Laurent Series and z-Transform
- Geometric Series
Applications A

20200204 Tue

Combinations of a and z -- common ratio

$$
\begin{tikzcd}
a^{n} & & a^{-n} \\
\boxed{a\, z} \arrow[d, shift right, leftrightarrow, "\boxed{-1}"'] \arrow[d, shift left, leftrightarrow, "\boxed{-1}"] & & \boxed{a^{\boxed{-1}}\, z} \arrow[d, shift right, leftrightarrow, "\boxed{-1}"'] \arrow[d, shift left, leftrightarrow, "\boxed{-1}"] \\
\boxed{a^{\boxed{-1}}\, z^{\boxed{-1}}} & & \boxed{a\, z^{\boxed{-1}}} \\
a^{n} & & a^{-n}
\end{tikzcd}
$$

the same formula,
different representations

$$az \qquad \frac{1}{1-az} = \frac{a^{-1}z^{-1}}{a^{-1}z^{-1}-1}$$

$$a^{\boxed{-1}}z^{\boxed{-1}} \qquad \frac{1}{1-a^{-1}z^{-1}} = \frac{az}{az-1}$$

$$a^{\boxed{-1}}z \qquad \frac{1}{1-a^{-1}z} = \frac{az^{-1}}{az^{-1}-1}$$

$$az^{\boxed{-1}} \qquad \frac{1}{1-az^{-1}} = \frac{a^{-1}z}{a^{-1}z-1}$$

the same formula
with different ROCs

different Geometric Series

$a\,z$	$\dfrac{1}{1-az}$ $\quad |z|<a^{-1}$	$-\dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\quad |z|>a^{-1}$
	$a^0z^0+a^1z^1+a^2z^2+\cdots$	$a^{-1}z^{-1}+a^{-2}z^{-2}+a^{-3}z^{-3}+\cdots$
$a^{\boxed{-1}}z^{\boxed{-1}}$	$\dfrac{1}{1-a^{-1}z^{-1}}$ $\quad |z|>a^{-1}$	$-\dfrac{az}{1-az}$ $\quad |z|<a^{-1}$
	$a^0z^0+a^{-1}z^{-1}+a^{-2}z^{-2}+\cdots$	$a^1z^1+a^2z^2+a^3z^3+\cdots$
$a^{\boxed{-1}}\,z$	$\dfrac{1}{1-a^{-1}z}$ $\quad |z|<a$	$-\dfrac{az^{-1}}{1-az^{-1}}$ $\quad |z|>a$
	$a^0z^0+a^{-1}z^1+a^{-2}z^2+\cdots$	$a^1z^{-1}+a^2z^{-2}+a^3z^{-3}+\cdots$
$a\,z^{\boxed{-1}}$	$\dfrac{1}{1-az^{-1}}$ $\quad |z|>a$	$-\dfrac{a^{-1}z}{1-a^{-1}z}$ $\quad |z|<a$
	$a^0z^0+a^1z^{-1}+a^2z^{-2}+\cdots$	$a^{-1}z^1+a^{-2}z^2+a^{-3}z^3+\cdots$
	geometric series starting with a unit term	geometric series starting with a non-unit term

Each representation has it own ROC
(Region of Convergence)

Geometric Power Series Property

the same formula with different ROCs

different Geometric Series -- Shifted Range

az	$\frac{1}{1-az}$ $\quad\|z\|<a^{-1}$ $a^0z^0 + a^1z^1 + a^2z^2 + \cdots$	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\quad\|z\|>a^{-1}$ $a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots$	left shifted range, inverted
$a^{-1}z^{-1}$	$\frac{1}{1-a^{-1}z^{-1}}$ $\quad\|z\|>a^{-1}$ $a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots$	$-\frac{az}{1-az}$ $\quad\|z\|<a^{-1}$ $a^1z^1 + a^2z^2 + a^3z^3 + \cdots$	right shifted range, inverted
$a^{-1}z$	$\frac{1}{1-a^{-1}z}$ $\quad\|z\|<a$ $a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots$	$-\frac{az^{-1}}{1-az^{-1}}$ $\quad\|z\|>a$ $a^1z^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$	left shifted range, inverted
az^{-1}	$\frac{1}{1-az^{-1}}$ $\quad\|z\|>a$ $a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots$	$-\frac{a^{-1}z}{1-a^{-1}z}$ $\quad\|z\|<a$ $a^{-1}z^1 + a^{-2}z^2 + a^{-3}z^3 + \cdots$	right shifted range, inverted
	geometric series starting with a unit term	geometric series starting with a non-unit term	

the same formula with different ROCs

different Geometric Series -- Complementary Relation

* inverted relation is ignored

az	causal: $\frac{1}{1-az}$, $\lvert z\rvert < a^{-1}$ $a^0z^0 + a^1z^1 + a^2z^2 + \cdots$	anti-causal: $\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$, $\lvert z\rvert > a^{-1}$ $a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots$	left shifted,
$a^{\boxed{-1}}z^{\boxed{-1}}$	anti-causal: $\frac{1}{1-a^{-1}z^{-1}}$, $\lvert z\rvert > a^{-1}$ $a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots$	causal: $\frac{az}{1-az}$, $\lvert z\rvert < a^{-1}$ $a^1z^1 + a^2z^2 + a^3z^3 + \cdots$	right shifted,
$a^{\boxed{-1}}z$	causal: $\frac{1}{1-a^{-1}z}$, $\lvert z\rvert < a$ $a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots$	anti-causal: $\frac{az^{-1}}{1-az^{-1}}$, $\lvert z\rvert > a$ $a^1z^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$	left shifted,
$az^{\boxed{-1}}$	anti-causal: $\frac{1}{1-az^{-1}}$, $\lvert z\rvert > a$ $a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots$	causal: $-\frac{a^{-1}z}{1-a^{-1}z}$, $\lvert z\rvert < a$ $a^{-1}z^1 + a^{-2}z^2 + a^{-3}z^3 + \cdots$	right shifted,
	geometric series starting with a unit term	geometric series starting with a common ratio term	

Common Ratio and ROC

	a^n		a^{-n}	
		$n = 0, 1, 2, \cdots$		$n = 0, 1, 2, \cdots$
left shifted range		$n = -1, -2, -3, \cdots$		$n = -1, -2, -3, \cdots$
	$\boxed{a\,z}$	$\lvert z\rvert < a^{-1}$ $\lvert z\rvert > a^{-1}$	$\boxed{a^{-1}\,z}$	$\lvert z\rvert < a$ $\lvert z\rvert > a$
	$\boxed{a^{-1}\,z^{-1}}$	$\lvert z\rvert > a^{-1}$ $\lvert z\rvert < a^{-1}$	$\boxed{a\,z^{-1}}$	$\lvert z\rvert > a$ $\lvert z\rvert < a$
	a^n	$n = 0, -1, -2, \cdots$	a^{-n}	$n = 0, -1, -2, \cdots$
right shifted range		$n = 1, 2, 3, \cdots$		$n = 1, 2, 3, \cdots$

Each common ratio has two represenations

Each representation has it own ROC

The two representations have complementary ROC's

each common ratio is associated with
2 different sequences (represenations)

$\boxed{a\,z}$

$\dfrac{1}{1-az}$ $\quad |z| < a^{-1}$

$a^0 z^0 + a^1 z^1 + a^2 z^2 + \cdots$

starting with a common ratio term

$-\dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\quad |z| > a^{-1}$

$a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \cdots$

left shifted range, inverted

$\boxed{a^{\boxed{-1}}\,z^{\boxed{-1}}}$

$\dfrac{1}{1-a^{-1}z^{-1}}$ $\quad |z| > a^{-1}$

$a^0 z^0 + a^{-1} z^{-1} + a^{-2} z^{-2} + \cdots$

starting with a common ratio term

$-\dfrac{az}{1-az}$ $\quad |z| < a^{-1}$

$a^1 z^1 + a^2 z^2 + a^3 z^3 + \cdots$

right shifted range, inverted

$\boxed{a^{\boxed{-1}}\,z}$

$\dfrac{1}{1-a^{-1}z}$ $\quad |z| < a$

$a^0 z^0 + a^{-1} z^1 + a^{-2} z^2 + \cdots$

starting with a common ratio term

$-\dfrac{az^{-1}}{1-az^{-1}}$ $\quad |z| > a$

$a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$

left shifted range, inverted

$\boxed{a\,z^{\boxed{-1}}}$

$\dfrac{1}{1-az^{-1}}$ $\quad |z| > a$

$a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \cdots$

starting with a common ratio term

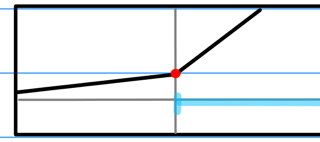
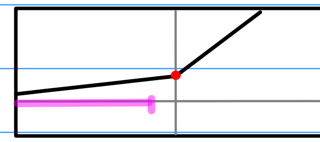
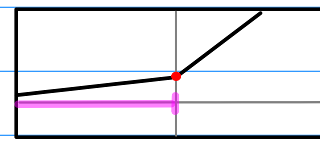
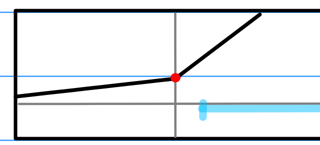
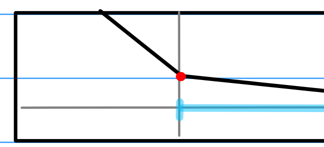
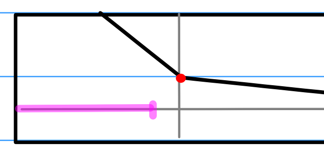
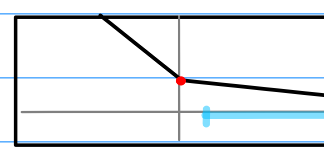
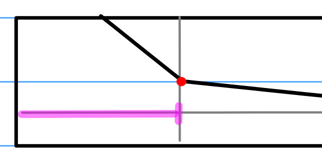
$-\dfrac{a^{-1}z}{1-a^{-1}z}$ $\quad |z| < a$

$a^{-1} z^1 + a^{-2} z^2 + a^{-3} z^3 + \cdots$

right shifted range, inverted

[Complementary Range & Inverted Relation]

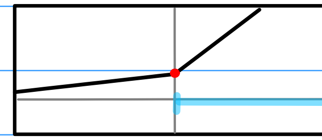
* inverted relation is ignored

$a\ z$	$\dfrac{1}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^0z^0 + a^1z^1 + a^2z^2 + \cdots$ a^n	
$a^{\boxed{-1}}\ z^{\boxed{-1}}$	$\dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots$ a^n	
$a^{\boxed{-1}}\ z^{\boxed{-1}}$	$\dfrac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots$ a^n	
$a\ z$	$\dfrac{az}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^1z^1 + a^2z^2 + a^3z^3 + \cdots$ a^n	
$a^{\boxed{-1}}\ z$	$\dfrac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$	$a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots$ a^{-n}	
$a\ z^{\boxed{-1}}$	$\dfrac{az^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$a^1z^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$ a^{-n}	
$a\ z^{\boxed{-1}}$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > a$	$a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots$ a^{-n}	
$a^{\boxed{-1}}\ z$	$\dfrac{a^{-1}z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$a^{-1}z^1 + a^{-2}z^2 + a^{-3}z^3 + \cdots$ a^{-n}	

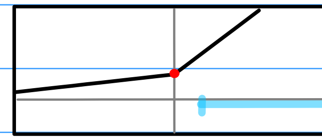
[Shifted Range Relation]

* inverted relation is ignored

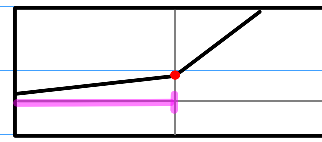
$a\ z$ | $\dfrac{1}{1-az}$ $\ |z|<a^{-1}$ | $a^0z^0 + a^1z^1 + a^2z^2 + \cdots$; a^n



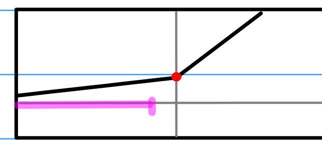
$a\ z$ | $\dfrac{az}{1-az}$ $\ |z|<a^{-1}$ | $a^1z^1 + a^2z^2 + a^3z^3 + \cdots$; a^n



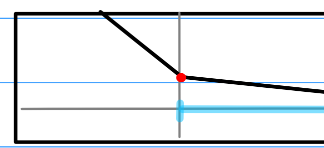
$a^{\boxed{-1}}z^{\boxed{-1}}$ | $\dfrac{1}{1-a^{-1}z^{-1}}$ $\ |z|>a^{-1}$ | $a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots$; a^n



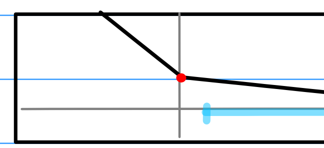
$a^{\boxed{-1}}z^{\boxed{-1}}$ | $\dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\ |z|>a^{-1}$ | $a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots$; a^n



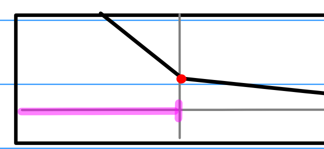
$a^{\boxed{-1}}\ z$ | $\dfrac{1}{1-a^{-1}z}$ $\ |z|<a$ | $a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots$; a^{-n}



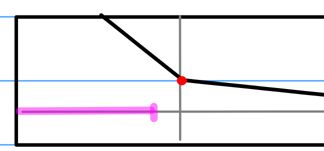
$a^{\boxed{-1}}\ z$ | $\dfrac{a^{-1}z}{1-a^{-1}z}$ $\ |z|<a$ | $a^{-1}z^1 + a^{-2}z^2 + a^{-3}z^3 + \cdots$; a^{-n}



$a\ z^{\boxed{-1}}$ | $\dfrac{1}{1-az^{-1}}$ $\ |z|>a$ | $a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots$; a^{-n}



