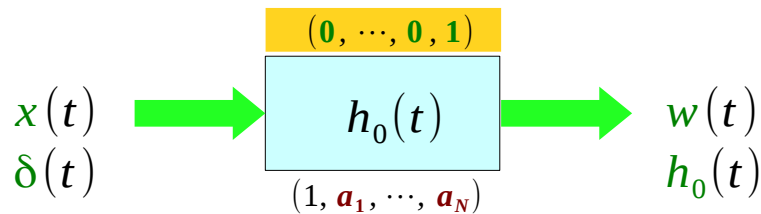
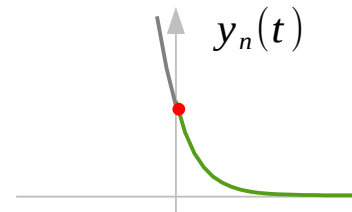
→ System S and S_0 have the same characteristic modes

$y_n(t)$: viewed as the zero input response of S_0

(cf) natural response all the lumped char modes
homogeneous response



an impulse function
generates energy storage
creates nonzero initial
condition at $t=0^+$



Superposition of inputs (1)

$$h^{(N)}(t) + \mathbf{a}_1 h^{(N-1)}(t) + \cdots + \mathbf{a}_{N-1} h^{(1)}(t) + \mathbf{a}_N h(t) = \mathbf{b}_0 \delta^{(N)}(t) + \mathbf{b}_1 \delta^{(N-1)}(t) + \cdots + \mathbf{b}_{N-1} \delta^{(1)}(t) + \mathbf{b}_N \delta(t)$$

$$\begin{aligned} h^{(N)} + \mathbf{a}_1 h^{(N-1)} + \cdots + \mathbf{a}_{N-1} h^{(1)} + \mathbf{a}_N h(t) &= \delta^{(N)} \\ h^{(N)} + \mathbf{a}_1 h^{(N-1)} + \cdots + \mathbf{a}_{N-1} h^{(1)} + \mathbf{a}_N h(t) &= \delta^{(N-1)} \\ \vdots & \\ h^{(N)} + \mathbf{a}_1 h^{(N-1)} + \cdots + \mathbf{a}_{N-1} h^{(1)} + \mathbf{a}_N h(t) &= \delta^{(1)} \\ h^{(N)} + \mathbf{a}_1 h^{(N-1)} + \cdots + \mathbf{a}_{N-1} h^{(1)} + \mathbf{a}_N h(t) &= \delta \end{aligned}$$

$$\begin{array}{l} N \\ N-1 \\ \vdots \\ 1 \\ 0 \end{array} \begin{array}{l} h(t) \\ h(t) \\ \vdots \\ h(t) \\ h(t) \end{array} \begin{array}{l} \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} \begin{array}{l} h_N(t) \\ h_{N-1}(t) \\ \vdots \\ h_1(t) \\ h_0(t) \end{array} = \begin{array}{l} h_0^{(N)}(t) \\ h_0^{(N-1)}(t) \\ \vdots \\ h_0^{(1)}(t) \\ h_0^{(0)}(t) \end{array}$$

$$\begin{aligned} h(t) &= \mathbf{b}_0 h_N + \mathbf{b}_1 h_{N-1} + \cdots + \mathbf{b}_{N-1} h_1 + \mathbf{b}_N h_0 \\ &= \mathbf{b}_0 h_0^{(N)} + \mathbf{b}_1 h_0^{(N-1)} + \cdots + \mathbf{b}_{N-1} h_0^{(1)} + \mathbf{b}_N h_0^{(0)} \end{aligned}$$

Superposition of inputs (2)

$$h^{(N)}(t) + \mathbf{a}_1 h^{(N-1)}(t) + \cdots + \mathbf{a}_{N-1} h^{(1)}(t) + \mathbf{a}_N h(t) = \mathbf{b}_0 \delta^{(N)}(t) + \mathbf{b}_1 \delta^{(N-1)}(t) + \cdots + \mathbf{b}_{N-1} \delta^{(1)}(t) + \mathbf{b}_N \delta(t)$$

$$h(t) = \mathbf{b}_0 h_0^{(N)} + \mathbf{b}_1 h_0^{(N-1)} + \cdots + \mathbf{b}_{N-1} h_0^{(1)} + \mathbf{b}_N h_0^{(0)}$$

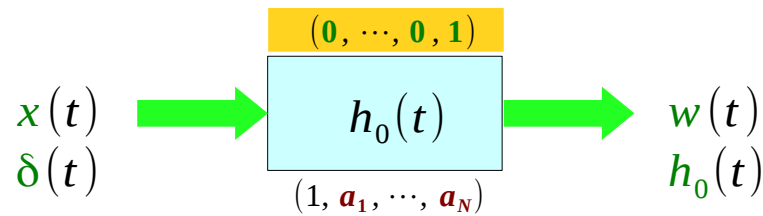
$$h^{(N)}(t) + \mathbf{a}_1 h^{(N-1)}(t) + \cdots + \mathbf{a}_{N-1} h^{(1)}(t) + \mathbf{a}_N h(t) = \delta(t)$$



$$h_0(t)$$

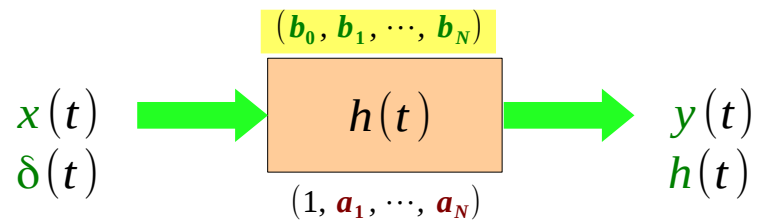


$h(t)$ of a General System S via $y_n(t)$



$$h_0 = \frac{1}{a_N} (\delta - h_0^{(N)} - a_1 h_0^{(N-1)} - \dots - a_{N-1} h_0^{(1)})$$

