Redundant CORDIC Tagaki

20160723

.

Lookahead Technique - CC Kao
Hybrid CORDIC - Wang & Swartzlander (1997)
Low Latency Time CORDIC Algorithms - Timmermann (1992)
Merged CORDIC Algorithms - Wang & Swartzlander (1995)
Merged Scaling Multiplication CORDIC Algorithms - Wang & Swartzlander (1997)
- Takagi (1987)
Redundant and on-line CORDIC - Ercegovac & Lang (1990)
Double Step Branching CORDIC - Phatak (1998)
- Duprat & Muller (1993)

Takagi : Redundant CORDIC

accelerate the CORDIC method

use a redundant number representation
the direction of the rotation
- determined by a few most significant digits of the remaining angle represented in the redundant number system.

- accelerated ← carry propogation eliminated

- but no rotation extension is performed for some angles
  → scale factor becomes a variable dependent on the operand
  → the scale factor has to be calculated during the computation and the result has to be corrected with it.

① double rotation
② correcting rotation.

① Double Rotation Method — Takagi

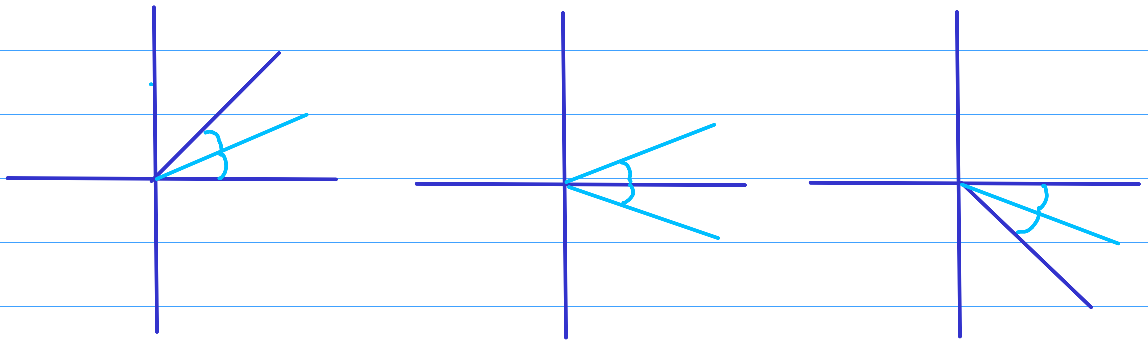
redundant binary representation.

digit set {$\bar{1}$, 0, 1}

[11] Chow, Robertson 1978 Logical Design of redundant binary adder

negative rotation − −
non-rotation ± ∓
positive rotation + +

2 sub rotation → const s.f.



$\tan^{-1} 2^{j}$

Takagi, Redundant CORDIC, const s.f. sin cos, 1991

$$X_j = X_{j-1} - q_j \cdot 2^{-j} \cdot Y_{j-1} - p_j \cdot 2^{-2(j+1)} \cdot X_{j-1}$$

$$Y_j = Y_{j-1} + q_j \cdot 2^{-j} \cdot X_{j-1} - p_j \cdot 2^{-2(j+1)} \cdot Y_{j-1}$$

$$z_j = Z_{j-1} - q_j \cdot 2 \cdot \tan^{-1} 2^{-(j+1)}$$

$$(q_j, p_j) = \begin{cases} (\bar{1}, 1) & z_{j-1}^{j-1} \; z_{j-1}^{j} \; z_{j-1}^{j+1} < 0 \\ (0, \bar{1}) & z_{j-1}^{j-1} \; z_{j-1}^{j} \; z_{j-1}^{j+1} = 0 \\ (1, 1) & z_{j-1}^{j-1} \; z_{j-1}^{j} \; z_{j-1}^{j+1} > 0 \end{cases}$$

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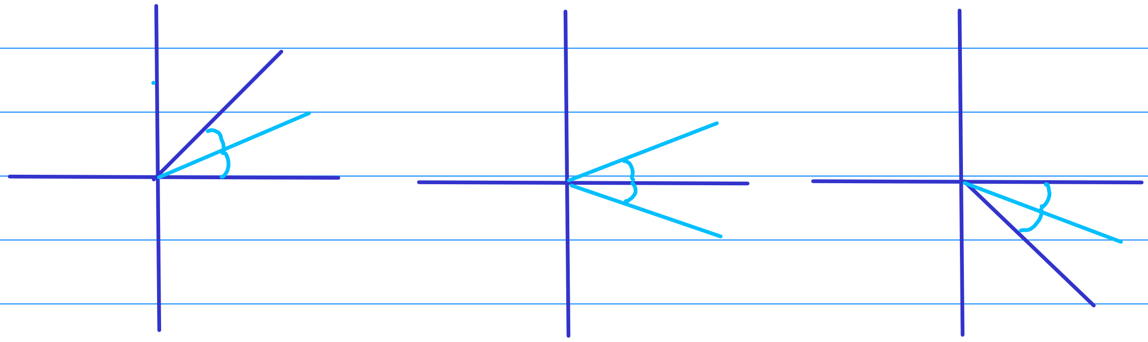
$$\boxed{\begin{aligned} X_j &= X_{j-1} - q_j \cdot 2^{-j} \cdot Y_{j-1} \\ Y_j &= Y_{j-1} + q_j \cdot 2^{-j} \cdot X_{j-1} \\ Z_j &= Z_{j-1} - q_j \cdot \tan^{-1} 2^{-j} \end{aligned}}$$

q_j : the direction of the j-th rotation

$\{\bar{1}, 1\}$



$$\boxed{\begin{aligned} X_j &= X_{j-1} - q_j \cdot 2^{-j} \cdot Y_{j-1} - p_j \cdot 2^{-2(j-1)} \cdot X_{j-1} \\ Y_j &= Y_{j-1} + q_j \cdot 2^{-j} \cdot X_{j-1} - p_j \cdot 2^{-2(j-1)} \cdot Y_{j-1} \\ Z_j &= Z_{j-1} - q_j \cdot 2 \cdot \tan^{-1} 2^{-j-1} \end{aligned}}$$