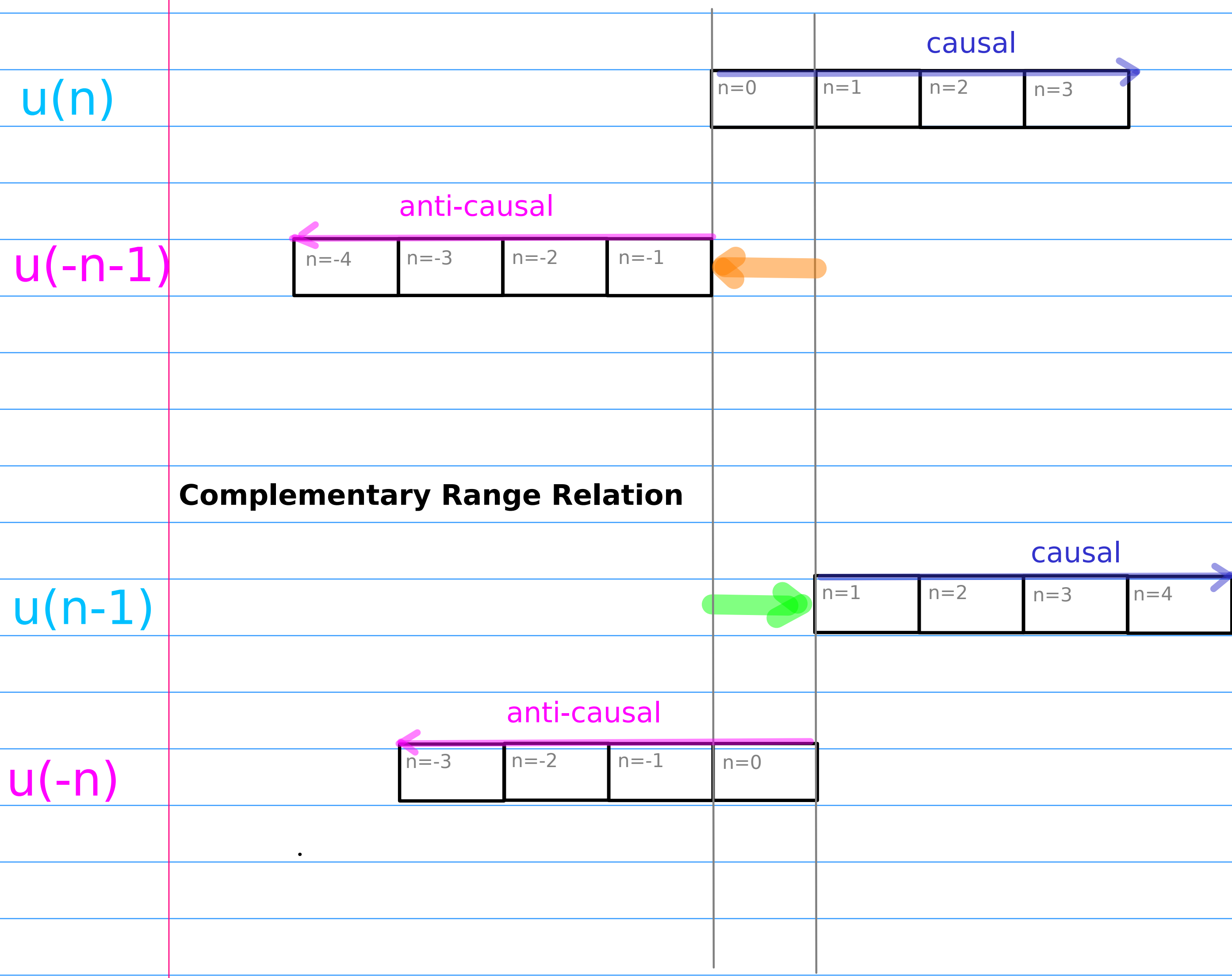
$a\ z^{\boxed{-1}}$ | $\dfrac{az^{-1}}{1-az^{-1}}$ $\ |z|>a$ | $a^1z^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$; a^{-n}



Complementary Relations of Ranges

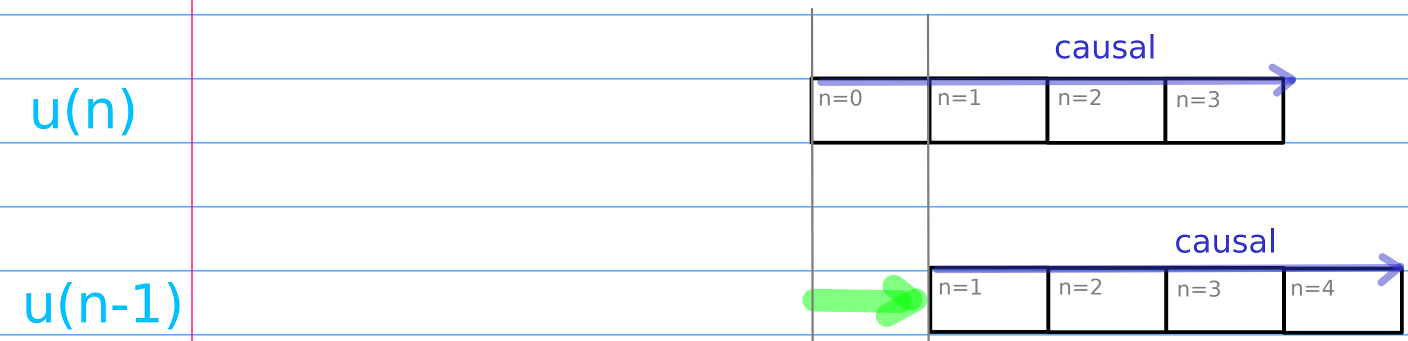
Complementary Range Relation

Complementary Range Relation

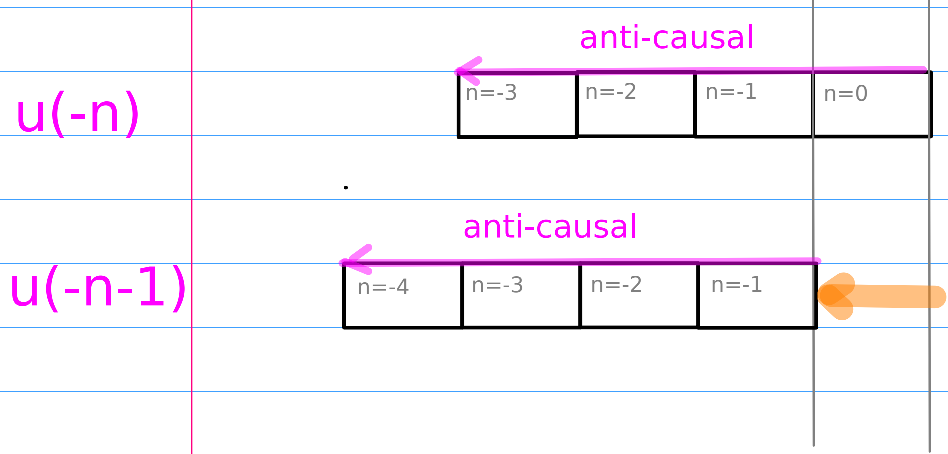


Shift Relations of Ranges

Right Shifted Range Relation



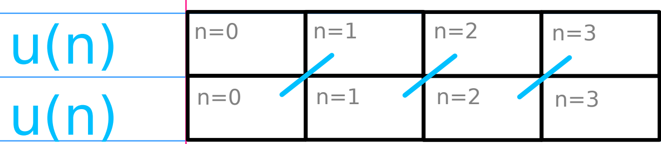
Left Shifted Range Relation



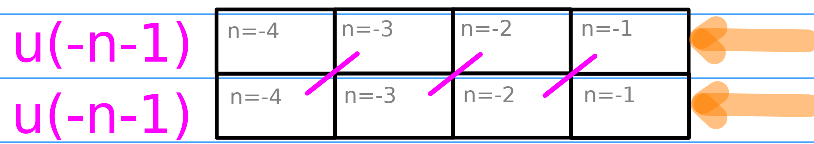
Making Shifted Sequences

making left shifted sequences

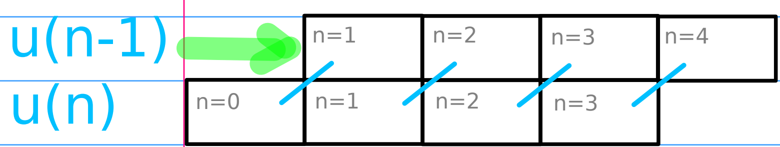
the same fixed slots,
left shifted samples



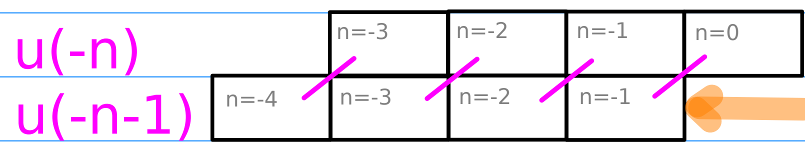
both left shifted slots
left shifted samples



right shifted pre-slot
the same fixed samples

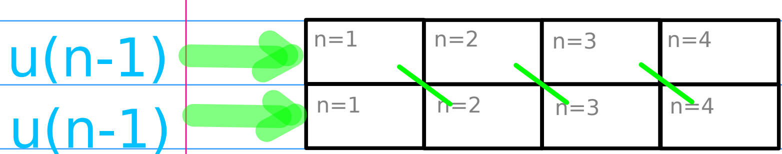


left shifted post-slot
the same fixed samples

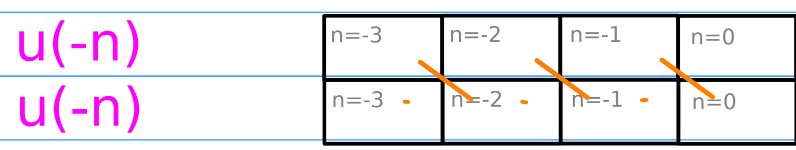


making right shifted sequences

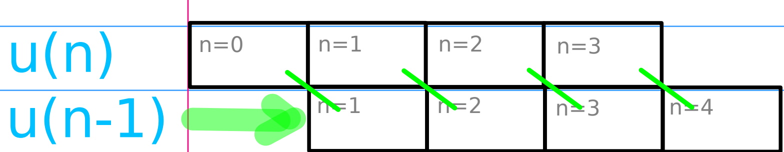
right shifted both slots
right shifted samples



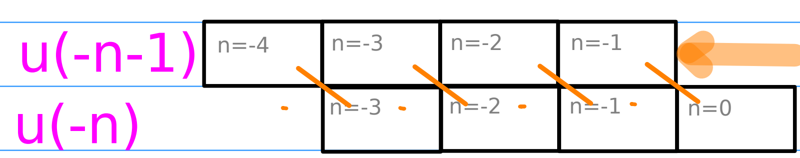
the same fixed slots
right shifted samples



right shifted post-slot
the same fixed samples



left shifted pre-slot
the same fixed samples



Two Types of Left-Shifted Causal Sequences

the same fixed slots

$u(n)$	n=0	n=1	n=2	n=3
$u(n)$	n=0	n=1	n=2	n=3

left-shift post-samples

a^0	a^1	a^2	a^3
a^1	a^2	a^3	a^4

a^n	a^0	a^1	a^2	a^3
a^{n+1}	a^1	a^2	a^3	a^4

left-shifted sequence (I)

right-shift pre-slot

$u(n-1)$		n=1	n=2	n=3	n=4
$u(n)$	n=0	n=1	n=2	n=3	

fixed samples

a^1	a^2	a^3	a^4
a^1	a^2	a^3	a^4

a^n		a^1	a^2	a^3	a^4
a^{n+1}	a^1	a^2	a^3	a^4	

left-shifted sequence (II)

Two Types of Left-Shifted Anti-Causal Sequences

left shift both slots

u(-n-1)	n=-4	n=-3	n=-2	n=-1
u(-n-1)	n=-4	n=-3	n=-2	n=-1

left-shift post-samples

a^{-4}	a^{-3}	a^{-2}	a^{-1}
a^{-3}	a^{-2}	a^{-1}	a^{0}

a^n	a^{-4}	a^{-3}	a^{-2}	a^{-1}
a^{n+1}	a^{-3}	a^{-2}	a^{-1}	a^{0}

left-shifted sequence (I)

left-shift post-slot

u(-n)		n=-3	n=-2	n=-1	n=0
u(-n-1)	n=-4	n=-3	n=-2	n=-1	

fixed samples

a^{-3}	a^{-2}	a^{-1}	a^{0}
a^{-3}	a^{-2}	a^{-1}	a^{0}

a^n		a^{-3}	a^{-2}	a^{-1}	a^{0}
a^{n+1}	a^{-3}	a^{-2}	a^{-1}	a^{0}	

left-shifted sequence (II)

Two Types of Right-Shifted Causal Sequences

right shift both slots

u(n-1)	n=1	n=2	n=3	n=4
u(n-1)	n=1	n=2	n=3	n=4

right shift post-samples

$a^1 \quad a^2 \quad a^3 \quad a^4$

$a^0 \quad a^1 \quad a^2 \quad a^3$

a^n	a^1	a^2	a^3	a^4
a^{n-1}	a^0	a^1	a^2	a^3

right-shifted sequence (I)

right shift post-slot

u(n)	n=0	n=1	n=2	n=3	
u(n-1)		n=1	n=2	n=3	n=4

fixed samples

$a^0 \quad a^1 \quad a^2 \quad a^3$

$a^0 \quad a^1 \quad a^2 \quad a^3$

a^n	a^0	a^1	a^2	a^3	
a^{n-1}		a^0	a^1	a^2	a^3

right-shifted sequence (II)

Two Types of Right-Shifted Anti-Causal Sequences

the same fixed slots

u(-n)	n=-3	n=-2	n=-1	n=0
u(-n)	n=-3	n=-2	n=-1	n=0

right shift post-samples

a^{-3} a^{-2} a^{-1} a^{0}

a^{-4} a^{-3} a^{-2} a^{-1}

a^n	a^{-3}	a^{-2}	a^{-1}	a^{0}
a^{n-1}	a^{-4}	a^{-3}	a^{-2}	a^{-1}

right-shifted sequence (I)

left shift pre-slot

u(-n-1)	n=-4	n=-3	n=-2	n=-1	
u(-n)		n=-3	n=-2	n=-1	n=0

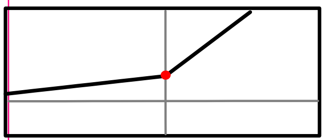
fixed samples

a^{-4} a^{-3} a^{-2} a^{-1}

a^{-4} a^{-3} a^{-2} a^{-1}

a^n	a^{-4}	a^{-3}	a^{-2}	a^{-1}	
a^{n-1}		a^{-4}	a^{-3}	a^{-2}	a^{-1}

right-shifted sequence (II)

a^n 

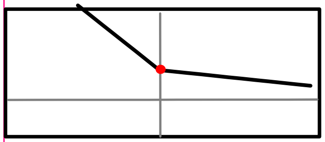
Shift Sequence Relations (1)

No Shift Ranges

a^n	u(n) $n = 0, 1, 2, \cdots$	a^{n+1}	u(n) $n = 0, 1, 2, \cdots$ **Left Shifted Sequence**
a^n	u(n-1) $n = 1, 2, 3, \cdots$	a^{n-1}	u(n-1) $n = 1, 2, 3, \cdots$ **Right Shifted Sequence**
a^n	u(-n) $n = 0, -1, -2, \cdots$	a^{n-1}	u(-n) $n = 0, -1, -2, \cdots$ **Left Shifted Sequence**
a^n	u(-n-1) $n = -1, -2, -3, \cdots$	a^{n+1}	u(-n-1) $n = -1, -2, -3, \cdots$ **Right Shifted Sequence**

With Shift Ranges

a^n	u(n) $n = 0, 1, 2, \cdots$	a^{n+1}	u(n) $n = 0, 1, 2, \cdots$ **Left Shifted Sequence**
a^n	u(n-1) $n = 1, 2, 3, \cdots$	a^{n-1}	u(n-1) $n = 1, 2, 3, \cdots$ **Right Shifted Sequence**
a^n	u(-n) $n = 0, -1, -2, \cdots$	a^{n-1}	u(-n) $n = 0, -1, -2, \cdots$ **Left Shifted Sequence**
a^n	u(-n-1) $n = -1, -2, -3, \cdots$	a^{n+1}	u(-n-1) $n = -1, -2, -3, \cdots$ **Right Shifted Sequence**

a^{-n} 

Shift Sequence Relations (2)