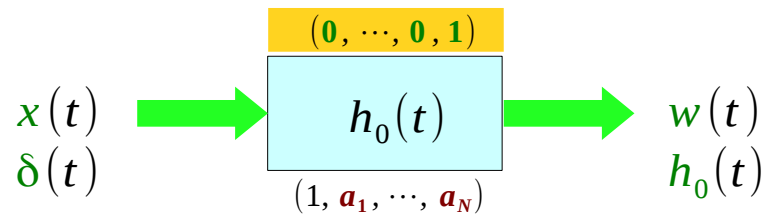
causal $h_0(t) = y_n(t)u(t)$



$$h(t) = b_0 h_0^{(N)}(t) + b_1 h_0^{(N-1)}(t) + \dots + b_N h_0^{(0)}(t)$$

causal $h(t) = b_0 \{y_n u\}^{(N)} + b_1 \{y_n u\}^{(N-1)} + \dots + b_N \{y_n u\}^{(0)}$

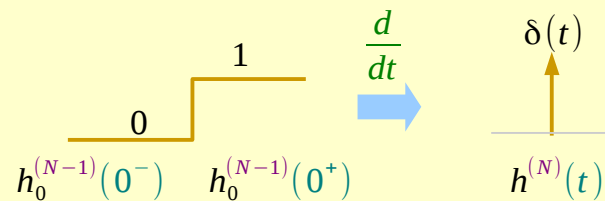
Impulse Matching of $h_0(t)$



$$h_0 = \frac{1}{a_N} (\delta - h_0^{(N)} - a_1 h_0^{(N-1)} - \dots - a_{N-1} h_0^{(1)})$$