2 rotation extensions with the angle $\tan^{-1} 2^{-j-1}$

Takagi, Redundant CORDIC, const s.f. sin cos, 1991

$$\boxed{\begin{aligned} X_j &= X_{j-1} - q_j \cdot 2^{-j} \cdot Y_{j-1} \\ Y_j &= Y_{j-1} + q_j \cdot 2^{-j} \cdot X_{j-1} \\ Z_j &= Z_{j-1} - q_j \cdot \tan^{-1} 2^{-j} \end{aligned}}$$

$$\boxed{a, b \in \{+1, -1\}}$$

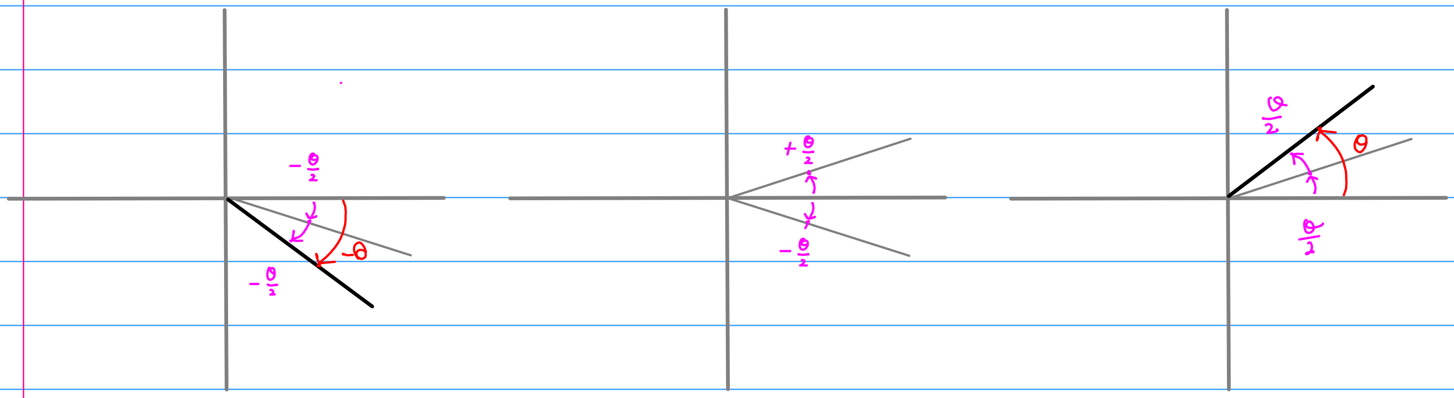
micro rotation (I) a

$$\left[\begin{aligned} X &= X_{j-1} - a \cdot 2^{-j-1} \cdot Y_{j-1} \\ Y &= Y_{j-1} + a \cdot 2^{-j-1} \cdot X_{j-1} \\ Z &= Z_{j-1} - a \cdot \tan^{-1} 2^{-j-1} \end{aligned}\right.$$

micro rotation (II) b

$$\left[\begin{aligned} X_j &= X - b \cdot 2^{-j+1} \cdot Y \\ Y_j &= Y + b \cdot 2^{-j+1} \cdot X \\ Z_j &= Z - b \cdot \tan^{-1} 2^{-j-1} \end{aligned}\right.$$

$$\begin{aligned} X_j &= \left(X_{j-1} - a \cdot 2^{-j-1} \cdot Y_{j-1} \right) - b \cdot 2^{-j-1} \cdot \left(Y_{j-1} + a \cdot 2^{-j-1} \cdot X_{j-1} \right) \\ Y_j &= \left(Y_{j-1} + a \cdot 2^{-j-1} \cdot X_{j-1} \right) + b \cdot 2^{-j-1} \left(X_{j-1} - a \cdot 2^{-j-1} \cdot Y_{j-1} \right) \\ Z_j &= \left(Z_{j-1} - a \cdot \tan^{-1} 2^{-j-1} \right) - b \cdot \tan^{-1} 2^{-j-1} \end{aligned}$$



$$\theta = \tan^{-1} 2^{-j} \qquad \frac{\theta}{2} = \tan^{-1} 2^{-j-1}$$

Takagi, Redundant CORDIC, const s.f. sin cos, 1991

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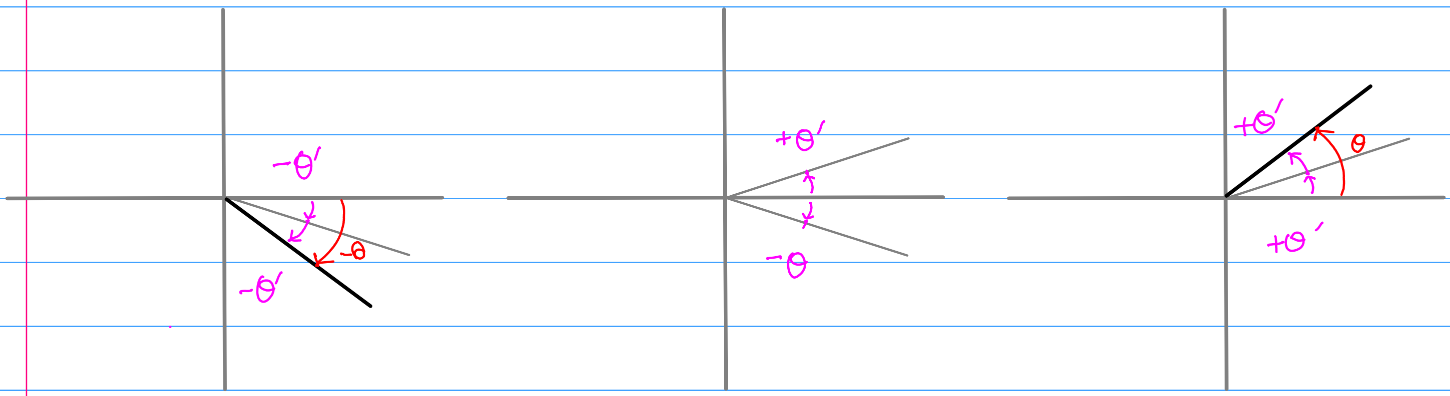
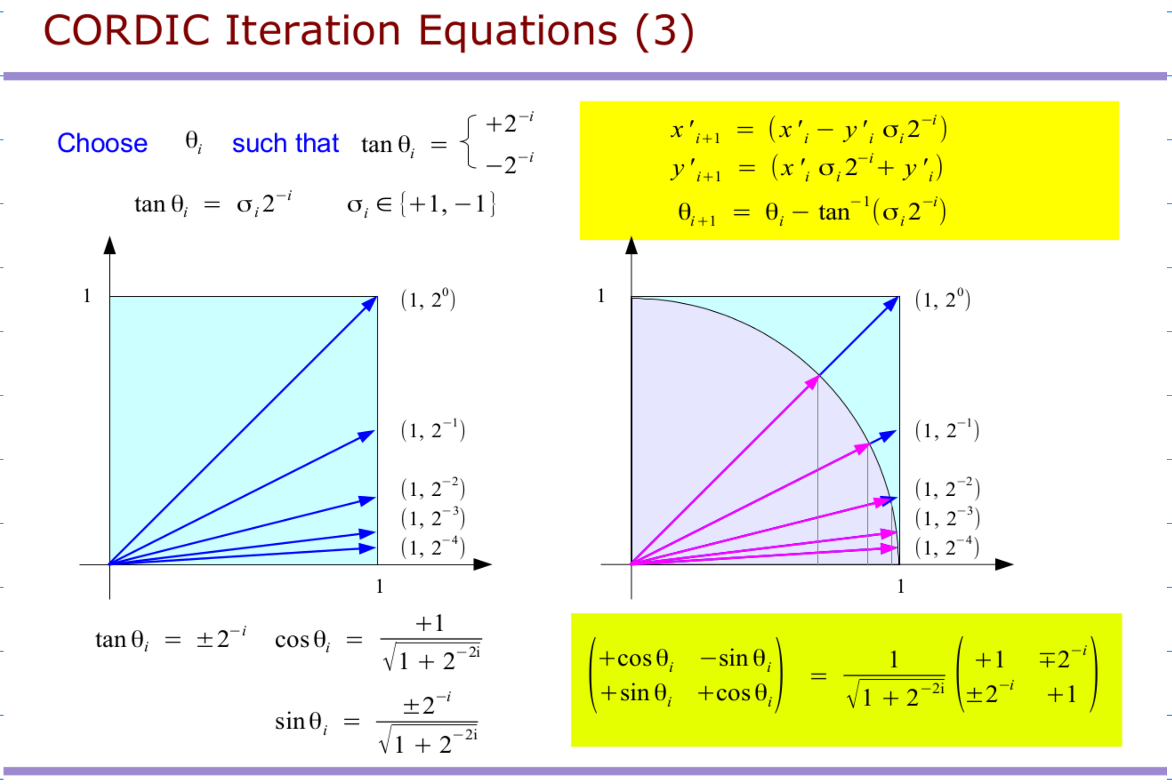