No Shift Ranges

u(n)

a^{-n} $n = 0, 1, 2, \cdots$

u(n)

a^{-n-1} $n = 0, 1, 2, \cdots$

Left Shifted Sequence

u(n-1)

a^{-n} $n = 1, 2, 3, \cdots$

u(n-1)

a^{-n+1} $n = 1, 2, 3, \cdots$

Right Shifted Sequence

u(-n)

a^{-n} $n = 0, -1, -2, \cdots$

u(-n)

a^{-n+1} $n = 0, -1, -2, \cdots$

Left Shifted Sequence

u(-n-1)

a^{-n} $n = -1, -2, -3, \cdots$

u(-n-1)

a^{-n-1} $n = -1, -2, -3, \cdots$

Right Shifted Sequence

With Shift Ranges

u(n)

a^{-n} $n = 0, 1, 2, \cdots$

u(n)

a^{-n-1} $n = 0, 1, 2, \cdots$

Left Shifted Sequence

u(n-1)

a^{-n} $n = 1, 2, 3, \cdots$

u(n-1)

a^{-n+1} $n = 1, 2, 3, \cdots$

Right Shifted Sequence

u(-n)

a^{-n} $n = 0, -1, -2, \cdots$

u(-n)

a^{-n+1} $n = 0, -1, -2, \cdots$

Left Shifted Sequence

u(-n-1)

a^{-n} $n = -1, -2, -3, \cdots$

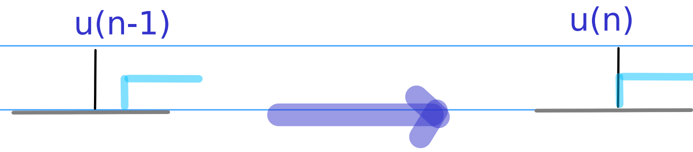
u(-n-1)

a^{-n-1} $n = -1, -2, -3, \cdots$

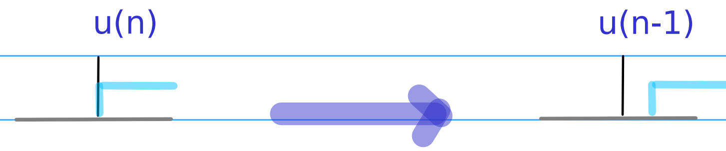
Right Shifted Sequence

Shifting of a Range

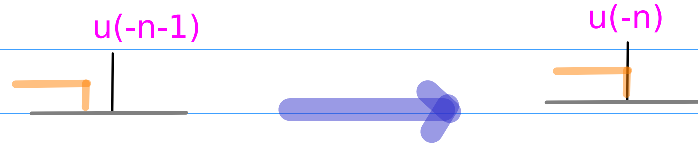
SHL.Rng



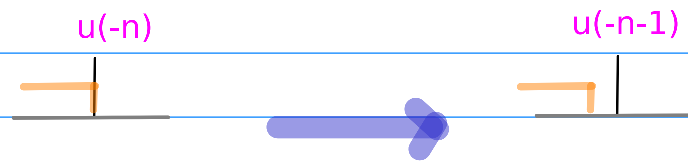
SHR.Rng



SHR.Rng

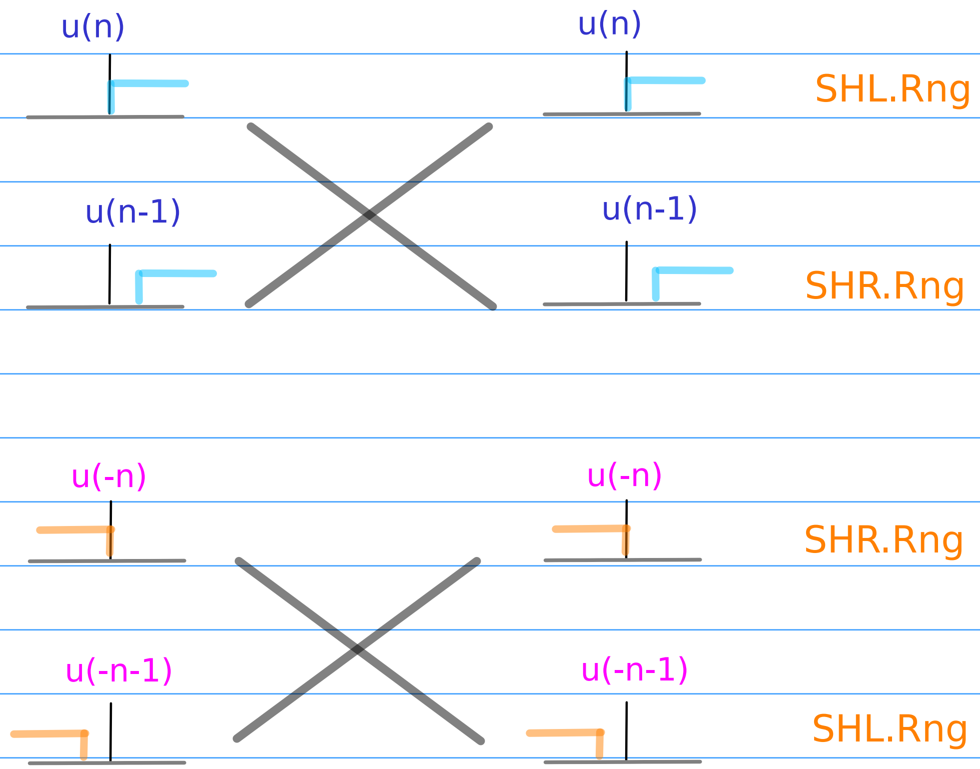


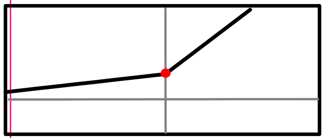
SHL.Rng



u(n)	u(n)	SHL.Rng
u(n-1)	u(n-1)	SHR.Rng

u(-n)	u(-n)	SHR.Rng
u(-n-1)	u(-n-1)	SHL.Rng


u(n)
u(n)
SHL.Rng
u(n-1)
u(n-1)
SHR.Rng
u(-n)
u(-n)
SHR.Rng
u(-n-1)
u(-n-1)
SHL.Rng

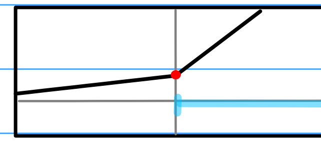
a^n 

Geometric Series Combinations (1)

* inverted relation is ignored

$\boxed{a \quad z}$

$$\boxed{\frac{1}{1-az} \quad |z| < a^{-1}}$$

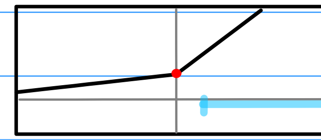


a^n

$a^n \quad n = 0, 1, 2, \cdots$

right shifted

$$\boxed{\frac{az}{1-az} \quad |z| < a^{-1}}$$

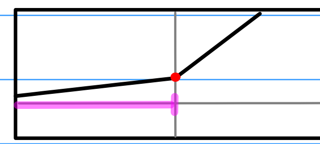


a^n

$a^n \quad n = 1, 2, 3, \cdots$

$\boxed{a^{\boxed{-1}} z^{\boxed{-1}}}$

$$\boxed{\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}}$$

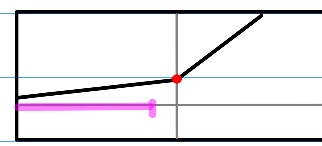


a^n

$a^n \quad n = 0, -1, -2, \cdots$

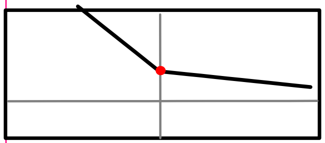
left shifted

$$\boxed{\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}}$$



a^n

$a^n \quad n = -1, -2, -3, \cdots$

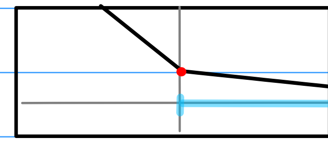
a^{-n} 

Geometric Series Combinations (2)

* inverted relation is ignored

$a^{\boxed{-1}} z$

$$\frac{1}{1-a^{-1}z} \quad |z|<a$$

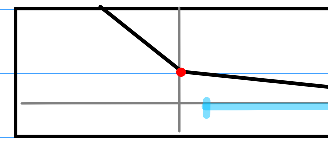


a^{-n}

$a^{-n} \quad n = 0, 1, 2, \cdots$

right shifted

$$\frac{a^{-1}z}{1-a^{-1}z} \quad |z|<a$$

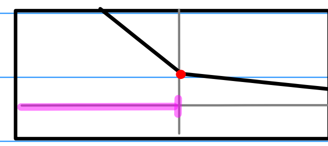


a^{-n}

$a^{-n} \quad n = 1, 2, 3, \cdots$

$a\, z^{\boxed{-1}}$

$$\frac{1}{1-a z^{-1}} \quad |z|>a$$

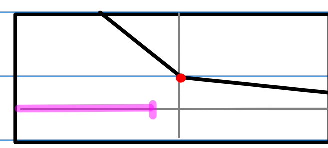


a^{-n}

$a^{-n} \quad n = 0, -1, -2, \cdots$

left shifted

$$\frac{a z^{-1}}{1-a z^{-1}} \quad |z|>a$$

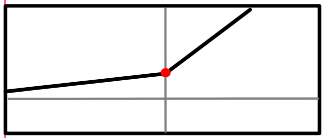


a^{-n}

$a^{-n} \quad n = -1, -2, -3, \cdots$

Shifting a sequence

a^n



*a SHL.Seq

/a SHR.Seq

SHL.Seq

$$\boxed{\frac{1}{1-az} \quad |z|<a^{-1}} \xrightarrow{*a} \boxed{\frac{a}{1-az} \quad |z|<a^{-1}}$$

$a^n \quad n = 0, 1, 2, \cdots$ $\qquad$ $a^{n+1} \quad n = 0, 1, 2, \cdots$

SHR.Seq

$$\boxed{\frac{az}{1-az} \quad |z|<a^{-1}} \xrightarrow{/a} \boxed{\frac{z}{1-az} \quad |z|<a^{-1}}$$

$a^n \quad n = 1, 2, 3, \cdots$ $\qquad$ $a^{n-1} \quad n = 1, 2, 3, \cdots$

SHR.Seq

$$\boxed{\frac{1}{1-a^{-1}z^{-1}} \quad |z|>a^{-1}} \xrightarrow{/a} \boxed{\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z|>a^{-1}}$$

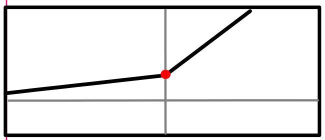
$a^n \quad n = 0, -1, -2, \cdots$ $\qquad$ $a^{n-1} \quad n = 0, -1, -2, \cdots$

SHL.Seq

$$\boxed{\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z|>a^{-1}} \xrightarrow{*a} \boxed{\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z|>a^{-1}}$$

$a^n \quad n = -1, -2, -3, \cdots$ $\qquad$ $a^{n+1} \quad n = -1, -2, -3, \cdots$

a^n



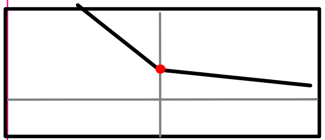
/z SHL.Seq, SHL.Rng

*z SHR.Seq, SHR.Rng

Shifting a sequence

$\dfrac{1}{1-az}$ $\quad |z| < a^{-1}$	/z	SHL.Seq, SHL.Rng $\dfrac{a}{1-az}$ $\quad |z| < a^{-1}$
$a^n \quad n = 0, 1, 2, \cdots$		$a^{n+1} \quad n = 0, 1, 2, \cdots$
	*z	SHR.Seq, SHR.Rng
$\dfrac{az}{1-az}$ $\quad |z| < a^{-1}$		$\dfrac{z}{1-az}$ $\quad |z| < a^{-1}$
$a^n \quad n = 1, 2, 3, \cdots$		$a^{n-1} \quad n = 1, 2, 3, \cdots$
		SHR.Seq, SHR.Rng
$\dfrac{1}{1-a^{-1}z^{-1}}$ $\quad |z| > a^{-1}$	*z	$\dfrac{a^{-1}}{1-a^{-1}z^{-1}}$ $\quad |z| > a^{-1}$
$a^n \quad n = 0, -1, -2, \cdots$		$a^{n-1} \quad n = 0, -1, -2, \cdots$
	/z	SHL.Seq, SHL.Rng
$\dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\quad |z| > a^{-1}$		$\dfrac{z^{-1}}{1-a^{-1}z^{-1}}$ $\quad |z| > a^{-1}$
$a^n \quad n = -1, -2, -3, \cdots$		$a^{n+1} \quad n = -1, -2, -3, \cdots$

* inverted relation is ignored

a^{-n} 

/a SHL.Seq

*a SHR.Seq

Shifting a sequence

$\frac{1}{1-a^{-1}z}$ $|z|<a$ $\xrightarrow{/a}$ SHL.Seq $\frac{a^{-1}}{1-a^{-1}z}$ $|z|<a$

a^{-n} $n = 0, 1, 2, \cdots$

a^{-n-1} $n = 0, 1, 2, \cdots$

$\frac{a^{-1}z}{1-a^{-1}z}$ $|z|<a$ $\xrightarrow{*a}$ SHR.Seq $\frac{z}{1-a^{-1}z}$ $|z|<a$

a^{-n} $n = 1, 2, 3, \cdots$

a^{-n+1} $n = 1, 2, 3, \cdots$

$\frac{1}{1-az^{-1}}$ $|z|>a$ $\xrightarrow{*a}$ SHR.Seq $\frac{a}{1-az^{-1}}$ $|z|>a$

a^{-n} $n = 0, -1, -2, \cdots$

a^{-n+1} $n = 0, -1, -2, \cdots$

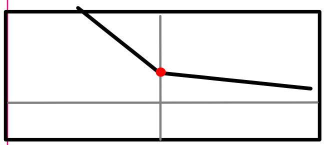
$\frac{az^{-1}}{1-az^{-1}}$ $|z|>a$ $\xrightarrow{/a}$ SHL.Seq $\frac{z^{-1}}{1-az^{-1}}$ $|z|>a$

a^{-n} $n = -1, -2, -3, \cdots$

a^{-n-1} $n = -1, -2, -3, \cdots$

Shifting a sequence

a^{-n}



/z SHL.Seq, SHL.Rng

*z SHR.Seq, SHR.Rng

$\boxed{\dfrac{1}{1-a^{-1}z} \quad |z|<a}$

$a^{-n} \quad n = 0, 1, 2, \cdots$

SHL.Seq, SHL.Rng

$\boxed{\dfrac{a^{-1}}{1-a^{-1}z} \quad |z|<a}$

$a^{-n-1} \quad n = 0, 1, 2, \cdots$

/z

*z

$\boxed{\dfrac{a^{-1}z}{1-a^{-1}z} \quad |z|<a}$

$a^{-n} \quad n = 1, 2, 3, \cdots$

SHR.Seq, SHR_Rng

$\boxed{\dfrac{z}{1-a^{-1}z} \quad |z|<a}$

$a^{-n+1} \quad n = 1, 2, 3, \cdots$

$\boxed{\dfrac{1}{1-az^{-1}} \quad |z|>a}$

$a^{-n} \quad n = 0, -1, -2, \cdots$

SHR.Seq, SHR.Rng

$\boxed{\dfrac{a}{1-az^{-1}} \quad |z|>a}$

$a^{-n+1} \quad n = 0, -1, -2, \cdots$

*z

/z

$\boxed{\dfrac{az^{-1}}{1-az^{-1}} \quad |z|>a}$

$a^{-n} \quad n = -1, -2, -3, \cdots$

SHL.Seq, SHL.Rng

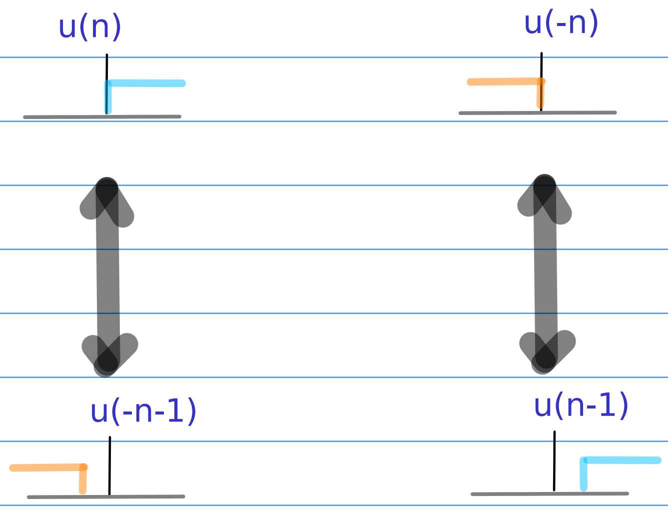
$\boxed{\dfrac{z^{-1}}{1-az^{-1}} \quad |z|>a}$