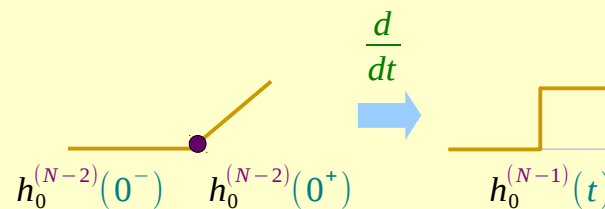
causal $h_0(t) = y_n(t)u(t)$

the only finite jump



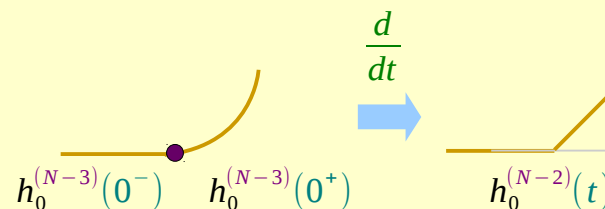
$$h_0^{(N-1)}(0^+) = 1$$

continuous



$$h_0^{(N-2)}(0^+) = 0$$

continuous

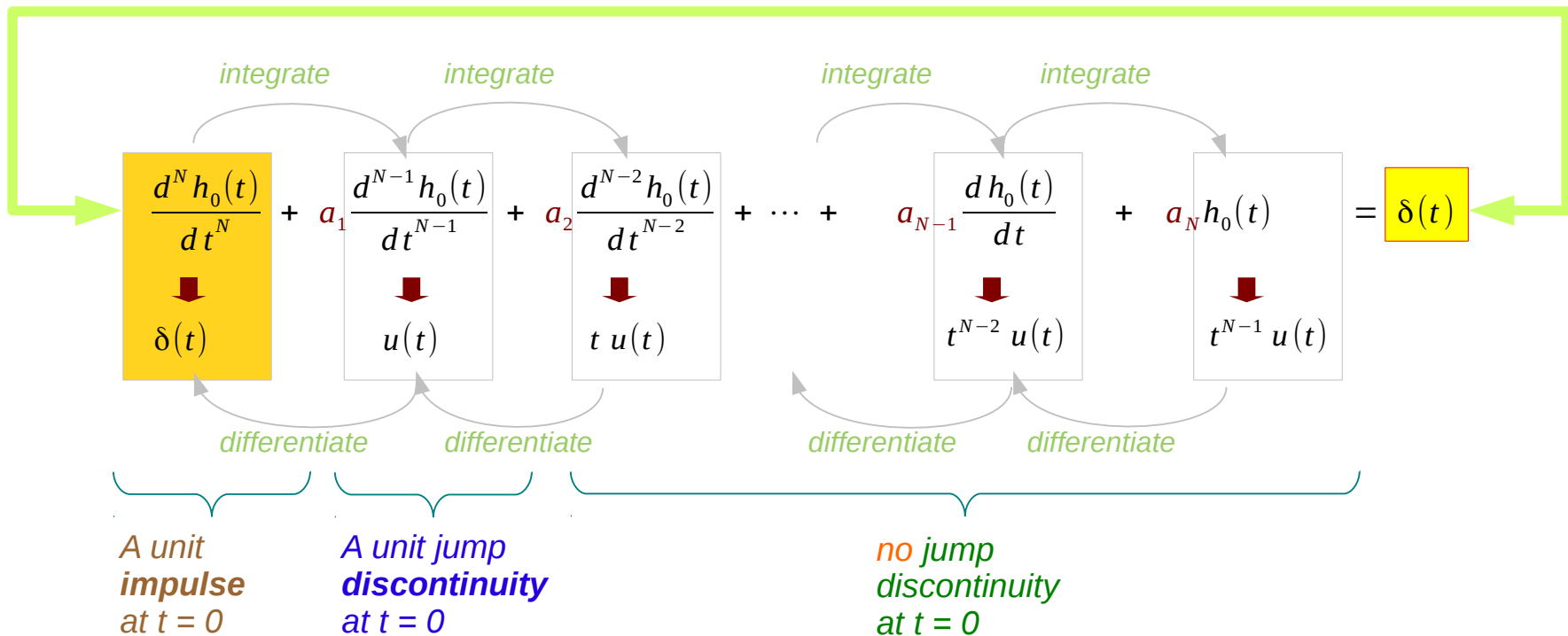


$$h_0^{(N-3)}(0^+) = 0$$

New Initial Condition created by $\delta(t)$

$$h^{(N)} + a_1 h^{(N-1)} + \dots + a_{N-1} h^{(1)} + a_N h(t) = \delta$$

Single Impulse at the right hand side creates a unique initial condition



$$h_0^{(N-1)}(0^+) = 1 \quad \underline{h_0^{(N-2)}(0^+) = h_0^{(N-1)}(0^+) = \dots = h_0^{(1)}(0^+) = h_0(0^+) = 0}$$

$h_0(t)$ and $y_n(t)$

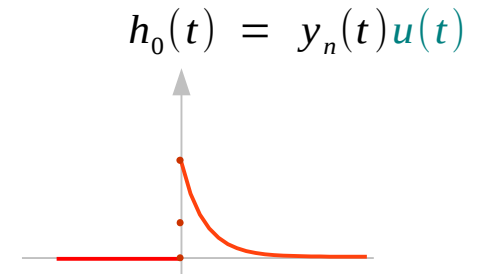
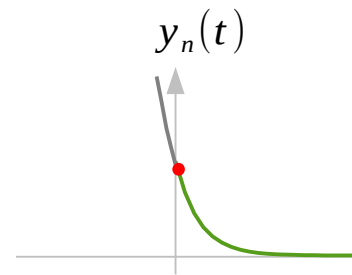
$$h^{(N)} + \mathbf{a}_1 h^{(N-1)} + \dots + \mathbf{a}_{N-1} h^{(1)} + \mathbf{a}_N h(t) = \delta \quad M=0$$

$$h_0^{(N-1)_n}(0^+) = 1 \quad h_0^{(N-2)}(0^+) = h_0^{(N-1)}(0^+) = \dots = h_0^{(1)}(0^+) = h_0(0^+) = 0$$

$N > M$: $h_0(t)$ include no impulse $\delta(t)$ ($t > 0$)

➔ $h_0(t)$ include characteristic modes only

➔ System **S** and **S0** have the same characteristic modes



$$y_n^{(N)} + \mathbf{a}_1 y_n^{(N-1)} + \dots + \mathbf{a}_{N-1} y_n^{(1)} + \mathbf{a}_N y_n(t) = 0 \quad \text{homogeneous solution}$$

$$y_n^{(N-1)_n}(0^+) = 1 \quad y_n^{(N-2)}(0^+) = y_n^{(N-1)}(0^+) = \dots = y_n^{(1)}(0^+) = y_n(0^+) = 0$$

Solve

$$y_n(t) = (c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \dots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N-1} t})$$



$$h_0(t) = y_n(t)u(t)$$

Simplified Impulse Matching

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) \cdot h(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \dots + b_{N-1} D + b_N) \cdot \delta(t)$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) \cdot h_0(t) = \delta(t)$$

$$Q(D) \cdot h(t) = P(D) \cdot \delta(t) \quad h(t) = P(D) \cdot h_0(t)$$

$$Q(D) \cdot h_0(t) = \delta(t)$$

$$t \geq 0^+ \quad h(t) = \text{characteristic mode terms}$$

$$t \geq 0 \quad h(t) = A_0 \delta(t) + \text{characteristic mode terms}$$

Simplified Impulse Matching Method



$$h(t) = b_0 \delta(t) + [P(D) h_0(t)] \cdot u(t)$$

$h_0(t)$ linear combination of characteristic modes with the following initial conditions

$$h_0(0^+) = h_0^{(1)}(0^+) = h_0^{(2)}(0^+) \dots = h_0^{(M-2)}(0^+) = 0 \quad h_0^{(M-1)}(0^+) = 1$$

Simplified Impulse Matching Method (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = \boxed{b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)}$$

Derived System **S**

shares the same characteristic modes

Base System **S0**

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = \boxed{x(t)}$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = \boxed{0}$$

$y_n(t)$ linear combination of characteristic modes with the following initial conditions

$$\boxed{y_n^{(N-1)}(0^+) = 1} \quad \boxed{y_n^{(N-2)}(0^+) = \dots = y_n^{(1)}(0^+) = y_n(0^+) = 0}$$

$y_h(t)$ Yet, another linear combination of characteristic modes

$$h(t) = b_0 \delta(t) + [P(D)y_n(t)]u(t)$$

becomes

$$h(t) = y_h(t)u(t)$$

Simplified Impulse Matching Method (2)