$$\theta = \tan^{-1} 2^{-j}$$

$$\theta' = \tan^{-1} 2^{-j-1}$$

$j >> 1$ 16



CORDIC Iteration Equations (3)

Choose θ_i such that $\tan\theta_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$\tan\theta_i = \sigma_i 2^{-i}$ $\sigma_i \in \{+1, -1\}$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$
$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$
$$\theta_{i+1} = \theta_i - \tan^{-1}(\sigma_i 2^{-i})$$

$\tan\theta_i = \pm 2^{-i}$ $\cos\theta_i = \dfrac{+1}{\sqrt{1 + 2^{-2i}}}$

$\sin\theta_i = \dfrac{\pm 2^{-i}}{\sqrt{1 + 2^{-2i}}}$

$$\begin{pmatrix} +\cos\theta_i & -\sin\theta_i \\ +\sin\theta_i & +\cos\theta_i \end{pmatrix} = \frac{1}{\sqrt{1 + 2^{-2i}}} \begin{pmatrix} +1 & \mp 2^{-i} \\ \pm 2^{-i} & +1 \end{pmatrix}$$

$$X_j = \left(X_{j-1} - a \cdot 2^{-j-1} \cdot Y_{j-1}\right) - b \cdot 2^{-j-1} \cdot \left(Y_{j-1} + a \cdot 2^{-j-1} \cdot X_{j-1}\right)$$
$$Y_j = \left(Y_{j-1} + a \cdot 2^{-j-1} \cdot X_{j-1}\right) + b \cdot 2^{-j-1} \left(X_{j-1} - a \cdot 2^{-j-1} \cdot Y_{j-1}\right)$$
$$Z_j = \left(Z_{j-1} - a \cdot \tan^{-1} 2^{-j-1}\right) - b \cdot \tan^{-1} 2^{-j-1}$$

$$X_j = X_{j-1} - (a+b) \cdot 2^{-j-1} \cdot Y_{j-1} - (ab)\, 2^{-2(j-1)} \cdot X_{j-1}$$
$$Y_j = Y_{j-1} + (a+b) \cdot 2^{-j-1} \cdot X_{j-1} - (ab)\, 2^{-2(j-1)} \cdot Y_{j-1}$$
$$Z_j = Z_{j-1} - (a+b) \cdot \tan^{-1} 2^{-j-1}$$

a	b	$a+b$	$a \cdot b$	
-1	-1	-2	1	if $Z_{j-1} < 0$
$+1$	-1	0	-1	if $Z_{j-1} = 0$
$+1$	$+1$	2	1	if $Z_{j-1} > 0$

$$X_j = X_{j-1} - \boxed{\frac{(a+b)}{2}} \cdot 2^{-j} \cdot Y_{j-1} - \boxed{(ab)} \cdot 2^{-2(j-1)} \cdot X_{j-1}$$

$$Y_j = Y_{j-1} + \boxed{\frac{(a+b)}{2}} \cdot 2^{-j} \cdot X_{j-1} - \boxed{(ab)} \cdot 2^{-2(j-1)} \cdot Y_{j-1}$$

$$Z_j = Z_{j-1} - \boxed{\frac{(a+b)}{2}} \cdot 2 \cdot \tan^{-1} 2^{-j-1}$$

$$X_j = X_{j-1} - q_j \cdot 2^{-j} \cdot Y_{j-1} - p_j \cdot 2^{-2(j-1)} \cdot X_{j-1}$$
$$Y_j = Y_{j-1} + q_j \cdot 2^{-j} \cdot X_{j-1} - p_j \cdot 2^{-2(j-1)} \cdot Y_{j-1}$$
$$Z_j = Z_{j-1} - q_j \cdot 2 \cdot \tan^{-1} 2^{-j-1}$$

$$\boxed{\frac{(a+b)}{2}} = q_j \qquad \boxed{(ab)} = p_j$$

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$$X_j = X_{j-1} - \boxed{\frac{(a+b)}{2}} \cdot 2^{-j} \cdot Y_{j-1} - \boxed{(ab)} \cdot 2^{-2(j-1)} \cdot X_{j-1}$$

$$Y_j = Y_{j-1} + \boxed{\frac{(a+b)}{2}} \cdot 2^{-j} \cdot X_{j-1} - \boxed{(ab)} \cdot 2^{-2(j-1)} \cdot Y_{j-1}$$

$$Z_j = Z_{j-1} - \boxed{\frac{(a+b)}{2}} \cdot 2 \cdot \tan^{-1} 2^{-j-1}$$

a	b	$a+b$	$a \cdot b$	q_j	p_j	
-1	-1	-2	1	$\bar{1}$	1	if $Z_{j-1} < 0$
$+1$	-1	0	-1	0	$\bar{1}$	if $Z_{j-1} = 0$
$+1$	$+1$	2	1	1	1	if $Z_{j-1} > 0$

$$X_j = X_{j-1} - q_j \cdot 2^{-j} \cdot Y_{j-1} - p_j \cdot 2^{-2(j-1)} \cdot X_{j-1}$$

$$Y_j = Y_{j-1} + q_j \cdot 2^{-j} \cdot X_{j-1} - p_j \cdot 2^{-2(j-1)} \cdot Y_{j-1}$$

$$Z_j = Z_{j-1} - q_j \cdot 2 \cdot \tan^{-1} 2^{-j-1}$$

$$(q_j, p_j) = \begin{cases} (\bar{1}, 1) & Z^{j-1}_{j-1} \; Z^{j}_{j-1} \; Z^{j+1}_{j-1} < 0 \\ (0, \bar{1}) & Z^{j-1}_{j-1} \; Z^{j}_{j-1} \; Z^{j+1}_{j-1} = 0 \\ (1, 1) & Z^{j-1}_{j-1} \; Z^{j}_{j-1} \; Z^{j+1}_{j-1} > 0 \end{cases}$$