$a^{-n-1} \quad n = -1, -2, -3, \cdots$

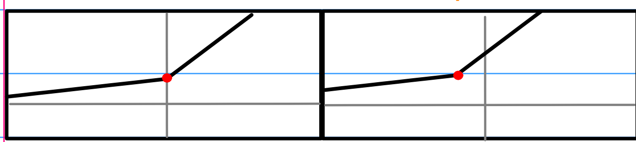
* inverted relation is ignored

SHL.Seq	Shift Right(Sequence Function)
SHR.Seq	Shift Right(Sequence Function)
SHL.ROC	Shift Right(Region of Convergence)
SHR.ROC	Shift Right(Region of Convergence)

Complement

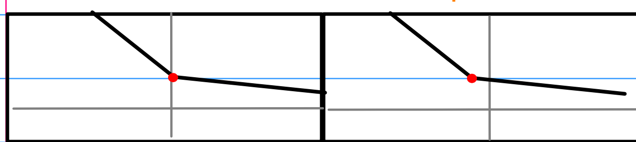


SHL.Seq



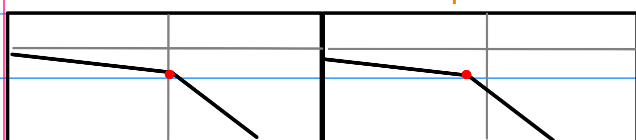
a^{n} a^{n+1}

SHL.Seq



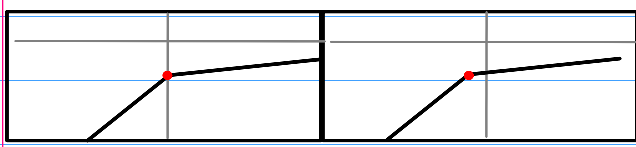
a^{-n} a^{-n-1}

SHL.Seq



$-a^{n}$ $-a^{n+1}$

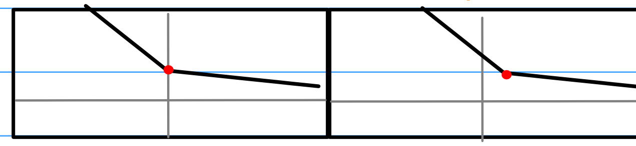
SHL.Seq



$-a^{-n}$ $-a^{-n-1}$

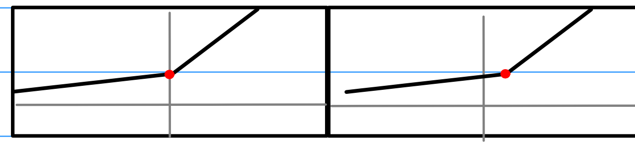
Left Shifted Sequence

SHR.Seq



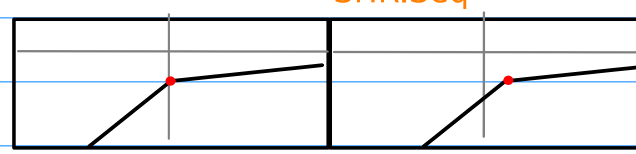
a^{-n} a^{-n+1}

SHR.Seq



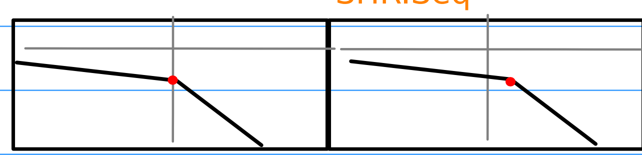
a^{n} a^{n-1}

SHR.Seq



$-a^{-n}$ $-a^{-n+1}$

SHR.Seq



$-a^{n}$ $-a^{n-1}$

Right Shifted Sequence

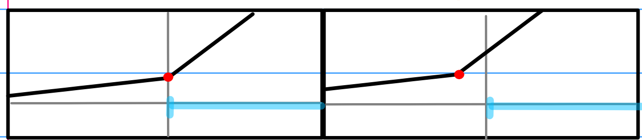
1-A	Left Shft	Causal	$u(n)$	(a, a^{-1})
1-B	Left Shft	Causal	$u(n-1)$	(z^{-1}, z^{-1})
1-C	Left Shft	Anti-Causal	$u(-n-1)$	(a, a^{-1})
1-D	Left Shft	Anti-Causal	$u(-n)$	(z^{-1}, z^{-1})
2-A	Right Shft	Causal	$u(n-1)$	(a^{-1}, a)
2-B	Right Shft	Causal	$u(n)$	(z, z)
2-C	Right Shft	Anti-Causal	$u(-n)$	(a^{-1}, a)
2-D	Right Shft	Anti-Causal	$u(-n-1)$	(z, z)

1-A $\boxed{\text{Left}}$ Shfted Sequence (a, a^{-1}) $\boxed{\text{Causal}}$

$$a^{n} \cdot u(n) \longrightarrow a^{n+1} \cdot u(n)$$
$$a^{-n} \cdot u(n) \longrightarrow a^{-n-1} \cdot u(n)$$

SHL.Seq

a



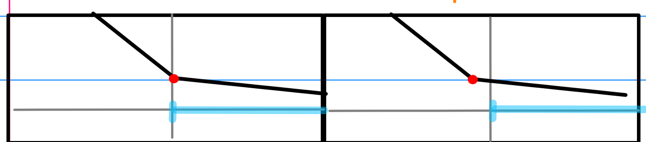
a^{n} $\qquad$ a^{n+1}

$$\frac{1}{1-az} \qquad \frac{a}{1-az}$$

$|z|<a^{-1}$ $\qquad$ $|z|<a^{-1}$

SHL.Seq

a^{-1}



a^{-n} $\qquad$ a^{-n-1}

$$\frac{1}{1-a^{-1}z} \qquad \frac{a^{-1}}{1-a^{-1}z}$$

$|z|<a$ $\qquad$ $|z|<a$

0 1 2

$$\frac{1}{1-az} = a^{0}z^{0} + a^{1}z^{1} + a^{2}z^{2} + \cdots \Rightarrow (a^{0}, a^{1}, a^{2}, \cdots) \quad (n \geq 0)$$

$|z|<a^{-1}$

$$\frac{a}{1-az} = a^{1}z^{0} + a^{2}z^{1} + a^{3}z^{2} + \cdots \Rightarrow (a^{1}, a^{2}, a^{3}, \cdots) \quad (n \geq 0)$$

0 1 2

$|z|<a^{-1}$

0 1 2

$$\frac{1}{1-a^{-1}z} = a^{0}z^{0} + a^{-1}z^{1} + a^{-2}z^{2} + \cdots \Rightarrow (a^{0}, a^{-1}, a^{-2}, \cdots) \quad (n \geq 0)$$

$|z|<a^{-1}$

$$\frac{a^{-1}}{1-a^{-1}z} = a^{-1}z^{0} + a^{-2}z^{1} + a^{-3}z^{2} + \cdots \Rightarrow (a^{-1}, a^{-2}, a^{-3}, \cdots) \quad (n \geq 0)$$

0 1 2

$|z|<a^{-1}$

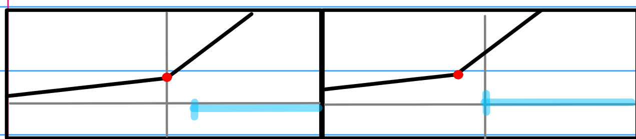
1-B $\boxed{\text{Left}}$ Shfted Sequence (z^{-1}, z^{-1}) $\boxed{\text{Causal}}$

$$a^{n} \cdot u(n-1) \longrightarrow a^{n+1} \cdot u(n)$$

$$a^{-n} \cdot u(n-1) \longrightarrow a^{-n-1} \cdot u(n)$$

SHL.Seq, SHL.Rng

z^{-1}



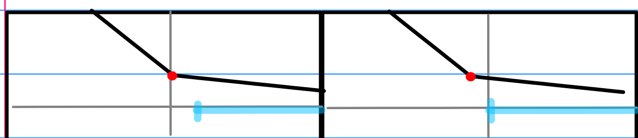
a^{n} a^{n+1}

$\dfrac{az}{1-az}$ $\quad |z| < a^{-1}$

$\dfrac{a}{1-az}$ $\quad |z| < a^{-1}$

SHL.Seq, SHL.Rng

z^{-1}



a^{-n} a^{-n-1}

$\dfrac{a^{-1}z}{1-a^{-1}z}$ $\quad |z| < a$

$\dfrac{a^{-1}}{1-a^{-1}z}$ $\quad |z| < a$

$$\frac{az}{1-az} = a^{1}z^{1} + a^{2}z^{2} + a^{3}z^{3} + \cdots \Rightarrow \overset{0}{(0}, \overset{1}{a^{1}}, \overset{2}{a^{2}}, \cdots) \quad (n \geq 1)$$

$|z| < a^{-1}$

$$\frac{a}{1-az} = a^{1}z^{0} + a^{2}z^{1} + a^{3}z^{2} + \cdots \Rightarrow (\underset{0}{a^{1}}, \underset{1}{a^{2}}, \underset{2}{a^{3}}, \cdots) \quad (n \geq 0)$$

$|z| < a^{-1}$

$$\frac{a^{-1}z}{1-a^{-1}z} = a^{-1}z^{1} + a^{-2}z^{2} + a^{-3}z^{3} + \cdots \Rightarrow \overset{0}{(0}, \overset{1}{a^{-1}}, \overset{2}{a^{-2}} \cdots) \quad (n \geq 1)$$

$|z| < a^{-1}$

$$\frac{a^{-1}}{1-a^{-1}z} = a^{-1}z^{0} + a^{-2}z^{1} + a^{-3}z^{2} + \cdots \Rightarrow (\underset{0}{a^{-1}}, \underset{1}{a^{-2}}, \underset{2}{a^{-3}}, \cdots) \quad (n \geq 0)$$

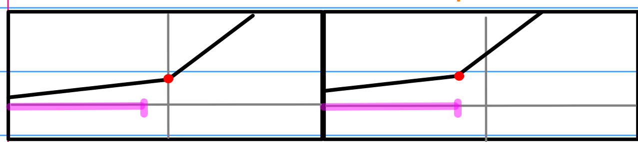
$|z| < a^{-1}$

1-C $\boxed{\text{Left}}$ Shfted Sequence (a, a^{-1}) $\boxed{\text{Anti-Causal}}$

$$a^{n} \cdot u(-n-1) \longrightarrow a^{n+1} \cdot u(-n-1)$$
$$a^{-n} \cdot u(-n-1) \longrightarrow a^{-n-1} \cdot u(-n-1)$$

SHL.Seq

a



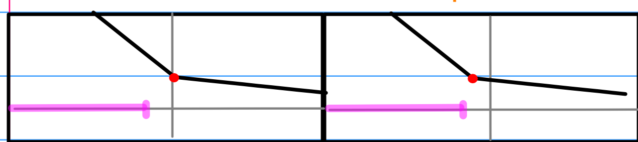
a^{n} $\quad$ a^{n+1}

$\dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\quad$ $\dfrac{z^{-1}}{1-a^{-1}z^{-1}}$

$|z|>a^{-1}$ $\quad$ $|z|>a^{-1}$

SHL.Seq

a^{-1}



a^{-n} $\quad$ a^{-n-1}

$\dfrac{az^{-1}}{1-az^{-1}}$ $\quad$ $\dfrac{z^{-1}}{1-az^{-1}}$

$|z|>a$ $\quad$ $|z|>a$

$$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} = \cdots + a^{-3}z^{-3} + a^{-2}z^{-2} + a^{-1}z^{-1} \Rightarrow (\cdots, \overset{-3}{a^{-3}}, \overset{-2}{a^{-2}}, \overset{-1}{a^{-1}}) \quad (n<0)$$

$|z|>a^{-1}$

$$\frac{z^{-1}}{1-a^{-1}z^{-1}} = \cdots + a^{-2}z^{-3} + a^{-1}z^{-2} + a^{0}z^{-1} \Rightarrow (\cdots, \underset{-3}{a^{-2}}, \underset{-2}{a^{-1}}, \underset{-1}{a^{0}}) \quad (n<0)$$

$|z|>a^{-1}$

$$\frac{az^{-1}}{1-az^{-1}} = \cdots + a^{3}z^{-3} + a^{2}z^{-2} + a^{1}z^{-1} \Rightarrow (\cdots, \overset{-3}{a^{3}}, \overset{-2}{a^{2}}, \overset{-1}{a^{1}}) \quad (n<0)$$

$|z|>a$

$$\frac{z^{-1}}{1-az^{-1}} = \cdots + a^{2}z^{-3} + a^{1}z^{-2} + a^{0}z^{-1} \Rightarrow (\cdots, \underset{-3}{a^{2}}, \underset{-2}{a^{1}}, \underset{-1}{a^{0}}) \quad (n<0)$$

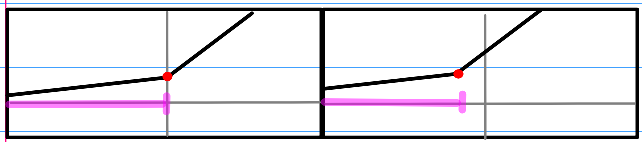
$|z|>a$

1-D $\boxed{\text{Left}}$ Shfted Sequence (z^{-1}, z^{-1}) $\boxed{\text{Anti-Causal}}$

$$a^{n} \cdot u(-n) \longrightarrow a^{n+1} \cdot u(-n-1)$$
$$a^{-n} \cdot u(-n) \longrightarrow a^{-n-1} \cdot u(-n-1)$$

SHL.Seq, SHL.Rng

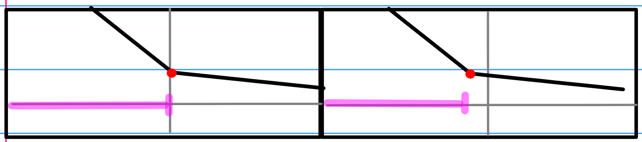
z^{-1}



$$\frac{1}{1-a^{-1}z^{-1}} \qquad \frac{z^{-1}}{1-a^{-1}z^{-1}}$$
$|z|>a^{-1}$ $\quad$ $|z|>a^{-1}$