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) w(t) = x(t)$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) y(t) = (b_0 D^M + \dots + b_{N-1} D + b_N) x(t)$$

$$y(t) = (b_0 D^M + \dots + b_{N-1} D + b_N) w(t)$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) y_n(t) = \delta(t)$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N) h(t) = (b_0 D^M + \dots + b_{N-1} D + b_N) \delta(t)$$

$$h(t) = (b_0 D^M + \dots + b_{N-1} D + b_N) y_n(t)$$

$$Q(D)w(t) = x(t)$$

$$Q(D)P(D)w(t) = P(D)x(t)$$

$$y(t) = P(D)w(t)$$

$$Q(D)y_n(t) = \delta(t)$$

$$Q(D)P(D)y_n(t) = P(D)\delta(t)$$

$$h(t) = P(D)y_n(t)$$

No interval restriction

Causality is considered *causal* $y_n(t)u(t)$

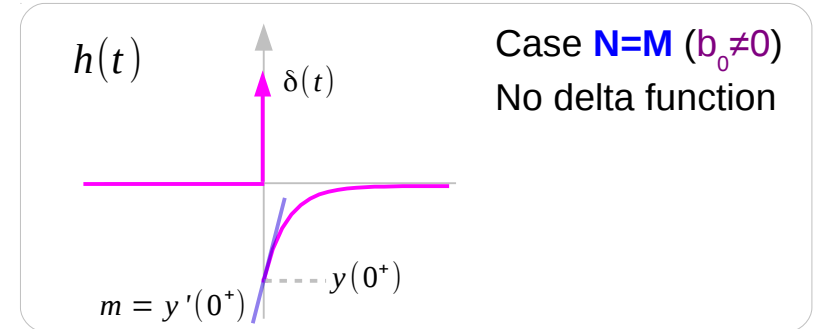
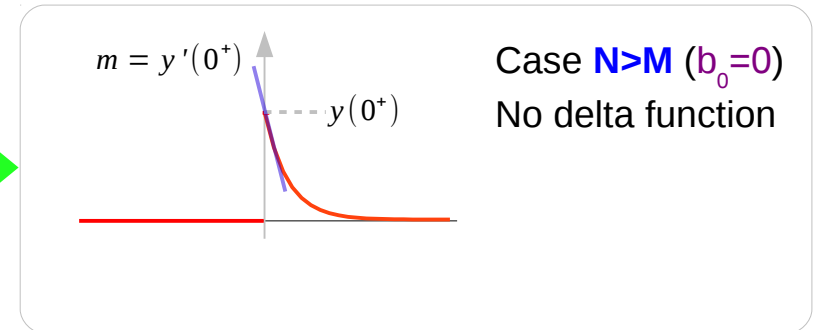
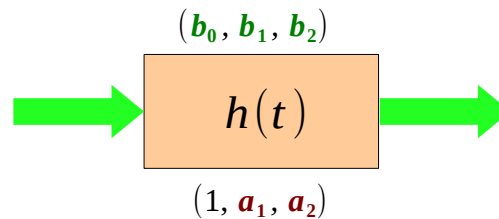
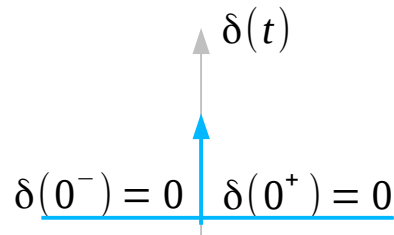
$$h(t) = P(D)[y_n(t)u(t)] \longrightarrow [P(D)y_n(t)]u(t)$$

$$h(t) = b_0 \delta(t) + P(D)y_n(t), \quad t \geq 0$$

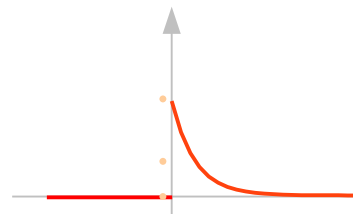
$$h(t) = b_0 \delta(t) + [P(D)y_n(t)]u(t)$$

Impulse Input

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_2 y(t) = b_0 \frac{d^2 x(t)}{dt^2} + b_1 \frac{dx(t)}{dt} + b_2 x(t)$$



Any two functions that have finite values everywhere and **differ** in value only at a finite number of points are **equivalent** in the system response or transform



All initial conditions are zero at $t=0^-$ generates energy storage creates nonzero initial condition at $t=0^+$

$$y(0^-) = 0$$

$$y'(0^-) = 0$$

$$y(0^+) = K_1$$

$$y'(0^+) = K_2$$

IVP (Initial Value Problem)

Derivatives of a delta function at the output

$$\frac{d^2 h(t)}{dt^2} + a_1 \frac{dh(t)}{dt} + a_2 h(t) = b_0 \frac{d^2 \delta(t)}{dt^2} + b_1 \frac{d\delta(t)}{dt} + b_2 \delta(t)$$