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Takagi, Redundant CORDIC, const s.f. sin cos, 1991

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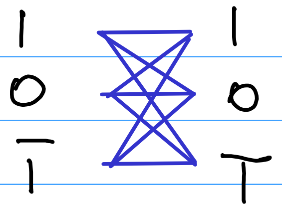
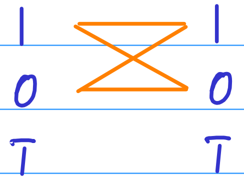


	p_i	q_i	c_i s_i
	1	1	(1 0)
⑥	1	0	(0 1) ($\bar{1}$ 1)
	1	$\bar{1}$	(0 0)
⑥	0	1	(0 1) ($\bar{1}$ 1)
	0	0	(0 0)
⑥	0	$\bar{1}$	(0 $\bar{1}$). ($\bar{1}$ 1)
	$\bar{1}$	1	(0 0)
⑥	$\bar{1}$	0	(0 $\bar{1}$). ($\bar{1}$ 1)
	$\bar{1}$	$\bar{1}$	($\bar{1}$ 0)

* Both non-negative

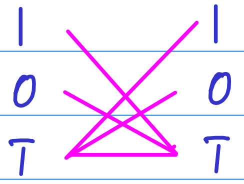
(1, 1) (1, 0) (0, 1) (0, 0)



* At least one negative

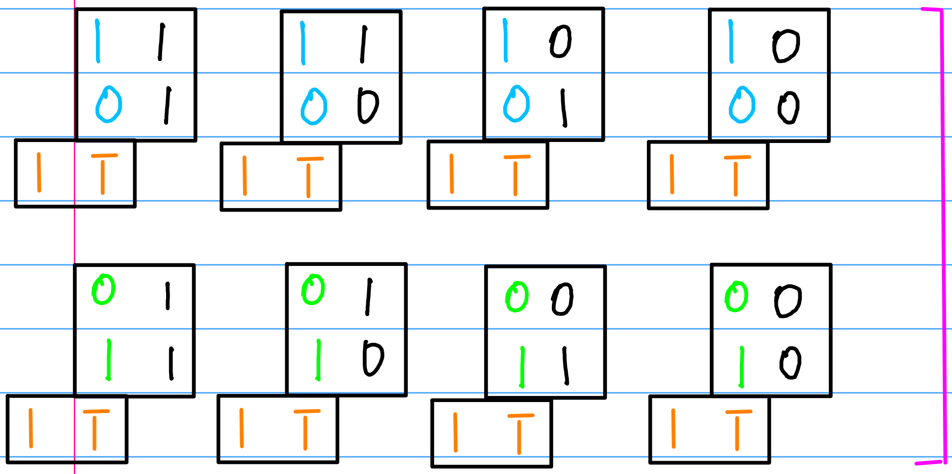
(1, $\bar{1}$) (0, $\bar{1}$), ($\bar{1}$ $\bar{1}$)

($\bar{1}$, 1) ($\bar{1}$, 0)



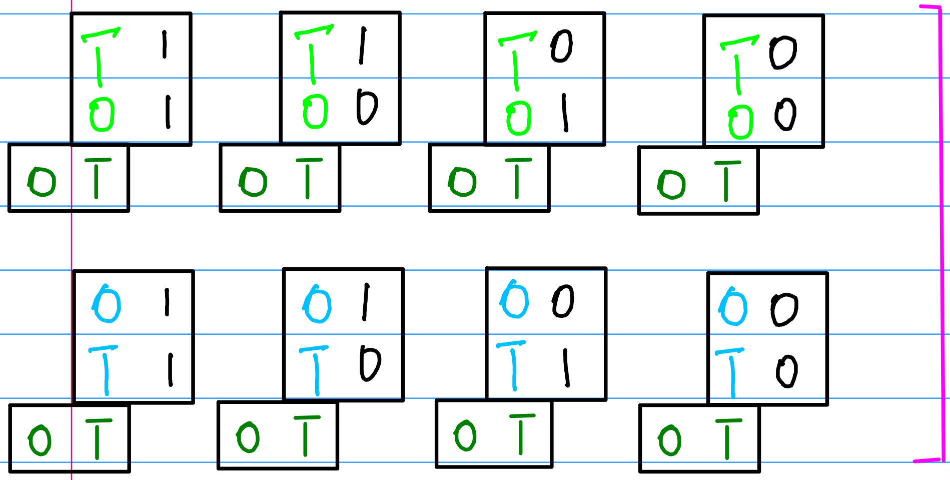
A Hardware Algorithm for Computing Sine and Cosing using Redundant Binary Representation, Takagi et al, Systems and Computers in Japan, 1987

p_{i+1}, q_{i+1} both non-negative



previous digits:
both non-negative
sum 1 ⇒ use 2-1

1 T



previous digits :
both non-negative
sum -1 ⇒ use -1

0 T

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p_{i+1}, q_{i+1} at least one $\bar{1}$