SHL.Seq, SHL.Rng

z^{-1}



$$\frac{1}{1-az^{-1}} \qquad \frac{z^{-1}}{1-az^{-1}}$$
$|z|>a^{-1}$ $\quad$ $|z|>a^{-1}$

$$\frac{1}{1-a^{-1}z^{-1}} = \cdots + a^{-2}z^{-2} + a^{-1}z^{-1} + a^{0}z^{0} \Rightarrow (\cdots, \overset{-2}{a^{-2}}, \overset{-1}{a^{-1}}, \overset{0}{a^{0}}) \quad (n<1)$$
$|z|>a^{-1}$

$$\frac{z^{-1}}{1-a^{-1}z^{-1}} = \cdots + a^{-2}z^{-3} + a^{-1}z^{-2} + a^{0}z^{-1} \Rightarrow (\cdots, \underset{-2}{a^{-1}}, \underset{-1}{a^{0}}, \underset{0}{0}) \quad (n<0)$$
$|z|>a^{-1}$

$$\frac{1}{1-az^{-1}} = \cdots + a^{2}z^{-2} + a^{1}z^{-1} + a^{0}z^{0} \Rightarrow (\cdots, \overset{-2}{a^{2}}, \overset{-1}{a^{1}}, \overset{0}{a^{0}}) \quad (n<1)$$
$|z|>a^{-1}$

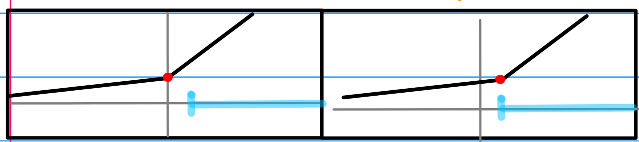
$$\frac{z^{-1}}{1-az^{-1}} = \cdots + a^{2}z^{-3} + a^{1}z^{-2} + a^{0}z^{-1} \Rightarrow (\cdots, \underset{-2}{a^{1}}, \underset{-1}{a^{0}}, \underset{0}{0}) \quad (n<0)$$
$|z|>a^{-1}$

2-A $\boxed{\text{Right}}$ Shfted Sequence (a^{-1}, a) $\boxed{\text{Causal}}$

$$a^{n} \cdot u(n-1) \longrightarrow a^{n-1} \cdot u(n-1)$$
$$a^{-n} \cdot u(n-1) \longrightarrow a^{-n+1} \cdot u(n-1)$$

SHR.Seq

a^{-1}



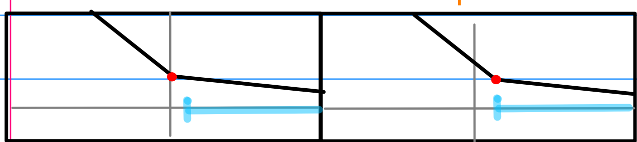
a^{n} $\quad$ a^{n-1}

$$\frac{az}{1-az} \qquad \frac{z}{1-az}$$

$|z| < a^{-1}$ $\quad$ $|z| < a^{-1}$

SHR.Seq

a



a^{-n} $\quad$ a^{-n+1}

$$\frac{a^{-1}z}{1-a^{-1}z} \qquad \frac{z}{1-a^{-1}z}$$

$|z| < a$ $\quad$ $|z| < a$

$$\frac{az}{1-az} = a^{1}z^{1} + a^{2}z^{2} + a^{3}z^{3} + \cdots \Rightarrow \underline{(\overset{1}{a^{1}}, \overset{2}{a^{2}}, \overset{3}{a^{3}}, \cdots)} \quad (n \geq 1)$$

$|z| < a^{-1}$

$$\frac{z}{1-az} = a^{0}z^{1} + a^{1}z^{2} + a^{2}z^{3} + \cdots \Rightarrow \underline{(\underset{1}{a^{0}}, \underset{2}{a^{1}}, \underset{3}{a^{2}}, \cdots)} \quad (n \geq 1)$$

$|z| < a^{-1}$

$$\frac{a^{-1}z}{1-a^{-1}z} = a^{-1}z^{1} + a^{-2}z^{2} + a^{-3}z^{3} + \cdots \Rightarrow \underline{(\overset{1}{a^{-1}}, \overset{2}{a^{-2}}, \overset{3}{a^{-3}}, \cdots)} \quad (n \geq 1)$$

$|z| < a^{-1}$

$$\frac{z}{1-a^{-1}z} = a^{0}z^{1} + a^{-1}z^{2} + a^{-2}z^{3} + \cdots \Rightarrow \underline{(\underset{1}{a^{0}}, \underset{2}{a^{-1}}, \underset{3}{a^{-2}}, \cdots)} \quad (n \geq 1)$$

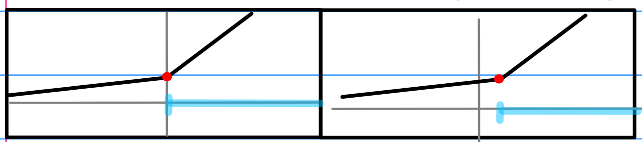
$|z| < a^{-1}$

2-B $\boxed{\text{Right}}$ Shfted Sequence (z, z) $\boxed{\text{Causal}}$

$$a^{n} \cdot u(n) \longrightarrow a^{n-1} \cdot u(n-1)$$
$$a^{-n} \cdot u(n) \longrightarrow a^{-n+1} \cdot u(n-1)$$

SHR.Seq, SHR.Rng

Z



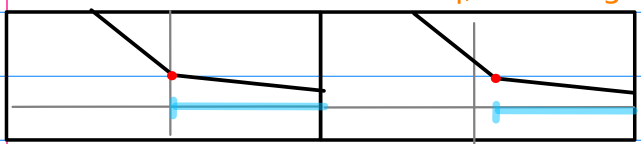
a^{n} $\quad$ a^{n-1}

$\dfrac{1}{1-az}$ $\quad$ $|z|<a^{-1}$

$\dfrac{z}{1-az}$ $\quad$ $|z|<a^{-1}$

SHR.Seq, SHR.Rng

Z



a^{-n} $\quad$ a^{-n+1}

$\dfrac{1}{1-a^{-1}z}$ $\quad$ $|z|<a$

$\dfrac{z}{1-a^{-1}z}$ $\quad$ $|z|<a$

$$\frac{1}{1-az} = a^0 z^0 + a^1 z^1 + a^2 z^2 + \cdots \Rightarrow \overset{0}{(a^0}, \overset{1}{a^1}, \overset{2}{a^2}, \cdots) \quad (n \geq 0)$$

$|z|<a^{-1}$

$$\frac{z}{1-az} = a^0 z^1 + a^1 z^2 + a^2 z^3 + \cdots \Rightarrow (\underset{0}{0}, \underset{1}{a^0}, \underset{2}{a^1}, \cdots) \quad (n \geq 1)$$

$|z|<a^{-1}$

$$\frac{1}{1-a^{-1}z} = a^0 z^0 + a^{-1} z^1 + a^{-2} z^2 + \cdots \Rightarrow \overset{0}{(a^0}, \overset{1}{a^{-1}}, \overset{2}{a^{-2}}, \cdots) \quad (n \geq 0)$$

$|z|<a^{-1}$

$$\frac{z}{1-a^{-1}z} = a^0 z^1 + a^{-1} z^2 + a^{-2} z^3 + \cdots \Rightarrow (\underset{0}{0}, \underset{1}{a^0}, \underset{2}{a^{-1}}, \cdots) \quad (n \geq 1)$$

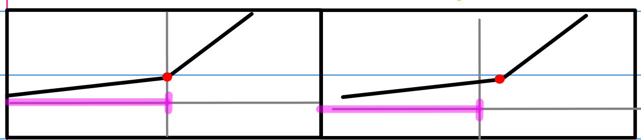
$|z|<a^{-1}$

2-c Right Shfted Sequence (a^{-1}, a) Anti-Causal

$$a^{n} \cdot u(-n) \longrightarrow a^{n-1} \cdot u(-n)$$
$$a^{-n} \cdot u(-n) \longrightarrow a^{-n+1} \cdot u(-n)$$

SHR.Seq

a^{-1}



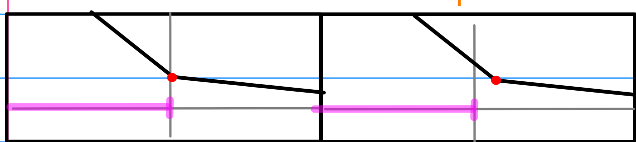
a^{n} a^{n-1}

$$\frac{1}{1-a^{-1}z^{-1}} \qquad \frac{a^{-1}}{1-a^{-1}z^{-1}}$$

$|z| > a^{-1}$ $|z| > a^{-1}$

SHR.Seq

a



a^{-n} a^{-n+1}

$$\frac{1}{1-a z^{-1}} \qquad \frac{a}{1-a z^{-1}}$$

$|z| > a$ $|z| > a$

$$\frac{1}{1-a^{-1}z^{-1}} = \cdots + a^{-2}z^{-2} + a^{-1}z^{-1} + a^{0}z^{0} \Rightarrow (\ldots, \overset{-2}{a^{-2}}, \overset{-1}{a^{-1}}, \overset{0}{a^{0}}) \quad (n<1)$$

$|z| > a^{-1}$

$$\frac{a^{-1}}{1-a^{-1}z^{-1}} = \cdots + a^{-3}z^{-2} + a^{-2}z^{-1} + a^{-1}z^{0} \Rightarrow (\ldots, \underset{-2}{a^{-3}}, \underset{-1}{a^{-2}}, \underset{0}{a^{-1}}) \quad (n<1)$$

$|z| > a^{-1}$

$$\frac{1}{1-a z^{-1}} = \cdots + a^{2}z^{-2} + a^{1}z^{-1} + a^{0}z^{0} \Rightarrow (\ldots, \overset{-2}{a^{2}}, \overset{-1}{a^{1}}, \overset{0}{a^{0}}) \quad (n<1)$$

$|z| > a$

$$\frac{a}{1-a z^{-1}} = \cdots + a^{3}z^{-2} + a^{2}z^{-1} + a^{1}z^{0} \Rightarrow (\ldots, \underset{-2}{a^{3}}, \underset{-1}{a^{2}}, \underset{0}{a^{1}}) \quad (n<1)$$

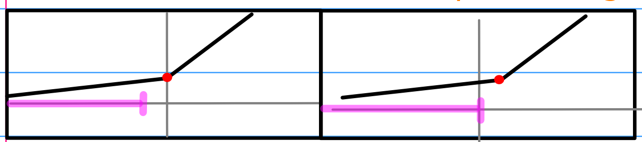
$|z| > a$

2-D $\boxed{\text{Right}}$ Shfted Sequence (z, z) Anti-Causal

$$a^{n}\cdot u(-n-1) \longrightarrow a^{n-1}\cdot u(-n)$$
$$a^{-n}\cdot u(-n-1) \longrightarrow a^{-n+1}\cdot u(-n)$$

SHR.Seq, SHR.Rng

z

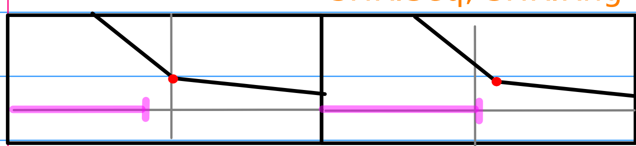


$$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \qquad |z|>a^{-1}$$

$$\frac{a^{-1}}{1-a^{-1}z^{-1}} \qquad |z|>a^{-1}$$

SHR.Seq, SHR.Rng

z



$$\frac{az^{-1}}{1-az^{-1}} \qquad |z|>a$$

$$\frac{a}{1-az^{-1}} \qquad |z|>a$$

$$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} = \cdots + a^{-3}z^{-3} + a^{-2}z^{-2} + a^{-1}z^{-1} \Rightarrow (\ldots, \overset{-3}{}\;\overset{-2}{a^{-2}}, \overset{-1}{a^{-1}}, 0) \quad (n<0)$$

$|z|>a^{-1}$

$$\frac{a^{-1}}{1-a^{-1}z^{-1}} = \cdots + a^{-3}z^{-2} + a^{-2}z^{-1} + a^{-1}z^{0} \Rightarrow (\ldots, \underset{-3}{a^{-3}}, \underset{-2}{a^{-2}}, \underset{-1}{a^{-1}}) \quad (n<1)$$

$|z|>a^{-1}$

$$\frac{az^{-1}}{1-az^{-1}} = \cdots + a^{3}z^{-3} + a^{2}z^{-2} + a^{1}z^{-1} \Rightarrow (\ldots, \overset{-2}{a^{2}}, \overset{-1}{a^{1}}, \overset{0}{0}) \quad (n<0)$$

$|z|>a$

$$\frac{a}{1-az^{-1}} = \cdots + a^{3}z^{-3} + a^{2}z^{-2} + a^{1}z^{-1} \Rightarrow (\ldots, \underset{-2}{a^{3}}, \underset{-1}{a^{2}}, \underset{0}{a^{1}}) \quad (n<1)$$

$|z|>a$

Original Sequence		Shifted Sequence	Original Sequence		Shifted Sequence
$a^{n} \cdot u(n)$	$\longrightarrow$	$a^{n+1} \cdot u(n)$	$a^{n} \cdot u(-n-1)$	$\longrightarrow$	$a^{n+1} \cdot u(-n-1)$
$a^{-n} \cdot u(n)$	$\longrightarrow$	$a^{-n-1} \cdot u(n)$	$a^{-n} \cdot u(-n-1)$	$\longrightarrow$	$a^{-n-1} \cdot u(-n-1)$
$a^{n} \cdot u(n-1)$	$\longrightarrow$	$a^{n+1} \cdot u(n)$	$a^{n} \cdot u(-n)$	$\longrightarrow$	$a^{n+1} \cdot u(-n-1)$
$a^{-n} \cdot u(n-1)$	$\longrightarrow$	$a^{-n-1} \cdot u(n)$	$a^{-n} \cdot u(-n)$	$\longrightarrow$	$a^{-n-1} \cdot u(-n-1)$
$a^{n} \cdot u(n-1)$	$\longrightarrow$	$a^{n-1} \cdot u(n-1)$	$a^{n} \cdot u(-n)$	$\longrightarrow$	$a^{n-1} \cdot u(-n)$
$a^{-n} \cdot u(n-1)$	$\longrightarrow$	$a^{-n+1} \cdot u(n-1)$	$a^{-n} \cdot u(-n)$	$\longrightarrow$	$a^{-n+1} \cdot u(-n)$
$a^{n} \cdot u(n)$	$\longrightarrow$	$a^{n-1} \cdot u(n-1)$	$a^{n} \cdot u(-n-1)$	$\longrightarrow$	$a^{n-1} \cdot u(-n)$
$a^{-n} \cdot u(n)$	$\longrightarrow$	$a^{-n+1} \cdot u(n-1)$	$a^{-n} \cdot u(-n-1)$	$\longrightarrow$	$a^{-n+1} \cdot u(-n)$