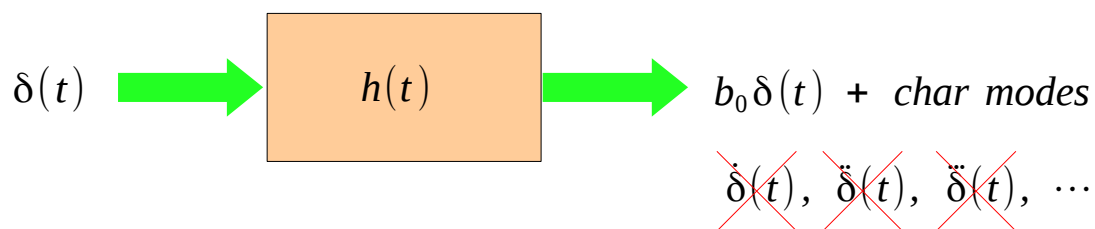
if $h(t)$ contains $\delta^{(1)}(t)$ ($b_0 \neq 0$)

$$\delta^{(3)}(t) + \dots \neq b_0 \delta^{(2)}(t) + b_1 \delta^{(1)}(t) + b_2 \delta(t)$$

the impulse response $h(t)$ can contain at most $\delta(t)$ ($b_0 \neq 0$)

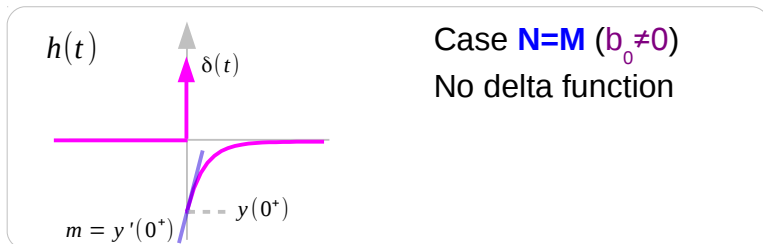
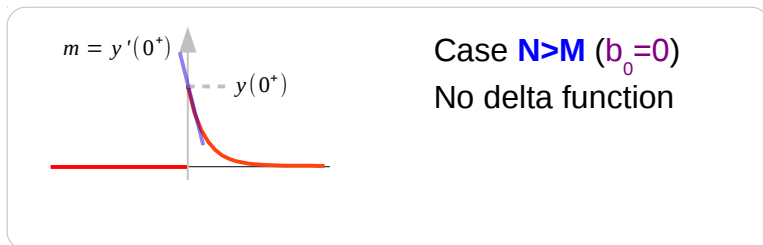
the impulse response $h(t)$ cannot contain any derivatives of $\delta(t)$

$$N \geq M$$



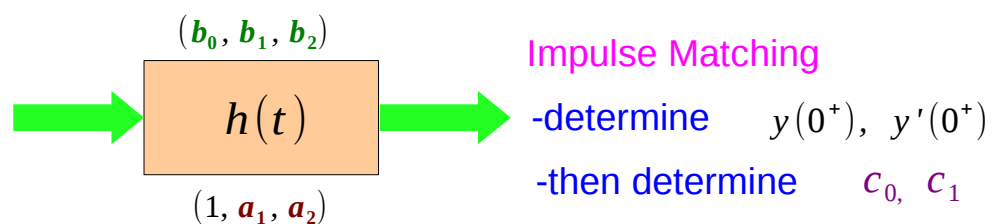
Impulse Response $h(t)$

$$\frac{d^2 h(t)}{dt^2} + a_1 \frac{dh(t)}{dt} + a_2 h(t) = b_0 \frac{d^2 \delta(t)}{dt^2} + b_1 \frac{d\delta(t)}{dt} + b_2 \delta(t)$$

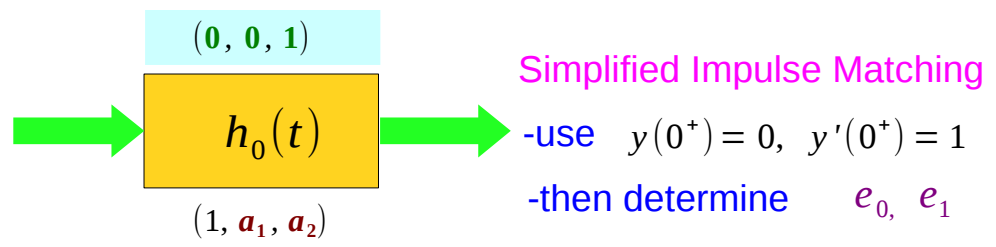


- $t \geq 0^+$ $h(t)$ = characteristic mode terms only
- $t = 0$ $h(t)$ can have at most an impulse $b_0 \delta(t)$ or finite jump