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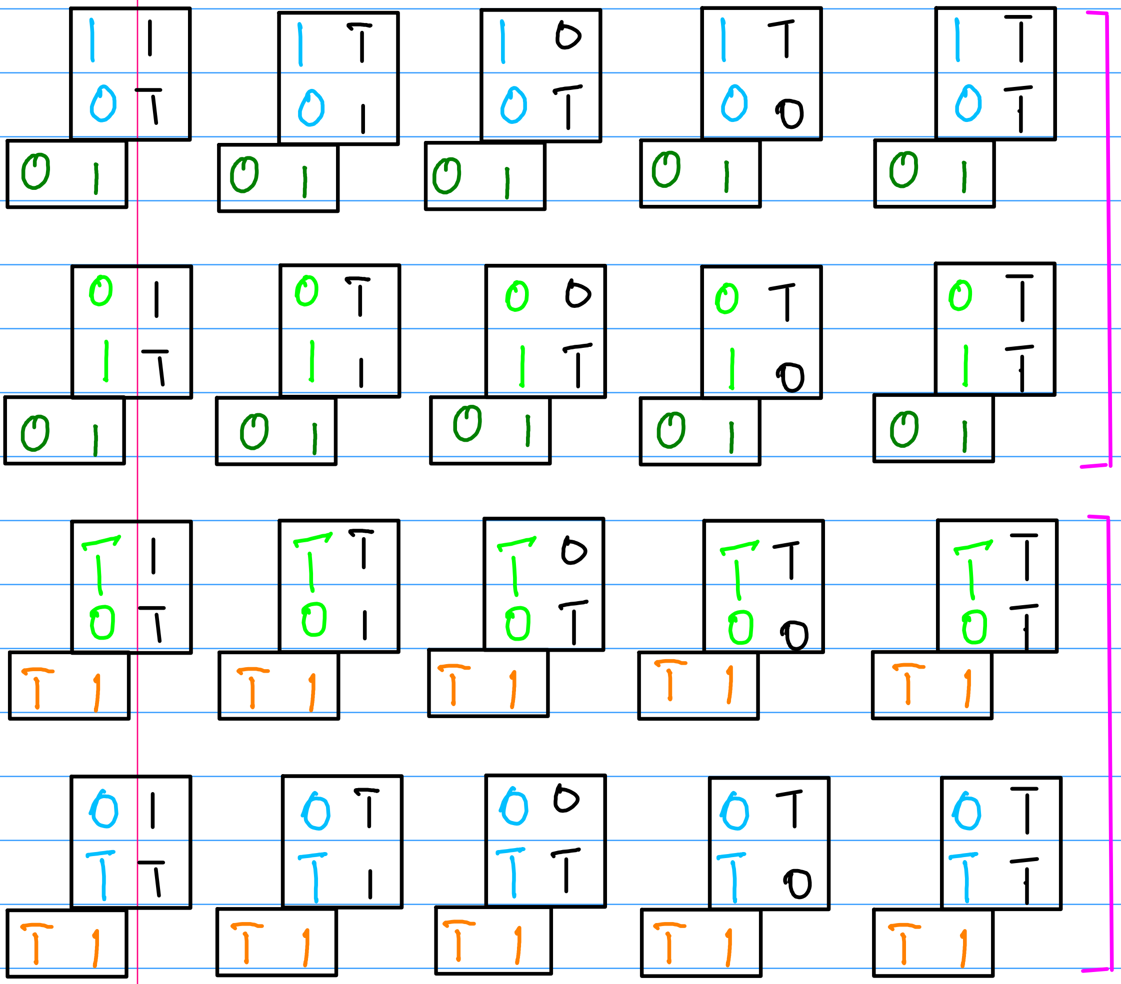

previous digits :
at least one negative
sum 1 ⇒ use 1

0 ①

previous digits :
at least one negative
sum -1 ⇒ use -2+1

$\bar{1}$ ①

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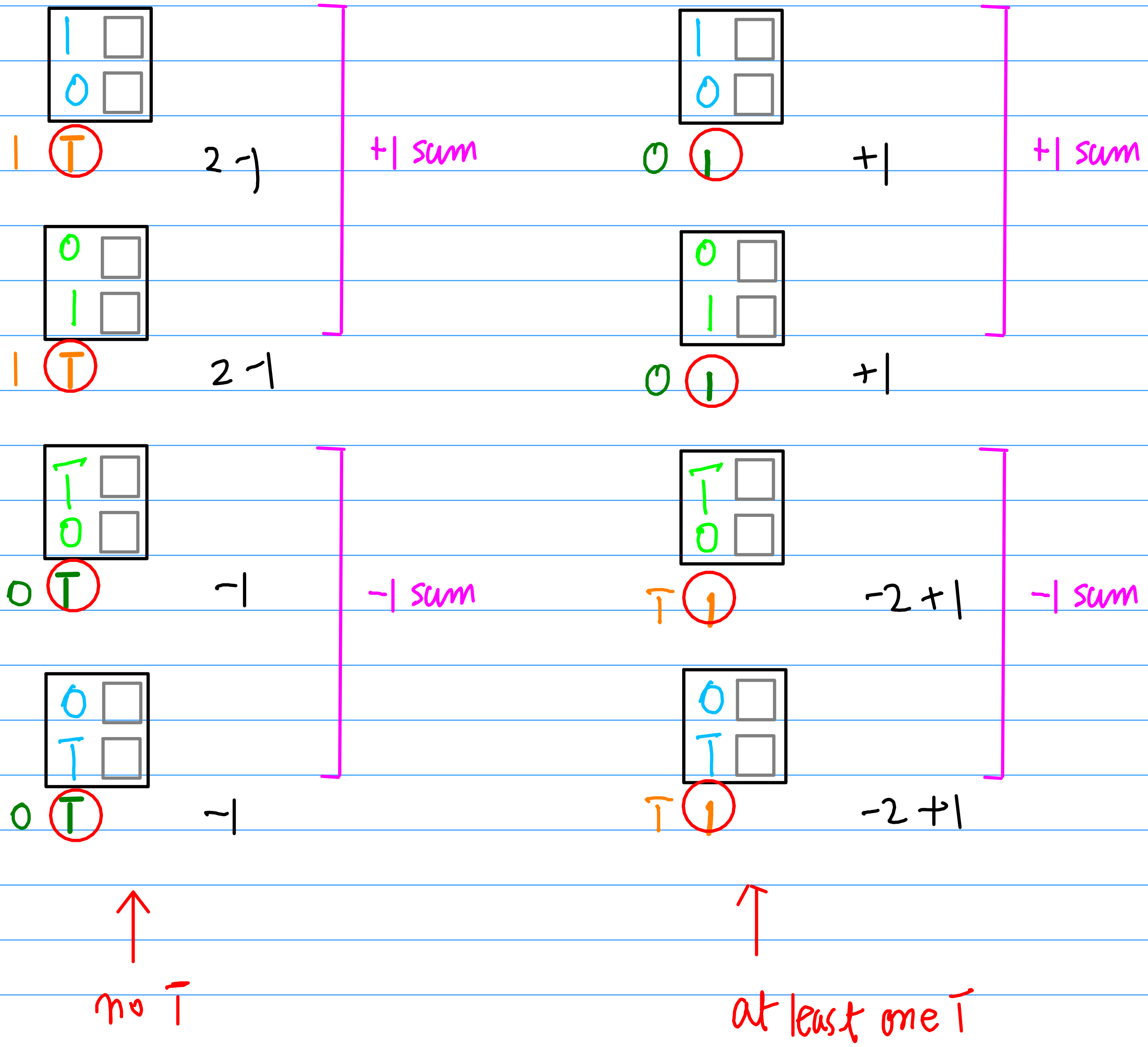
*** Both non-negative**

1	1	0	0
1	0	1	0

*** At least one negative**

1	0	$\bar{1}$	$\bar{1}$	$\bar{1}$
$\bar{1}$	$\bar{1}$	1	0	$\bar{1}$



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	p_i	q_i	c_i	s_i	
	1	1	1	0	
+1 sum	1	0	1	$\bar{1}$	p_{i+1}, q_{i+1} both non-negatives
	0	1	0	1	p_{i+1}, q_{i+1} at least one negative
	0	0	0	0	
	1	$\bar{1}$	0	0	
	$\bar{1}$	1	0	0	
−1 sum	$\bar{1}$	0	0	$\bar{1}$	p_{i+1}, q_{i+1} both non-negatives
	0	$\bar{1}$	$\bar{1}$	1	p_{i+1}, q_{i+1} at least one negative
	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	