Complementary Ranges

$u(n)$ $u(-n-1)$

$u(n-1)$ $u(-n)$

Original Sequence	Shifted Sequence	Original Sequence	Shifted Sequence
$a^{n} \cdot u(n) \longrightarrow$	$a^{n+1} \cdot u(n)$	$a^{n} \cdot u(-n-1) \longrightarrow$	$a^{n+1} \cdot u(-n-1)$
$a^{-n} \cdot u(n) \longrightarrow$	$a^{-n-1} \cdot u(n)$	$a^{-n} \cdot u(-n-1) \longrightarrow$	$a^{-n-1} \cdot u(-n-1)$
$\ll(a^0, a^1, a^2, \cdots)$	$(a^1, a^2, a^3, \cdots)$	$(\cdots, a^{-3}, a^{-2}, a^{-1})$	$(\cdots, a^{-2}, a^{-1}, a^0)\ll$
$\ll(a^0, a^{-1}, a^{-2}, \cdots)$	$(a^{-1}, a^{-2}, a^{-3}, \cdots)$	$(\cdots, a^{3}, a^{2}, a^{1})$	$(\cdots, a^{2}, a^{1}, a^0)\ll$
shift out			shift in
$a^{n} \cdot u(n-1) \longrightarrow$	$a^{n+1} \cdot u(n)$	$a^{n} \cdot u(-n) \longrightarrow$	$a^{n+1} \cdot u(-n-1)$
$a^{-n} \cdot u(n-1) \longrightarrow$	$a^{-n-1} \cdot u(n)$	$a^{-n} \cdot u(-n) \longrightarrow$	$a^{-n-1} \cdot u(-n-1)$
$\ll(0, a^1, a^2, \cdots)$	$(a^1, a^2, a^3, \cdots)$	$(\cdots, a^{-2}, a^{-1}, a^0)$	$(\cdots, a^{-1}, a^0, 0)\ll$
$\ll(0, a^{-1}, a^{-2}, \cdots)$	$(a^{-1}, a^{-2}, a^{-3}, \cdots)$	$(\cdots, a^{2}, a^{1}, a^0)$	$(\cdots, a^{1}, a^0, 0)\ll$
shift out			shift in
$a^{n} \cdot u(n-1) \longrightarrow$	$a^{n-1} \cdot u(n-1)$	$a^{n} \cdot u(-n) \longrightarrow$	$a^{n-1} \cdot u(-n)$
$a^{-n} \cdot u(n-1) \longrightarrow$	$a^{-n+1} \cdot u(n-1)$	$a^{-n} \cdot u(-n) \longrightarrow$	$a^{-n+1} \cdot u(-n)$
$(a^1, a^2, a^3, \cdots)$	$\gg(a^0, a^1, a^2, \cdots)$	$(\cdots, a^{-2}, a^{-1}, a^0)\gg$	$(\cdots, a^{-3}, a^{-2}, a^{-1})$
$(a^{-1}, a^{-2}, a^{-3}, \cdots)$	$\gg(a^0, a^{-1}, a^{-2}, \cdots)$	$(\cdots, a^{2}, a^{1}, a^0)\gg$	$(\cdots, a^{3}, a^{2}, a^{1})$
	shift in	shift out	
$a^{n} \cdot u(n) \longrightarrow$	$a^{n-1} \cdot u(n-1)$	$a^{n} \cdot u(-n-1) \longrightarrow$	$a^{n-1} \cdot u(-n)$
$a^{-n} \cdot u(n) \longrightarrow$	$a^{-n+1} \cdot u(n-1)$	$a^{-n} \cdot u(-n-1) \longrightarrow$	$a^{-n+1} \cdot u(-n)$
$(a^0, a^1, a^2, \cdots)$	$\gg(0, a^0, a^1, \cdots)$	$(\cdots, a^{-2}, a^{-1}, 0)\gg$	$(\cdots, a^{-3}, a^{-2}, a^{-1})$
$(a^0, a^{-1}, a^{-2}, \cdots)$	$\gg(0, a^0, a^{-1}, \cdots)$	$(\cdots, a^{2}, a^{1}, 0)\gg$	$(\cdots, a^{3}, a^{2}, a^{1})$
	shift in	shift out	

Complementary and Symmetric Relations

a^n	$\longrightarrow$	a^{n+1}
a^{-n}	$\longrightarrow$	a^{-n-1}

$u(n)$	$\longrightarrow$	$u(n)$
$u(-n-1)$	$\longrightarrow$	$u(-n-1)$
$u(n-1)$	$\longrightarrow$	$u(n)$
$u(-n)$	$\longrightarrow$	$u(-n-1)$

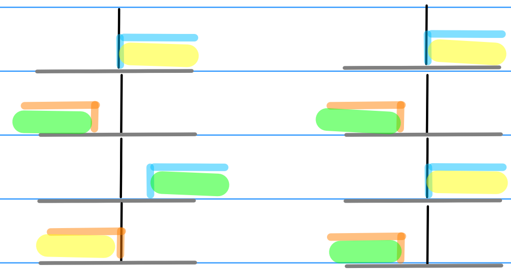
a^n	$\longrightarrow$	a^{n-1}
a^{-n}	$\longrightarrow$	a^{-n+1}

$u(n-1)$	$\longrightarrow$	$u(n-1)$
$u(-n)$	$\longrightarrow$	$u(-n)$
$u(n)$	$\longrightarrow$	$u(n-1)$
$u(-n-1)$	$\longrightarrow$	$u(-n)$

$u(n)$ complementary $u(-n-1)$ symmetric $u(n-1)$

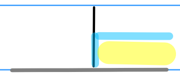
$u(-n)$ complementary $u(n-1)$ symmetric $u(-n-1)$

$u(n) \longrightarrow u(n)$	no shift	
$u(-n-1) \longrightarrow u(-n-1)$	no shift	
$u(n-1) \longrightarrow u(n)$	left shift	
$u(-n) \longrightarrow u(-n-1)$	left shift	

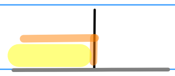


$u(n-1) \longrightarrow u(n-1)$	no shift	
$u(-n) \longrightarrow u(-n)$	no shift	
$u(n) \longrightarrow u(n-1)$	right shift	
$u(-n-1) \longrightarrow u(-n)$	right shift	

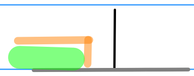
u(n)



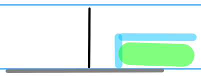
u(-n)



u(-n-1)

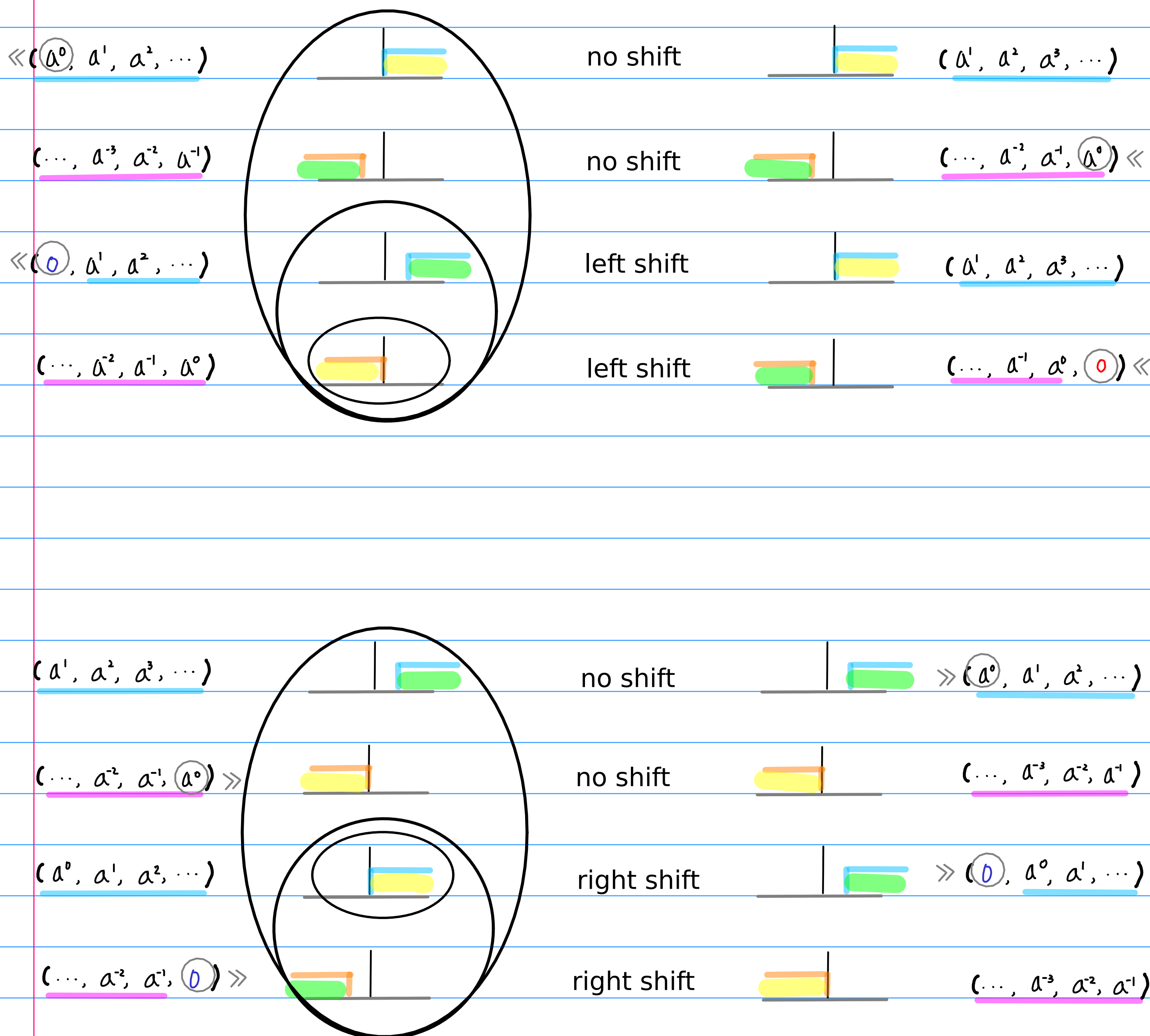


u(n-1)



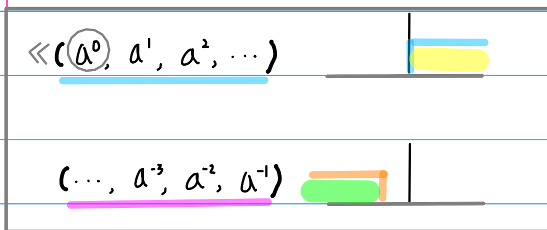
Original
Sequence

Shifted
Sequence

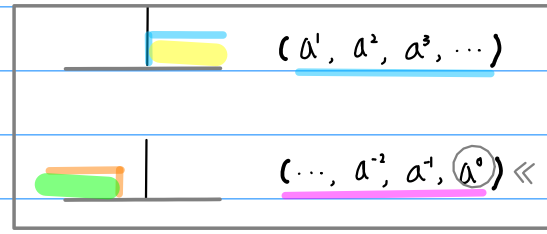


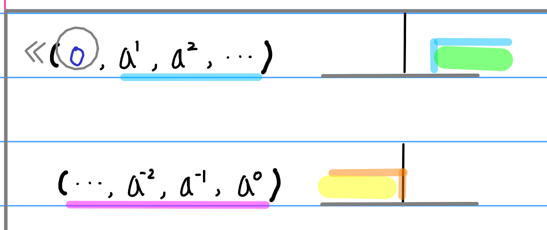
Original Sequence

Shifted Sequence

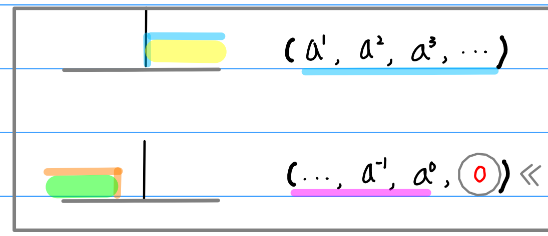


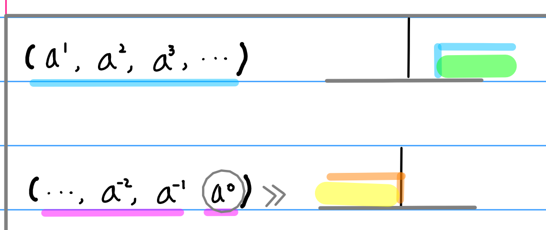
* no shift
* non-zero shift in
* a new value introduced



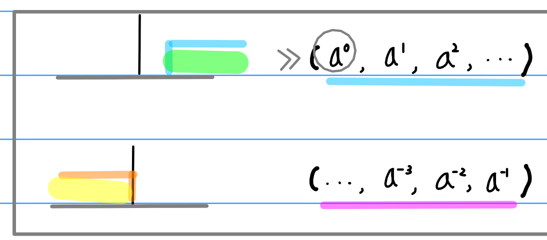


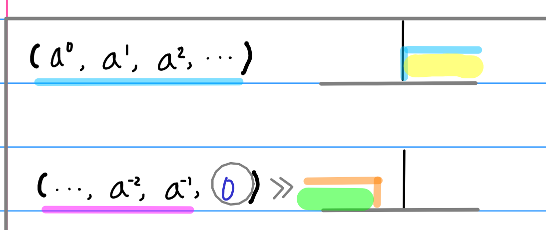
* left shift
* zero shift in
* the same set of values



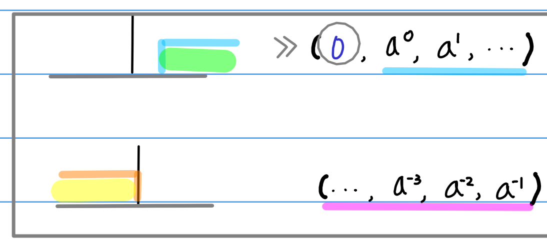


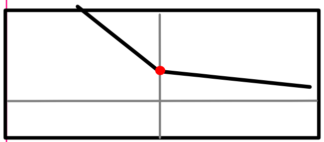
* no shift
* non-zero shift in
* a new value introduced