$$h(t) = b_0 \delta(t) + \text{char mode terms } t \geq 0$$



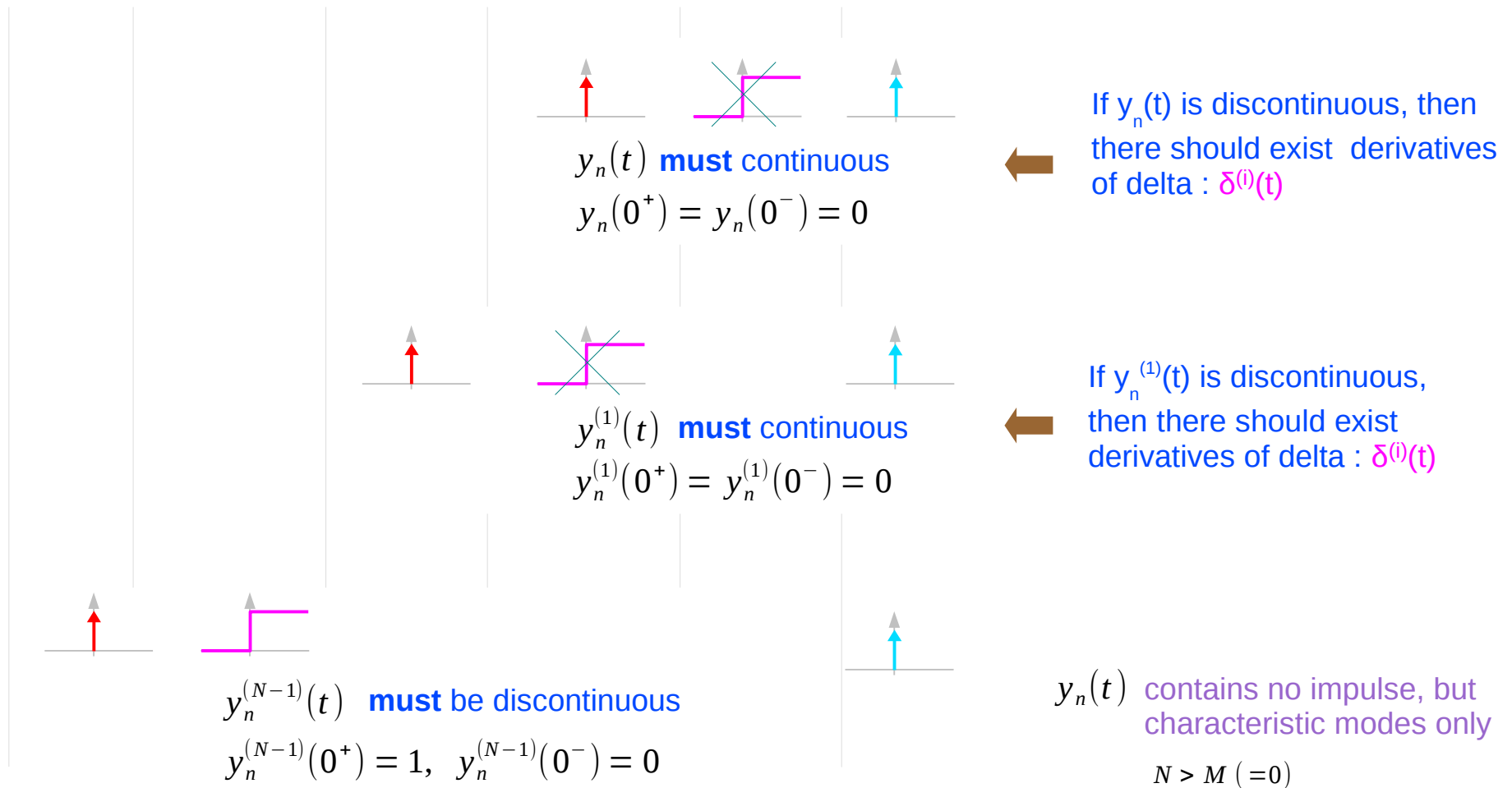
$$h(t) = b_0 \delta(t) + \left(\sum_i c_i e^{-\lambda_i t} \right) u(t)$$



$$h(t) = b_0 \delta(t) + \left[(b_0 D^2 + b_1 D + b_2) \left(\sum_i e_i e^{-\lambda_i t} \right) \right] u(t)$$

Base System Impulse Matching

$$y_n^{(N)}(t) + a_1 y_n^{(N-1)}(t) + \dots + a_{N-1} y_n^{(1)}(t) + a_N y_n(t) = \delta(t)$$



Impulse Response & Particular Solution

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \dots + b_{M-1} \frac{dx(t)}{dt} + b_M x(t)$$

$y(t) = h(t)$: impulse response

when $x(t) = \delta(t)$: forcing function

	causal system	interval of existence / uniqueness		
	$t < 0$	$t = 0$	$t > 0$	
forcing function	$x(t) = 0$	$\delta(t)$	0	
particular solution	$y_p(t) = 0$	$A\delta(t)$	0	$\left\{ \begin{array}{l} A=0 \quad (N>M) \\ A \neq 0 \quad (N=M) \end{array} \right.$
homogeneous solution	$y_h(t) = 0$	$y_h(t)$	$y_h(t)$	

impulse response

$$\left\{ \begin{array}{l} h(t) = y_h(t)u(t) \\ h(t) = y_h(t)u(t) + A\delta(t) \text{ only when } N=M \end{array} \right.$$

Impulse Response of Differential Equations

(t > 0) Homogeneous solution

$$a_0 \frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = 0$$

$y_h(t)$ **homogeneous solution**
- characteristic modes only

$$\begin{cases} h(t) = y_h(t)u(t) \\ h(t) = y_h(t)u(t) + A\delta(t) \text{ only when } N=M \end{cases}$$

linear combination of all the derivatives of $h(t)$ **must add to zero** for any time $t \neq 0$

(t = 0) Particular solution

$$\begin{aligned} & a_0 \frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) \\ &= b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \dots + b_{M-1} \frac{d\delta(t)}{dt} + b_M \delta(t) \end{aligned}$$

$y_p(t)$ **particular solution**
- linear combination of the *forcing function* $x(t)$ and *all its unique derivatives* $x^{(i)}(t)$

$$y_p(t) \leftarrow \begin{cases} y(t) = b_0 \delta(t) \\ y^{(i)}(t) = d_i \delta^{(i)}(t) \end{cases} \leftarrow \begin{cases} x(t) = \delta(t) \\ x^{(i)}(t) = \delta^{(i)}(t) \end{cases}$$

linear combination of an **impulse** and **its unique derivatives** (the doublet, the triplet, etc) : all these exist at time $t = 0$