+1 sum		−1 sum		
1	$\bar{1}$	0	$\bar{1}$	p_{i+1} q_{i+1} both non-negatives
0	1	$\bar{1}$	1	p_{i+1}, q_{i+1} at least one negative

A Hardware Algorithm for Computing Sine and Cosing using Redundant Binary Representation, Takagi et al, Systems and Computers in Japan, 1987

Addition

p_i	q_i	c_i	s_i
1	1	1	0
1	0	1	$\bar{1}$
0	1	0	1
0	0	0	0
1	$\bar{1}$	0	0
$\bar{1}$	1	0	0
$\bar{1}$	0	0	$\bar{1}$
0	$\bar{1}$	$\bar{1}$	1
$\bar{1}$	$\bar{1}$	$\bar{1}$	0

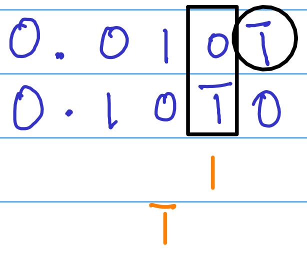
augend $0.010\bar{1}$ [SD2] } step 1

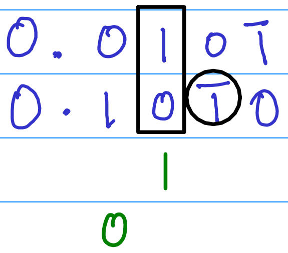
addend $0.10\bar{1}0$ [SD2]

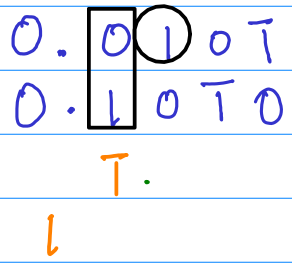
intermediate sum → $0.\bar{1}11\bar{1}$ } step 2

intermediate carry → $01.0\bar{1}0$

Sum $01.\bar{1}01\bar{1}$







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Subtraction

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① reverse the sign of the subtrahend

$1 \rightarrow \bar{1}$

$0 \rightarrow 0$

$\bar{1} \rightarrow 1$

② add it to the minuend

parallel addition-subtraction
by a combinational circuit
→ in a fixed time
regardless of the length
of operands

A Hardware Algorithm for Computing Sine and Cosing using Redundant Binary Representation,
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Adding/Subtracting ordinary binary numbers

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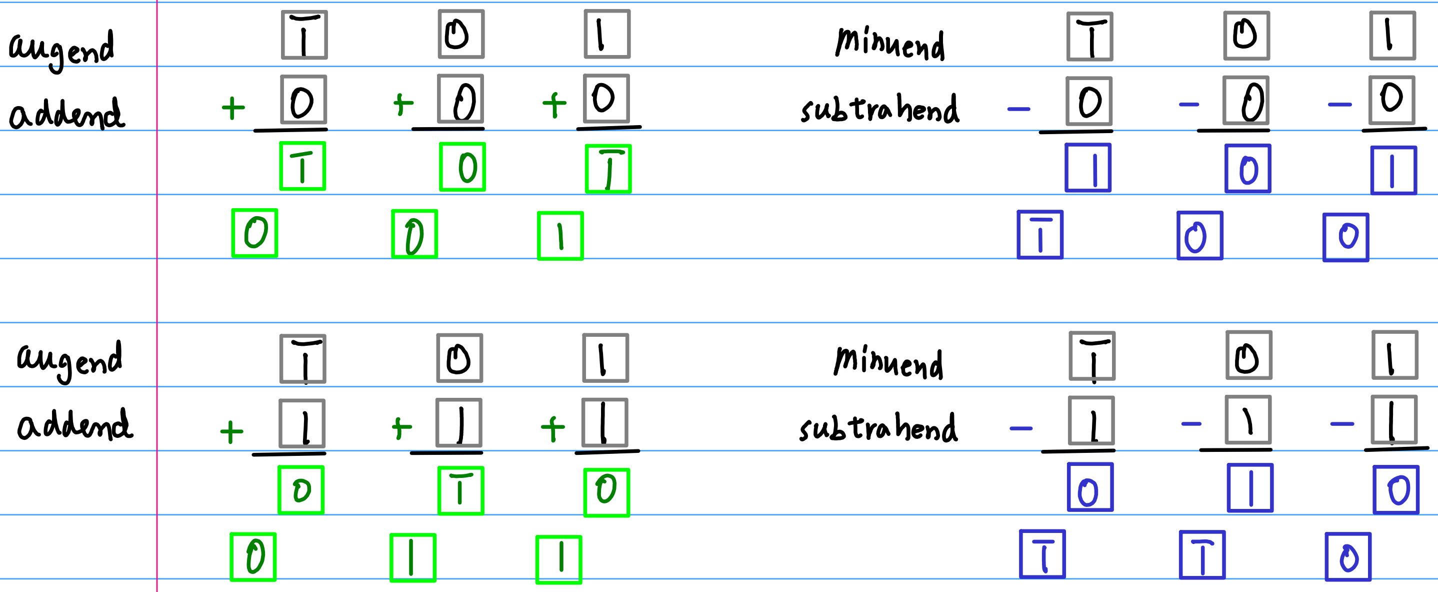
When addend or subtrahend
redundant binary numbers
with non-negative digit only
(=> ordinary binary number)

intermediate sum & carry
no need to check the next
digit (one lower position)



A Hardware Algorithm for Computing Sine and Cosing using Redundant Binary Representation, Takagi et al, Systems and Computers in Japan, 1987

$$\begin{array}{cc} & \boxed{T} \\ + & \boxed{0} \\ \hline & \boxed{T} \\ \boxed{0} & \end{array} \qquad \begin{array}{cc} & \boxed{0} \\ + & \boxed{0} \\ \hline & \boxed{0} \\ \boxed{0} & \end{array} \qquad \begin{array}{cc} & \boxed{1} \\ + & \boxed{0} \\ \hline & \boxed{T} \\ \boxed{1} & \end{array} \qquad\qquad \begin{array}{cc} & \boxed{T} \\ - & \boxed{0} \\ \hline & \boxed{1} \\ \boxed{T} & \end{array} \qquad \begin{array}{cc} & \boxed{0} \\ - & \boxed{0} \\ \hline & \boxed{0} \\ \boxed{0} & \end{array} \qquad \begin{array}{cc} & \boxed{1} \\ - & \boxed{0} \\ \hline & \boxed{1} \\ \boxed{0} & \end{array}$$

(-1) (0) (+1) (-1) (0) (+1)

$$\begin{array}{cc} & \boxed{T} \\ + & \boxed{1} \\ \hline & \boxed{0} \\ \boxed{0} & \end{array} \qquad \begin{array}{cc} & \boxed{0} \\ + & \boxed{1} \\ \hline & \boxed{T} \\ \boxed{1} & \end{array} \qquad \begin{array}{cc} & \boxed{1} \\ + & \boxed{1} \\ \hline & \boxed{0} \\ \boxed{1} & \end{array} \qquad\qquad \begin{array}{cc} & \boxed{T} \\ - & \boxed{1} \\ \hline & \boxed{0} \\ \boxed{T} & \end{array} \qquad \begin{array}{cc} & \boxed{0} \\ - & \boxed{1} \\ \hline & \boxed{1} \\ \boxed{T} & \end{array} \qquad \begin{array}{cc} & \boxed{1} \\ - & \boxed{1} \\ \hline & \boxed{0} \\ \boxed{0} & \end{array}$$