* right shift
* zero shift in
* the same set of values



a^{-n} 

			scale(1/a)		SHL.Seq
$-\frac{1}{1-a^{-1}z}$	$\lvert z\rvert<a$	$-\frac{a^{-1}}{1-a^{-1}z}$	$\lvert z\rvert<a$	$-\left(\frac{1}{a}\right)^n\ (n\geq 0)$ $-\left(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots\right)$	$-\left(\frac{1}{a}\right)^{n+1}\ (n\geq 0)$ $-\left(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots\right)$
$\frac{az^{-1}}{1-az^{-1}}$	$\lvert z\rvert>a$	$\frac{z^{-1}}{1-az^{-1}}$	$\lvert z\rvert>a$	$\left(\frac{1}{a}\right)^n\ (n<0)$ $(\cdots, a^3, a^2, a^1)$	$\left(\frac{1}{a}\right)^{n+1}\ (n<0)$ $(\cdots, a^2, a^1, a^0)$

			scale(1/z)		SHL.Seq, SHL.Rng
$\frac{1}{1-az^{-1}}$	$\lvert z\rvert>a$	$\frac{z^{-1}}{1-az^{-1}}$	$\lvert z\rvert>a$	$\left(\frac{1}{a}\right)^n\ (n<1)$ $(\cdots, a^2, a^1, a^0)$	$\left(\frac{1}{a}\right)^{n+1}\ (n<0)$ $(\cdots, a^2, a^1, a^0)$
$-\frac{a^{-1}z}{1-a^{-1}z}$	$\lvert z\rvert<a$	$-\frac{a^{-1}}{1-a^{-1}z}$	$\lvert z\rvert<a$	$-\left(\frac{1}{a}\right)^n\ (n\geq 1)$ $-\left(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots\right)$	$-\left(\frac{1}{a}\right)^{n+1}\ (n\geq 0)$ $-\left(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots\right)$

			scale(a)		SHR.Seq
$-\frac{1}{1-az^{-1}}$	$\lvert z\rvert>a$	$-\frac{a}{1-az^{-1}}$	$\lvert z\rvert>a$	$-\left(\frac{1}{a}\right)^n\ (n<1)$ $-(\cdots, a^2, a^1, a^0)$	$-\left(\frac{1}{a}\right)^{n-1}\ (n<1)$ $-(\cdots, a^3, a^2, a^1)$
$\frac{a^{-1}z}{1-a^{-1}z}$	$\lvert z\rvert<a$	$\frac{z}{1-a^{-1}z}$	$\lvert z\rvert<a$	$\left(\frac{1}{a}\right)^n\ (n\geq 1)$ $\left(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots\right)$	$\left(\frac{1}{a}\right)^{n-1}\ (n\geq 1)$ $\left(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots\right)$

			scale(z)		SHR.Seq, SHR.Rng
$\frac{1}{1-a^{-1}z}$	$\lvert z\rvert<a$	$\frac{z}{1-a^{-1}z}$	$\lvert z\rvert<a$	$\left(\frac{1}{a}\right)^n\ (n\geq 0)$ $\left(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots\right)$	$\left(\frac{1}{a}\right)^{n-1}\ (n\geq 1)$ $\left(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots\right)$
$\frac{az^{-1}}{1-az^{-1}}$	$\lvert z\rvert>a$	$\frac{a}{1-az^{-1}}$	$\lvert z\rvert>a$	$-\left(\frac{1}{a}\right)^n\ (n<0)$ $(\cdots, a^3, a^2, a^1)$	$-\left(\frac{1}{a}\right)^{n-1}\ (n<1)$ $(\cdots, a^3, a^2, a^1)$

Original Series | Scaled Series | Original Sequence | Shifted Sequence

2 formulas   Simple Pole Form

$$\boxed{\frac{1}{z-p}} \qquad \boxed{\frac{1}{z^{-1}-p}}$$

2 representations each   Geometric Series Form

causal   anti- causal

$$\boxed{\frac{1}{z-p}} \to \begin{cases} \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \triangleq f(z) = X(z^{-1}) \\ \qquad\qquad\qquad\quad \| \qquad\quad\ \ \| \\ \boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \triangleq Y(z) = g(z^{-1}) \end{cases}$$

causal   anti- causal

causal   anti- causal

$$\boxed{\frac{1}{z^{-1}-p}} \to \begin{cases} \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} \triangleq X(z) = f(z^{-1}) \\ \qquad\qquad\qquad\quad \| \qquad\quad\ \ \| \\ \boxed{\dfrac{z}{1-pz}} \triangleq g(z) = Y(z^{-1}) \end{cases}$$

causal   anti- causal

Simple Pole Form | Geometric Series Form

Geometric Series Form Combinations
with a unit start term

$$-\frac{1}{1-\boxed{az}}$$

$$-\frac{1}{1-\boxed{az^{-1}}}$$

$$+\frac{1}{1-\boxed{a^{-1}z^{-1}}}$$

$$+\frac{1}{1-\boxed{a^{-1}z}}$$

$$-\frac{1}{1-\boxed{a^{-1}z}}$$

$$-\frac{1}{1-\boxed{a^{-1}z^{-1}}}$$

$$+\frac{1}{1-\boxed{az^{-1}}}$$

$$+\frac{1}{1-\boxed{az}}$$

Geometric Series Form Combinations
with non-unit start term

$$+\frac{a^{-1}z^{-1}}{1-\boxed{a^{-1}z^{-1}}}$$

$$+\frac{a^{-1}z}{1-\boxed{a^{-1}z}}$$

$$-\frac{az}{1-\boxed{az}}$$

$$-\frac{az^{-1}}{1-\boxed{az^{-1}}}$$

$$+\frac{az^{-1}}{1-\boxed{az^{-1}}}$$

$$+\frac{az}{1-\boxed{az}}$$

$$-\frac{a^{-1}z}{1-\boxed{a^{-1}z}}$$

$$-\frac{a^{-1}z^{-1}}{1-\boxed{a^{-1}z^{-1}}}$$

Geometric Series with a unit start term

Laurent Series

$\boxed{a\,z}$ $\boxed{|z|<a^{-1}}$

$-\left(a^0z^0 + a^1z^1 + a^2z^2 + \cdots\right)$

$\boxed{a_n = -a^n \quad (n \geq 0)}$

$\boxed{a\,z^{\boxed{-1}}}$ $\boxed{|z|>a}$

$-\left(a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots\right)$

$-\left(\left(\frac{1}{a}\right)^0z^0 + \left(\frac{1}{a}\right)^{-1}z^{-1} + \left(\frac{1}{a}\right)^{-2}z^{-2} + \cdots\right)$

$\boxed{a_n = -\left(\frac{1}{a}\right)^n \quad (n<1)}$

$\boxed{a^{\boxed{-1}}z^{\boxed{-1}}}$ $\boxed{|z|>a^{-1}}$

$\left(a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots\right)$

$\boxed{a_n = a^n \quad (n<1)}$

$\boxed{a^{\boxed{-1}}z}$ $\boxed{|z|<a}$

$\left(a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots\right)$

$\left(\left(\frac{1}{a}\right)^0z^0 + \left(\frac{1}{a}\right)^1z^1 + \left(\frac{1}{a}\right)^2z^2 + \cdots\right)$

$\boxed{a_n = \left(\frac{1}{a}\right)^n \quad (n \geq 0)}$

$\boxed{a^{\boxed{-1}}z}$ $\boxed{|z|<a}$

$-\left(a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots\right)$

$-\left(\left(\frac{1}{a}\right)^0z^0 + \left(\frac{1}{a}\right)^1z^1 + \left(\frac{1}{a}\right)^2z^2 + \cdots\right)$

$\boxed{a_n = -\left(\frac{1}{a}\right)^n \quad (n \geq 0)}$

$\boxed{a^{\boxed{-1}}z^{\boxed{-1}}}$ $\boxed{|z|>a^{-1}}$

$-\left(a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots\right)$

$\boxed{a_n = -a^n \quad (n<1)}$

$\boxed{a\,z^{\boxed{-1}}}$ $\boxed{|z|>a}$

$\left(a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots\right)$

$\left(\left(\frac{1}{a}\right)^0z^0 + \left(\frac{1}{a}\right)^{-1}z^{-1} + \left(\frac{1}{a}\right)^{-2}z^{-2} + \cdots\right)$

$\boxed{a_n = \left(\frac{1}{a}\right)^n \quad (n<1)}$

$\boxed{a\,z}$ $\boxed{|z|<a^{-1}}$

$\left(a^0z^0 + a^1z^1 + a^2z^2 + \cdots\right)$

$\boxed{a_n = a^n \quad (n \geq 0)}$

Geometric Series with a unit start term

z-Transform

$a\,z$	$\lvert z\rvert < a^{-1}$

$-\left(a^0 z^0 + a^1 z^1 + a^2 z^2 + \cdots\right)$

$-\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^{-1} z^1 + \left(\frac{1}{a}\right)^{-2} z^2 + \cdots\right)$

$a_n = -a^{-n}$	$(-n \geq 0)$
$a_n = -\left(\frac{1}{a}\right)^n$	$(n < 1)$

$a\,z^{-1}$	$\lvert z\rvert > a$

$-\left(a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \cdots\right)$

$-\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^{-1} z^{-1} + \left(\frac{1}{a}\right)^{-2} z^{-2} + \cdots\right)$

$a_n = -\left(\frac{1}{a}\right)^{-n}$	$(-n < 1)$
$a_n = -a^n$	$(n \geq 0)$

$a^{-1} z^{-1}$	$\lvert z\rvert > a^{-1}$

$\left(a^0 z^0 + a^{-1} z^{-1} + a^{-2} z^{-2} + \cdots\right)$

$\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^{-1} + \left(\frac{1}{a}\right)^2 z^{-2} + \cdots\right)$

$a_n = a^{-n}$	$(-n < 1)$
$a_n = \left(\frac{1}{a}\right)^n$	$(n \geq 0)$

$a^{-1} z$	$\lvert z\rvert < a$

$\left(a^0 z^0 + a^{-1} z^1 + a^{-2} z^2 + \cdots\right)$

$\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \cdots\right)$

$a_n = \left(\frac{1}{a}\right)^{-n}$	$(-n \geq 0)$
$a_n = a^n$	$(n < 1)$

$a^{-1} z$	$\lvert z\rvert < a$

$-\left(a^0 z^0 + a^{-1} z^1 + a^{-2} z^2 + \cdots\right)$

$-\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \cdots\right)$

$a_n = -\left(\frac{1}{a}\right)^{-n}$	$(-n \geq 0)$
$a_n = -a^n$	$(n < 1)$

$a^{-1} z^{-1}$	$\lvert z\rvert > a^{-1}$

$-\left(a^0 z^0 + a^{-1} z^{-1} + a^{-2} z^{-2} + \cdots\right)$

$-\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^{-1} + \left(\frac{1}{a}\right)^2 z^{-2} + \cdots\right)$

$a_n = -a^{-n}$	$(-n < 1)$
$a_n = -\left(\frac{1}{a}\right)^n$	$(n \geq 0)$

$a\,z^{-1}$	$\lvert z\rvert > a$

$\left(a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \cdots\right)$

$\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^{-1} z^{-1} + \left(\frac{1}{a}\right)^{-2} z^{-2} + \cdots\right)$

$a_n = \left(\frac{1}{a}\right)^{-n}$	$(-n < 1)$
$a_n = a^n$	$(n \geq 0)$

$a\,z$	$\lvert z\rvert < a^{-1}$

$\left(a^0 z^0 + a^1 z^1 + a^2 z^2 + \cdots\right)$

$\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^{-1} z^1 + \left(\frac{1}{a}\right)^{-2} z^2 + \cdots\right)$

$a_n = a^{-n}$	$(-n \geq 0)$
$a_n = \left(\frac{1}{a}\right)^n$	$(n < 1)$

Geometric Series with a unit start term

Laurent Series vs. z-Transform

$a\,z$ $\quad |z| < a^{-1}$

$-\left(a^0 z^0 + a^1 z^1 + a^2 z^2 + \cdots\right)$

$-\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^{-1} z^1 + \left(\tfrac{1}{a}\right)^{-2} z^2 + \cdots\right)$

nt Series	$a_n = -a^n$	$(n \geq 0)$
sform	$a_n = -\left(\tfrac{1}{a}\right)^n$	$(n < 1)$

$a\,z^{\boxed{-1}}$ $\quad |z| > a$

$-\left(a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \cdots\right)$

$-\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^{-1} z^{-1} + \left(\tfrac{1}{a}\right)^{-2} z^{-2} + \cdots\right)$

$a_n = -\left(\tfrac{1}{a}\right)^n$	$(n < 1)$
$a_n = -a^n$	$(n \geq 0)$

$a^{\boxed{-1}} z^{\boxed{-1}}$ $\quad |z| > a^{-1}$

$\left(a^0 z^0 + a^{-1} z^{-1} + a^{-2} z^{-2} + \cdots\right)$

$\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^1 z^{-1} + \left(\tfrac{1}{a}\right)^2 z^{-2} + \cdots\right)$

nt Series	$a_n = a^n$	$(n < 1)$
nsform	$a_n = \left(\tfrac{1}{a}\right)^n$	$(n \geq 0)$

$a^{\boxed{-1}} z$ $\quad |z| < a$

$\left(a^0 z^0 + a^{-1} z^1 + a^{-2} z^2 + \cdots\right)$

$\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^1 z^1 + \left(\tfrac{1}{a}\right)^2 z^2 + \cdots\right)$

$a_n = \left(\tfrac{1}{a}\right)^n$	$(n \geq 0)$
$a_n = a^n$	$(n < 1)$

$a^{\boxed{-1}} z$ $\quad |z| < a$

$-\left(a^0 z^0 + a^{-1} z^1 + a^{-2} z^2 + \cdots\right)$

$-\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^1 z^1 + \left(\tfrac{1}{a}\right)^2 z^2 + \cdots\right)$

ent Series	$a_n = -\left(\tfrac{1}{a}\right)^n$	$(n \geq 0)$
nsform	$a_n = -a^n$	$(n < 1)$

$a^{\boxed{-1}} z^{\boxed{-1}}$ $\quad |z| > a^{-1}$

$-\left(a^0 z^0 + a^{-1} z^{-1} + a^{-2} z^{-2} + \cdots\right)$

$-\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^1 z^{-1} + \left(\tfrac{1}{a}\right)^2 z^{-2} + \cdots\right)$

$a_n = -a^n$	$(n < 1)$
$a_n = -\left(\tfrac{1}{a}\right)^n$	$(n \geq 0)$

$a\,z^{\boxed{-1}}$ $\quad |z| > a$

$\left(a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \cdots\right)$

$\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^{-1} z^{-1} + \left(\tfrac{1}{a}\right)^{-2} z^{-2} + \cdots\right)$

ent Series	$a_n = \left(\tfrac{1}{a}\right)^n$	$(n < 1)$
nsform	$a_n = a^n$	$(n \geq 0)$

$a\,z$ $\quad |z| < a^{-1}$

$\left(a^0 z^0 + a^1 z^1 + a^2 z^2 + \cdots\right)$

$\left(\left(\tfrac{1}{a}\right)^0 z^0 + \left(\tfrac{1}{a}\right)^{-1} z^1 + \left(\tfrac{1}{a}\right)^{-2} z^2 + \cdots\right)$