Particular Solution at $t=0$

$$\frac{d^N y(t)}{dt^N} + \mathbf{a}_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + \mathbf{a}_{N-1} \frac{dy(t)}{dt} + \mathbf{a}_N y(t) = \mathbf{b}_0 \frac{d^M x(t)}{dt^M} + \mathbf{b}_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + \mathbf{b}_{M-1} \frac{dx(t)}{dt} + \mathbf{b}_M x(t)$$

when $x(t) = \delta(t)$: forcing function

$$y_p(t) = \mathbf{b}_0 \delta^{(M)}(t) + \mathbf{b}_1 \delta^{(M-1)}(t) + \cdots + \mathbf{b}_{M-1} \delta^{(1)}(t) + \mathbf{b}_M \delta(t)$$

$$\frac{d^N y(t)}{dt^N} + \mathbf{a}_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + \mathbf{a}_{N-1} \frac{dy(t)}{dt} + \mathbf{a}_N y(t) = x(t)$$

when $x(t) = \delta(t)$: forcing function

$$y_p(t) = \delta(t)$$

Particular Solutions for $x(t)=\delta(t)$

Method of Undetermined coefficients

Particular solution

$$\mathbf{a}_0 \frac{d^N h(t)}{dt^N} + \mathbf{a}_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + \mathbf{a}_{N-1} \frac{dh(t)}{dt} + \mathbf{a}_N h(t) = \mathbf{b}_0 \frac{d^M \delta(t)}{dt^M} + \mathbf{b}_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \dots + \mathbf{b}_{M-1} \frac{d\delta(t)}{dt} + \mathbf{b}_M \delta(t)$$

$h(t)$ is differentiated up to n times

← must match →

$\delta(t)$ is differentiated up to m times

Finding particular solution

We can determine whether $h(t)$ can contain an impulse or its unique derivatives

$$\int_{0^-}^{0^+} h(t) dt = 0 \quad \text{when } h(t) \text{ does not have an impulse or its derivatives}$$

$$\int_{0^-}^{0^+} h(t) dt \neq 0 \quad \text{Otherwise}$$

$h(t)$ can have at most a $\delta(t)$

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \dots + a_{N-1} \frac{dh(t)}{dt} + a_N h(t) = b_0 \frac{d^N \delta(t)}{dt^N} + b_1 \frac{d^{N-1} \delta(t)}{dt^{N-1}} + \dots + b_{N-1} \frac{d\delta(t)}{dt} + b_N \delta(t)$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N)h(t) = (b_0 D^N + b_1 D^{N-1} + \dots + b_{N-1} D + b_N)\delta(t)$$

$$M = N$$

$$(D^N + a_1 D^{N-1} + \dots + a_{N-1} D + a_N)h(t) = (b_0 D^N + b_1 D^{N-1} + \dots + b_{N-1} D + b_N)\delta(t)$$

If $\delta^{(1)}(t)$ is included in $h(t)$, then the highest order term

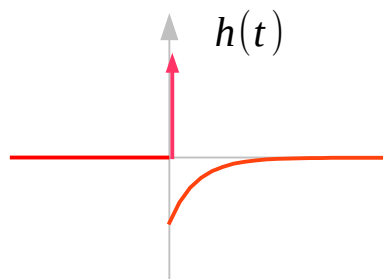
$$D^N \delta^{(1)}(t) \longrightarrow \delta^{(N+1)}(t)$$

$$b_0 D^N \delta(t) \longrightarrow \delta^{(N)}(t)$$

Contradiction!!!

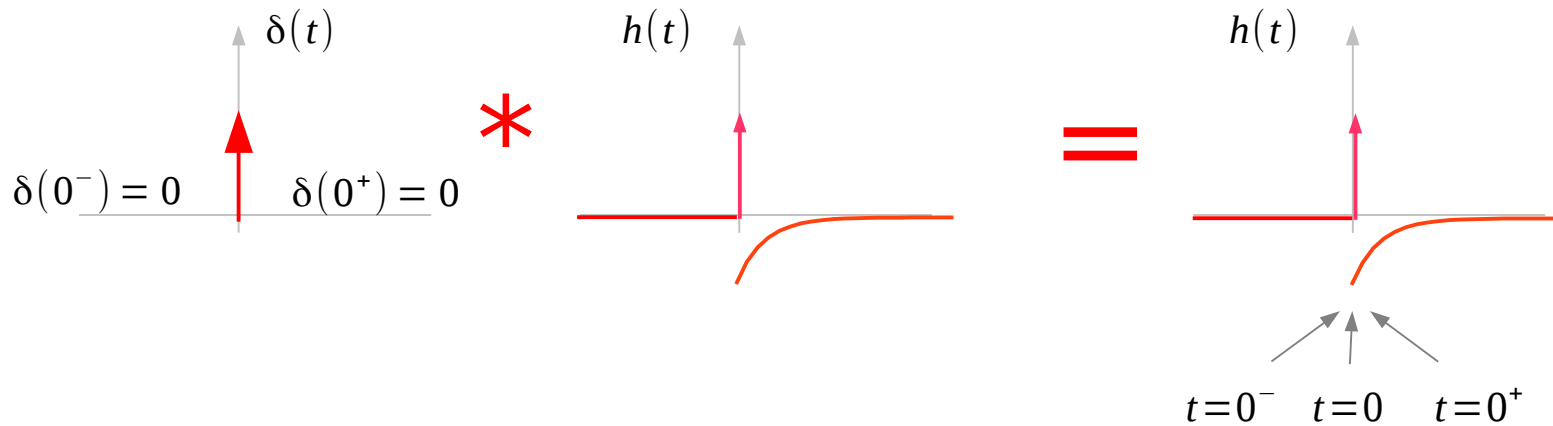
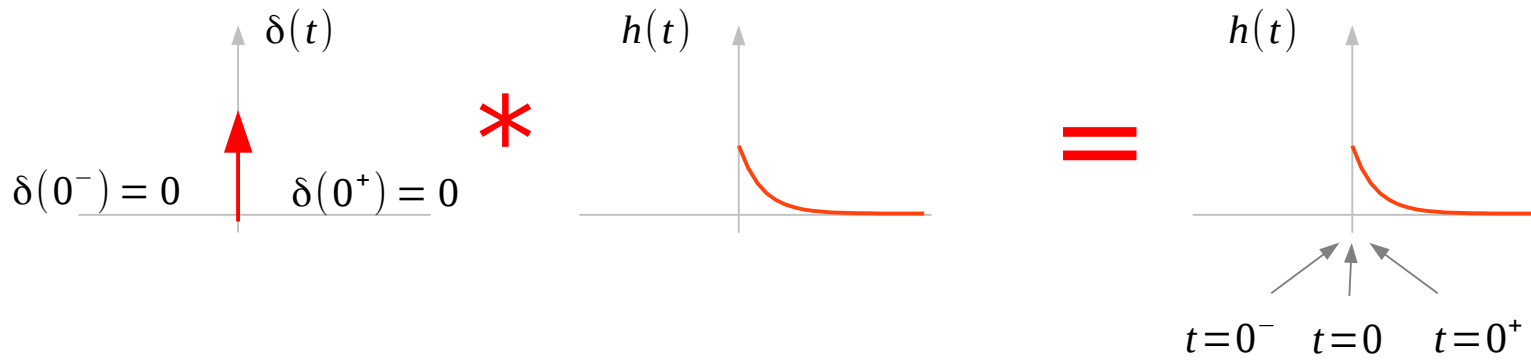
$h(t)$ cannot contain $\delta^{(i)}(t)$ at all \rightarrow