(0) (+1) (+2) (-2) (-1) (0)

unique	+2	1	0	
2 choices	+1	1	T	(0 1)
unique	0	0	0	
2 choices	-1	0	T	(T 1)

unique	+1	T	0	(0 1)
2 choices	0	0	0	
unique	-1	T	1	(0 T)
2 choices	-2	T	0	

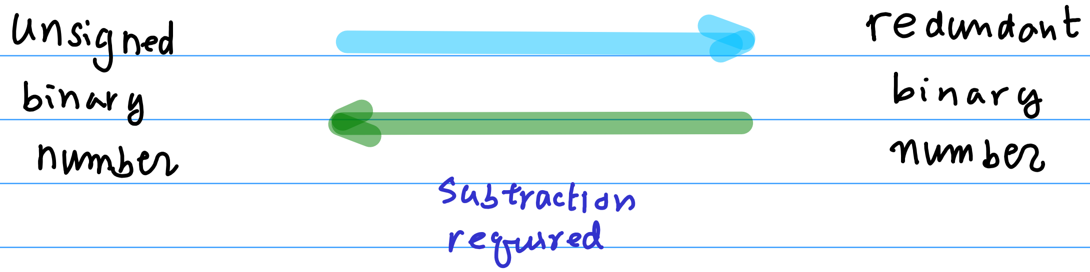
$$\boxed{T} \curvearrowleft Y_0$$

carry propagation stops

$$Y_0 \rightarrow \boxed{T}$$

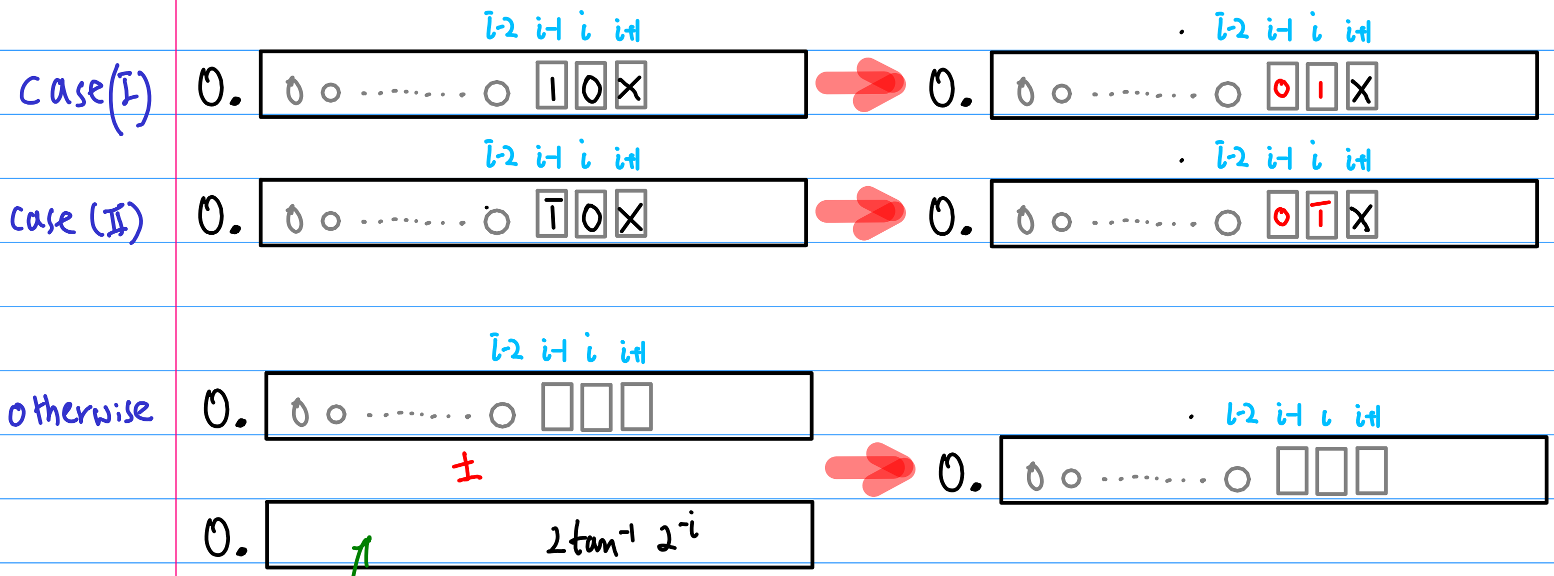
borrow propagation stops

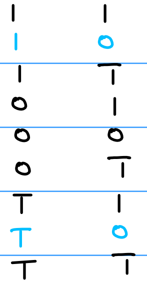
Conversion



$1\bar{1}00\bar{1}011$

$$\begin{array}{r} 10000011 \\ -\ 01001000 \end{array}$$





Ordinary binary numbers

(redundant binary number with non-negative digit)

n constants necessary

x_i, y_i, z_i

comp time $O(1)$
area $O(n)$

n times loop
time $O(n)$
area $O(n^2)$

Conversion to ordinary binary number

RCA
time $O(n)$
area $O(n)$

CLA
time $O(\log n)$
area $O(n)$

---

n-bit cos/sin computation

time	$O(n)$	$O(n \log n)$	$O(n^2)$
area	$O(n^2)$	$O(n^2)$	$O(n^2)$
		↑	↑
		all CLA's	all RCA's

$\left.\begin{array}{l} X_j \\ Y_j \end{array}\right\}$ redundant binary numbers

Z_j redundant binary fractions
most significant digit is located
j th binary position

determine the direction of the rotation
by evaluating the three most significant digits
of Z_{j-1}

z_{j-1}^{j-1} z_{j-1}^{j} z_{j-1}^{j+1}

q_j : the direction of the j-th rotation

$$(q_j, p_j) = \begin{cases} (\bar{1}, 1) & [z^{j-1}_{j-1} \ z^{j}_{j-1} \ z^{j+1}_{j-1}] < 0 \\ (0, 1) & [z^{j-1}_{j-1} \ z^{j}_{j-1} \ z^{j+1}_{j-1}] = 0 \\ (1, 1) & [z^{j-1}_{j-1} \ z^{j}_{j-1} \ z^{j+1}_{j-1}] > 0 \end{cases}$$