$a_n = a^n$	$(n \geq 0)$
$a_n = \left(\tfrac{1}{a}\right)^n$	$(n < 1)$

Geometric Series with a non-unit start term

Laurent Series

$$a^{\boxed{-1}} z^{\boxed{-1}} \qquad |z| > a^{-1}$$

$$(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots)$$

$$a_n = a^n \qquad (n < 0)$$

$$a^{\boxed{-1}} z \qquad |z| < a$$

$$(a^{-1}z^{1} + a^{-2}z^{2} + a^{-3}z^{3} + \cdots)$$

$$\left(\left(\tfrac{1}{a}\right)^{1}z^{1} + \left(\tfrac{1}{a}\right)^{2}z^{2} + \left(\tfrac{1}{a}\right)^{3}z^{3} + \cdots\right)$$

$$a_n = \left(\tfrac{1}{a}\right)^n \qquad (n \geq 1)$$

$$a\, z \qquad |z| < a^{-1}$$

$$-(a^{1}z^{1} + a^{2}z^{2} + a^{3}z^{3} + \cdots)$$

$$a_n = -a^n \qquad (n \geq 1)$$

$$a\, z^{\boxed{-1}} \qquad |z| > a$$

$$-(a^{1}z^{-1} + a^{2}z^{-2} + a^{3}z^{-3} + \cdots)$$

$$-\left(\left(\tfrac{1}{a}\right)^{-1}z^{-1} + \left(\tfrac{1}{a}\right)^{-2}z^{-2} + \left(\tfrac{1}{a}\right)^{-3}z^{-3} + \cdots\right)$$

$$a_n = -\left(\tfrac{1}{a}\right)^n \qquad (n < 0)$$

$$a\, z^{\boxed{-1}} \qquad |z| > a$$

$$(a^{1}z^{-1} + a^{2}z^{-2} + a^{3}z^{-3} + \cdots)$$

$$\left(\left(\tfrac{1}{a}\right)^{-1}z^{-1} + \left(\tfrac{1}{a}\right)^{-2}z^{-2} + \left(\tfrac{1}{a}\right)^{-3}z^{-3} + \cdots\right)$$

$$a_n = \left(\tfrac{1}{a}\right)^n \qquad (n < 0)$$

$$a\, z \qquad |z| < a^{-1}$$

$$(a^{1}z^{1} + a^{2}z^{2} + a^{3}z^{3} + \cdots)$$

$$a_n = a^n \qquad (n \geq 1)$$

$$a^{\boxed{-1}} z \qquad |z| < a$$

$$-(a^{-1}z^{1} + a^{-2}z^{2} + a^{-3}z^{3} + \cdots)$$

$$-\left(\left(\tfrac{1}{a}\right)^{1}z^{1} + \left(\tfrac{1}{a}\right)^{2}z^{2} + \left(\tfrac{1}{a}\right)^{3}z^{3} + \cdots\right)$$

$$a_n = -\left(\tfrac{1}{a}\right)^n \qquad (n \geq 1)$$

$$a^{\boxed{-1}} z^{\boxed{-1}} \qquad |z| > a^{-1}$$

$$-(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots)$$

$$a_n = -a^n \qquad (n < 0)$$

Geometric Series with a non-unit start term

z-Transform

$a^{-1} z^{-1}$ | $|z| > a^{-1}$

$$(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots)$$

$$\left(\left(\tfrac{1}{a}\right)^{1}z^{-1} + \left(\tfrac{1}{a}\right)^{2}z^{-2} + \left(\tfrac{1}{a}\right)^{3}z^{-3} + \cdots\right)$$

$a_n = a^{-n}$	$(-n < 0)$
$a_n = \left(\tfrac{1}{a}\right)^{n}$	$(n \geq 1)$

$a^{-1} z$ | $|z| < a$

$$(a^{-1}z^{1} + a^{-2}z^{2} + a^{-3}z^{3} + \cdots)$$

$$\left(\left(\tfrac{1}{a}\right)^{1}z^{1} + \left(\tfrac{1}{a}\right)^{2}z^{2} + \left(\tfrac{1}{a}\right)^{3}z^{3} + \cdots\right)$$

$a_n = \left(\tfrac{1}{a}\right)^{-n}$	$(-n \geq 1)$
$a_n = a^{n}$	$(n < 0)$

$a z$ | $|z| < a^{-1}$

$$-(a^{1}z^{1} + a^{2}z^{2} + a^{3}z^{3} + \cdots)$$

$$-\left(\left(\tfrac{1}{a}\right)^{-1}z^{1} + \left(\tfrac{1}{a}\right)^{-2}z^{2} + \left(\tfrac{1}{a}\right)^{-3}z^{3} + \cdots\right)$$

$a_n = -a^{-n}$	$(-n \geq 1)$
$a_n = -\left(\tfrac{1}{a}\right)^{n}$	$(n < 0)$

$a z^{-1}$ | $|z| > a$

$$-(a^{1}z^{-1} + a^{2}z^{-2} + a^{3}z^{-3} + \cdots)$$

$$-\left(\left(\tfrac{1}{a}\right)^{-1}z^{-1} + \left(\tfrac{1}{a}\right)^{-2}z^{-2} + \left(\tfrac{1}{a}\right)^{-3}z^{-3} + \cdots\right)$$

$a_n = -\left(\tfrac{1}{a}\right)^{-n}$	$(-n < 0)$
$a_n = -a^{n}$	$(n \geq 1)$

$a z^{-1}$ | $|z| > a$

$$(a^{1}z^{-1} + a^{2}z^{-2} + a^{3}z^{-3} + \cdots)$$

$$\left(\left(\tfrac{1}{a}\right)^{-1}z^{-1} + \left(\tfrac{1}{a}\right)^{-2}z^{-2} + \left(\tfrac{1}{a}\right)^{-3}z^{-3} + \cdots\right)$$

$a_n = \left(\tfrac{1}{a}\right)^{-n}$	$(-n < 0)$
$a_n = a^{n}$	$(n \geq 1)$

$a z$ | $|z| < a^{-1}$

$$(a^{1}z^{1} + a^{2}z^{2} + a^{3}z^{3} + \cdots)$$

$$\left(\left(\tfrac{1}{a}\right)^{-1}z^{1} + \left(\tfrac{1}{a}\right)^{-2}z^{2} + \left(\tfrac{1}{a}\right)^{-3}z^{3} + \cdots\right)$$

$a_n = a^{-n}$	$(-n \geq 1)$
$a_n = \left(\tfrac{1}{a}\right)^{n}$	$(n < 0)$

$a^{-1} z$ | $|z| < a$

$$-(a^{-1}z^{1} + a^{-2}z^{2} + a^{-3}z^{3} + \cdots)$$

$$-\left(\left(\tfrac{1}{a}\right)^{1}z^{1} + \left(\tfrac{1}{a}\right)^{2}z^{2} + \left(\tfrac{1}{a}\right)^{3}z^{3} + \cdots\right)$$

$a_n = -\left(\tfrac{1}{a}\right)^{-n}$	$(-n \geq 1)$
$a_n = -a^{n}$	$(n < 0)$

$a^{-1} z^{-1}$ | $|z| > a^{-1}$

$$-(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots)$$

$$-\left(\left(\tfrac{1}{a}\right)^{1}z^{-1} + \left(\tfrac{1}{a}\right)^{2}z^{-2} + \left(\tfrac{1}{a}\right)^{3}z^{-3} + \cdots\right)$$

$a_n = -a^{-n}$	$(-n < 0)$
$a_n = -\left(\tfrac{1}{a}\right)^{n}$	$(n \geq 1)$

Geometric Series with a non-unit start term

Laurent Series vs. z-Transform

$a^{-1} z^{-1}$ | $|z| > a^{-1}$

$(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \cdots)$

$((\frac{1}{a})^{1} z^{-1} + (\frac{1}{a})^{2} z^{-2} + (\frac{1}{a})^{3} z^{-3} + \cdots)$

nt Series	$a_n = a^n$	$(n < 0)$
nsform	$a_n = (\frac{1}{a})^n$	$(n \geq 1)$

$a^{-1} z$ | $|z| < a$

$(a^{-1} z^{1} + a^{-2} z^{2} + a^{-3} z^{3} + \cdots)$

$((\frac{1}{a})^{1} z^{1} + (\frac{1}{a})^{2} z^{2} + (\frac{1}{a})^{3} z^{3} + \cdots)$

nt Series	$a_n = (\frac{1}{a})^n$	$(n \geq 1)$
nsform	$a_n = a^n$	$(n < 0)$

$a z$ | $|z| < a^{-1}$

$-(a^{1} z^{1} + a^{2} z^{2} + a^{3} z^{3} + \cdots)$

$-((\frac{1}{a})^{-1} z^{1} + (\frac{1}{a})^{-2} z^{2} + (\frac{1}{a})^{-3} z^{3} + \cdots)$

nt Series	$a_n = -a^n$	$(n \geq 1)$
nsform	$a_n = -(\frac{1}{a})^n$	$(n < 0)$

$a z^{-1}$ | $|z| > a$

$-(a^{1} z^{-1} + a^{2} z^{-2} + a^{3} z^{-3} + \cdots)$

$-((\frac{1}{a})^{-1} z^{-1} + (\frac{1}{a})^{-2} z^{-2} + (\frac{1}{a})^{-3} z^{-3} + \cdots)$

	$a_n = -(\frac{1}{a})^n$	$(n < 0)$
	$a_n = -a^n$	$(n \geq 1)$

$a z^{-1}$ | $|z| > a$

$(a^{1} z^{-1} + a^{2} z^{-2} + a^{3} z^{-3} + \cdots)$

$((\frac{1}{a})^{-1} z^{-1} + (\frac{1}{a})^{-2} z^{-2} + (\frac{1}{a})^{-3} z^{-3} + \cdots)$

nt Series	$a_n = (\frac{1}{a})^n$	$(n < 0)$
nsform	$a_n = a^n$	$(n \geq 1)$

$a z$ | $|z| < a^{-1}$

$(a^{1} z^{1} + a^{2} z^{2} + a^{3} z^{3} + \cdots)$

$((\frac{1}{a})^{-1} z^{1} + (\frac{1}{a})^{-2} z^{2} + (\frac{1}{a})^{-3} z^{3} + \cdots)$

	$a_n = a^n$	$(n \geq 1)$
	$a_n = (\frac{1}{a})^n$	$(n < 0)$

$a^{-1} z$ | $|z| < a$

$-(a^{-1} z^{1} + a^{-2} z^{2} + a^{-3} z^{3} + \cdots)$

$-((\frac{1}{a})^{1} z^{1} + (\frac{1}{a})^{2} z^{2} + (\frac{1}{a})^{3} z^{3} + \cdots)$

nt Series	$a_n = -(\frac{1}{a})^n$	$(n \geq 1)$
sform	$a_n = -a^n$	$(n < 0)$

$a^{-1} z^{-1}$ | $|z| > a^{-1}$

$-(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \cdots)$

$-((\frac{1}{a})^{1} z^{-1} + (\frac{1}{a})^{2} z^{-2} + (\frac{1}{a})^{3} z^{-3} + \cdots)$

	$a_n = -a^n$	$(n < 0)$
	$a_n = -(\frac{1}{a})^n$	$(n \geq 1)$

Complemnt ROC Pairs -
Original Geometric Series Form Combinations

unit	$-\frac{1}{1-az}$	$\lvert z\rvert<a^{-1}$	$-a^n \quad (n \geq 0)$	$-\frac{1}{1-az^{-1}}$	$\lvert z\rvert>a$	$-(\frac{1}{a})^n \quad (n<1)$
non-unit	$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert>a^{-1}$	$a^n \quad (n<0)$	$\frac{a^{-1}z}{1-a^{-1}z}$	$\lvert z\rvert<a$	$(\frac{1}{a})^n \quad (n\geq 1)$

unit	$\frac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert>a^{-1}$	$a^n \quad (n<1)$	$\frac{1}{1-a^{-1}z}$	$\lvert z\rvert<a$	$(\frac{1}{a})^n \quad (n\geq 0)$
non-unit	$-\frac{az}{1-az}$	$\lvert z\rvert<a^{-1}$	$-a^n \quad (n\geq 1)$	$\frac{az^{-1}}{1-az^{-1}}$	$\lvert z\rvert>a$	$-(\frac{1}{a})^n \quad (n<0)$

unit	$-\frac{1}{1-a^{-1}z}$	$\lvert z\rvert<a$	$-(\frac{1}{a})^n \quad (n\geq 0)$	$-\frac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert>a^{-1}$	$-a^n \quad (n<1)$
non-unit	$\frac{az^{-1}}{1-az^{-1}}$	$\lvert z\rvert>a$	$(\frac{1}{a})^n \quad (n<0)$	$\frac{az}{1-az}$	$\lvert z\rvert<a^{-1}$	$a^n \quad (n\geq 1)$

unit	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert>a$	$(\frac{1}{a})^n \quad (n<1)$	$\frac{1}{1-az}$	$\lvert z\rvert<a^{-1}$	$a^n \quad (n\geq 0)$
non-unit	$-\frac{a^{-1}z}{1-a^{-1}z}$	$\lvert z\rvert<a$	$-(\frac{1}{a})^n \quad (n\geq 1)$	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert>a^{-1}$	$-a^n \quad (n<0)$

start term

Complemnt ROC Pairs -
Shifted Geometric Series Form Combinations

$-\frac{a}{1-az}$ $\quad |z|<a^{-1}$	$-a^{n+1}$ $\quad (n\geq 0)$	$-\frac{a}{1-az^{-1}}$ $\quad |z|>a$	$-(\frac{1}{a})^{n-1}$ $\quad (n<1)$
$\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\quad |z|>a^{-1}$	a^{n+1} $\quad (n<0)$	$\frac{z}{1-a^{-1}z}$ $\quad |z|<a$	$(\frac{1}{a})^{n-1}$ $\quad (n\geq 1)$

$\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\quad |z|>a^{-1}$	a^{n+1} $\quad (n<0)$	$\frac{z}{1-a^{-1}z}$ $\quad |z|<a$	$(\frac{1}{a})^{n-1}$ $\quad (n\geq 1)$
$-\frac{a}{1-az}$ $\quad |z|<a^{-1}$	$-a^{n+1}$ $\quad (n\geq 0)$	$\frac{a}{1-az^{-1}}$ $\quad |z|>a$	$-(\frac{1}{a})^{n-1}$ $\quad (n<1)$

$-\frac{a^{-1}}{1-a^{-1}z}$ $\quad |z|<a$	$-(\frac{1}{a})^{n+1}$ $\quad (n\geq 0)$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\quad |z|>a^{-1}$	$-a^{n-1}$ $\quad (n<1)$
$\frac{z^{-1}}{1-az^{-1}}$ $\quad |z|>a$	$(\frac{1}{a})^{n+1}$ $\quad (n<0)$	$\frac{z}{1-az}$ $\quad |z|<a^{-1}$	a^{n-1} $\quad (n\geq 1)$

$\frac{z^{-1}}{1-az^{-1}}$ $\quad |z|>a$	$(\frac{1}{a})^{n+1}$ $\quad (n<0)$	$\frac{z}{1-az}$ $\quad |z|<a^{-1}$	a^{n-1} $\quad (n\geq 1)$
$-\frac{a^{-1}}{1-a^{-1}z}$ $\quad |z|<a$	$-(\frac{1}{a})^{n+1}$ $\quad (n\geq 0)$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\quad |z|>a^{-1}$	$-a^{n-1}$ $\quad (n<1)$

Complemnt ROC Pairs - Reduced
Shifted Geometric Series Form Combinations

$-\dfrac{a}{1-az}$	$\lvert z\rvert < a^{-1}$	$-a^{n+1}$	$(n \geq 0)$
$\dfrac{z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	a^{n+1}	$(n < 0)$

$-\dfrac{a}{1-az^{-1}}$	$\lvert z\rvert > a$	$-\left(\frac{1}{a}\right)^{n-1}$	$(n < 1)$
$\dfrac{z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\left(\frac{1}{a}\right)^{n-1}$	$(n \geq 1)$

$-\dfrac{a^{-1}}{1-a^{-1}z}$	$\lvert z\rvert < a$	$-\left(\frac{1}{a}\right)^{n+1}$	$(n \geq 0)$
$\dfrac{z^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$\left(\frac{1}{a}\right)^{n+1}$	$(n < 0)$

$-\dfrac{a^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$-a^{n-1}$	$(n < 1)$
$\dfrac{z}{1-az}$	$\lvert z\rvert < a^{-1}$	a^{n-1}	$(n \geq 1)$