$h(t)$ can contain at most $\delta(t)$ $M \leq N$

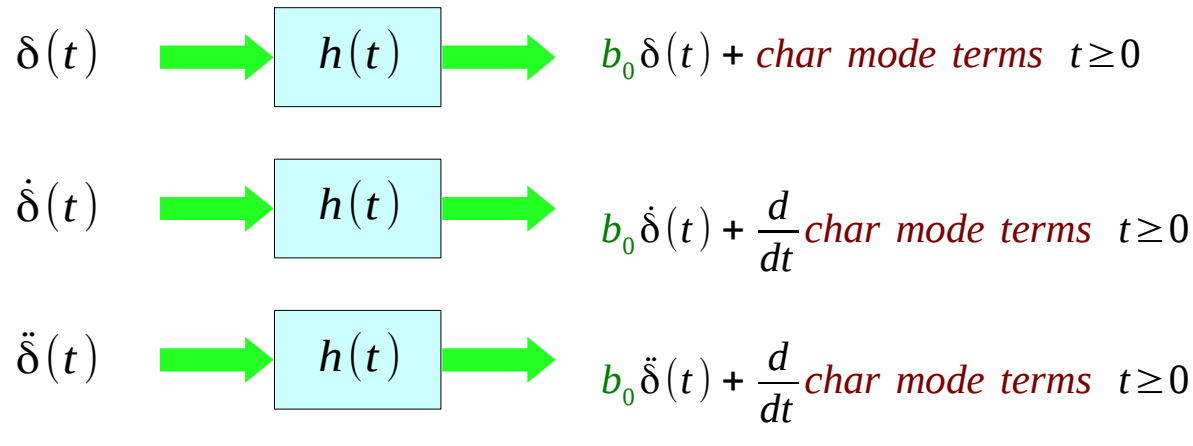


$$h(t) = b_0 \delta(t) + \text{char mode terms } t \geq 0$$

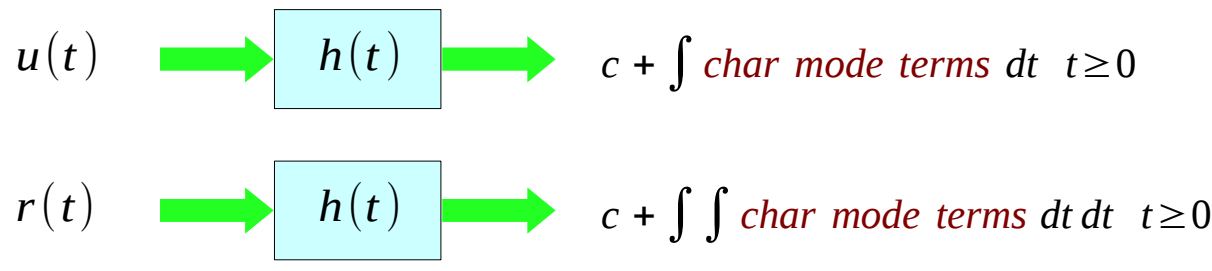
Impulse Response $h(t)$



Initial Conditions & Total Response



linear combination of an **impulse** and **its unique derivatives** (the doublet, the triplet, etc) : all these exist at time $t = 0$



no **impulse** and **its unique derivatives** at time $t = 0$

Generally, the interval of interest for $y(t)$ is $t > 0$

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] B.P. Lathi, Linear Systems and Signals (2nd Ed)