Lemma 1 for all j $(0 \leq j \leq n)$

$$-2^{-j} < z_j < 2^{-j}$$

Theorem 1 the rounding off errors are not considered

the error of X from $\cos\theta$ $< 2^{-n}$
the error of Y from $\sin\theta$ $< 2^{-n}$

$$|X - \cos\theta| < 2^{-n}$$
$$|Y - \sin\theta| < 2^{-n}$$

For all j's $(0 \le j \le n)$

$-2^{-j} < z_j < 2^{-j}$

$j=1$ $[-2^{-1}, +2^{-1}]$ $1.\overset{\overset{j}{\downarrow}}{\boxed{1}}000 < z_1 < 0.\overset{\overset{j}{\downarrow}}{\boxed{1}}000$

$j=2$ $[-2^{-2}, +2^{-2}]$ $1.1\boxed{1}00 < z_2 < 0.0\boxed{1}00$

$j=3$ $[-2^{-3}, +2^{-3}]$ $1.11\boxed{1}0 < z_3 < 0.00\boxed{1}0$

$1.1xxx$ $0.0xxx$

$1.11xx$ $0.00xx$

$1.111x$ $0.000x$

z_j can be represented redundant binary fraction

msd : j-th binary position

$z_1 = 0.\boxed{1}000 - 0.0\square\square\square = 0.0xxx$

$z_2 = 0.0\boxed{1}00 - 0.00\square\square = 0.00xx$

$z_3 = 0.00\boxed{1}0 - 0.000\square = 0.000x$

$z_1 = 0.\boxed{\bar{1}}000 + 0.0\square\square\square = 1.1xxx$

$z_2 = 0.0\boxed{\bar{1}}00 + 0.00\square\square = 1.11xx$

$z_3 = 0.00\boxed{\bar{1}}0 + 0.000\square = 1.111x$

msd (most significant digit)

$$(q_j , p_j) = \begin{cases} (\bar{1} , 1) & [z^{j-1}_{j-1} \;\; z^{j}_{j-1} \;\; z^{j+1}_{j-1}] < 0 \\ (0 , 1) & [z^{j-1}_{j-1} \;\; z^{j}_{j-1} \;\; z^{j+1}_{j-1}] = 0 \\ (1 , 1) & [z^{j-1}_{j-1} \;\; z^{j}_{j-1} \;\; z^{j+1}_{j-1}] > 0 \end{cases}$$

$$\boxed{\begin{aligned} X_j &= X_{j-1} - q_j \cdot 2^{-j} \cdot Y_{j-1} \\ Y_j &= Y_{j-1} + q_j \cdot 2^{-j} \cdot X_{j-1} \; . \\ Z_j &= Z_{j-1} - q_j \cdot \tan^{-1} 2^{-j} \end{aligned}}$$

$$Z_j \leftarrow Z_{j-1}$$

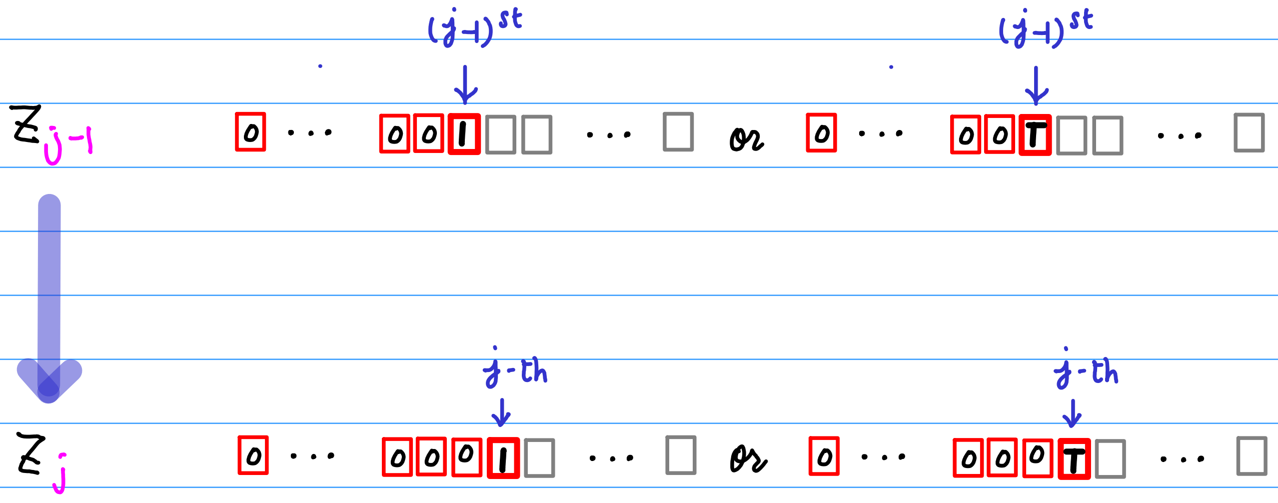
$$\begin{array}{lccc} & [z^{j-1}_{j-1} & z^{j}_{j-1} & z^{j+1}_{j-1}] \\ & \downarrow & \downarrow & \downarrow \\ \text{bit position :} & j-1 & j & j+1 \end{array}$$

> 0 ...... $q_j = 1$ Subtraction

$= 0$ ..... $q_j = 0$ no action

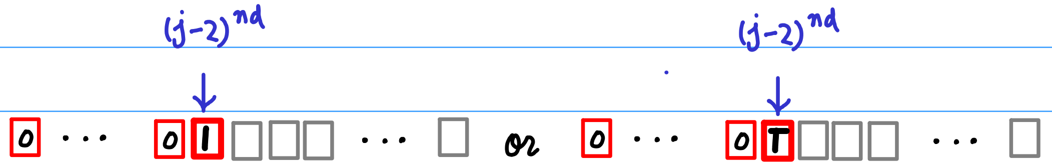
< 0 .... $q_j = -1$ addition

$\tan^{-1} 2^{-j}$ ordinary binary number



but

Z_j could be ⇒



because there may be a carry from the (j-1)st bit position

Special computation rule
at the bit position (j-1) & (j)
in adding / subtracting

* perform ordinary redundant binary addition / subtraction

$$\boxed{z_j = z_{j-1} - q_j \cdot \tan^{-1} 2^{-j}}$$

$$z_j \leftarrow z_{j-1}$$

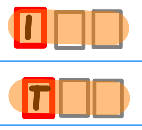
> 0 ..... $q_j = 1$ subtraction
= 0 ..... $q_j = 0$ no action
< 0 .... $q_j = -1$ addition

$$\begin{array}{lccc} & [\, z^{j-1}_{j-1} & z^{j}_{j-1} & z^{j+1}_{j-1} \,] \\ & \downarrow & \downarrow & \downarrow \\ \text{bit position :} & j-1 & j & j+1 \end{array}$$

but

z_j could be ⇒

* evaluate the 3 most significant digits of the obtained fraction



$$\Big\}\quad \begin{array}{ccccc} +1 & \text{or} & 0 & \text{or} & -1 \\ \downarrow & & \downarrow & & \downarrow \\ z_j = 1 & & z_j = 0 & & z_j = \bar{1} \end{array}$$