Propositional Logic– Logical Implication (4A)

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Based on

Contemporary Artificial Intelligence,
R.E. Neapolitan & X. Jiang

Logic and Its Applications,
Burkey & Foxley

What a proposition denotes

A: It is raining.

B: Professor N is 5 feet tall.

Proposition A denotes it is raining *currently*.

Proposition A does not denote it is raining an hour ago, tomorrow, a year ago…

Proposition B denotes Professor N is 6 feet

Proposition B does not denote Professor N is 7 feet, 5 feet, …

When both A and B are false

A: It is raining.

B: Professor N is 5 feet tall.

Suppose that

It is **not** raining (A is false)

Professor N is **6 feet** actually (B is false)

$A \rightarrow B$ is true according to **the truth table**

What A → B denotes

A: It is raining. It is **not** raining (A is false)

B: Professor N is 5 feet tall. Professor N is **6 feet** actually (B is false)

A → B is true according to **the truth table**

A → B : does not mean that $\neg A \to \neg B$

If it rains some day (false A),

then Professor N will be 5 feet (false B).

first concerns

the proposition that it is raining currently (false)

Suppose A → B is true

After finding A is true, then B must be true

After finding A is false, then we **do not know** whether B is true or false

Semantics of Implication Rule

Suppose A → B is true
If A is true then B must be true

Suppose A → B is false
If A is true then B must be false

Suppose A → B is true
If A is false then we **do not know** whether B is true or false

A	B	A→B
T	T	T
T	F	F
F	T	T
F	F	T

A different truth table

A	B	A→B
T	T	T
T	F	F
F	T	F
F	F	F

Suppose A → B is true
If B is true then A must be true

B implies A
This is *not* what we want

B → A (X)

Conventional Truth Table

A	B	A→B
T	T	T
T	F	F
F	T	T
F	F	T

Another different truth table

A	B	A→B
T	T	T
T	F	F
F	T	F
F	F	T

Suppose A → B is true
If A is false then B must be false

false A implies false B
This is not what we want

¬A → ¬B (X)

Conventional Truth Table

A	B	A→B
T	T	T
T	F	F
F	T	T
F	F	T

Logical Implication & Equivalence

A proposition is called a **tautology**
If and only if it is **true** **in all possible world** **(every interpretation)**
cf) in every model

A proposition is called a **contradiction**
If and only if it is **false** **in all possible world** **(every interpretation)**
cf) in every model

Given two propositions A and B,
If $A \Rightarrow B$ is a tautology
It is said that A **logically implies** B (A $\Rrightarrow$ B)

Given two propositions A and B,
If $A \Leftrightarrow B$ is a tautology,
It is said that A and B are **logically equivalent** $(A \equiv B)$

Material Implication & Logical Implication

Given two propositions **A** and **B**,

If **A** ⇒ **B** is a tautology

It is said that **A** **logically implies** **B** (**A** ⇛ **B**)

Material Implication **A** ⇒ **B** (not a tautology)
Logical Implication **A** ⇛ **B** (a tautology)

A	B	A⇒B
T	T	T
T	F	F
F	T	T
F	F	T

A	B	A∧B	A∧B ⇒ A
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

tautology

A∧B ⇛ A

Interpretations and Models

	A	B	A⇒B	
Interpretation **I1**	T	T	T	a **Model** of A⇒B
Interpretation **I2**	T	F	F	~~a **Model** of A⇒B~~
Interpretation **I3**	F	T	T	a **Model** of A⇒B
Interpretation **I4**	F	F	T	a **Model** of A⇒B

	A	B	A∧B	A∧B ⇒ A	
Interpretation **I1**	T	T	T	T	a **Model** of A∧B
Interpretation **I2**	T	F	F	T	~~a **Model** of A∧B~~
Interpretation **I3**	F	T	F	T	~~a **Model** of A∧B~~
Interpretation **I4**	F	F	F	T	~~a **Model** of A∧B~~

Entailment **A∧B ⊨ A**, or **A∧B ⇛ A**

Entailment

if **A→B** holds **in every model** then **A ⊨ B**,
and conversely if **A ⊨ B** then **A→B** is **true in every model**

any model that makes **A∧B** true

also makes **A** true **A∧B ⊨ A**

No case : True ⇒ False

A	B	A⇒B
T	T	T
T	F	F
F	T	T
F	F	T

A	B	A∧B	A∧B ⇒ A
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Entailment **A∧B ⊨ A**, or **A∧B ⇛ A**

Logical Equivalences

Excluded Middle Law	$A \lor \neg A \equiv \text{True}$
Contradiction Law	$A \land \neg A \equiv \text{False}$
Identity Law	$A \lor \text{False} \equiv A, \quad A \land \text{False} \equiv A$
Dominion Law	$A \land \text{False} \equiv \text{False}, \quad A \lor \text{True} \equiv \text{True}$
Idempotent Law	$A \lor A \equiv A, \quad A \land A \equiv A$
Commutativity Law	$A \land B \equiv B \land A, \quad A \lor B \equiv B \lor A$
Associativity Law	$(A \land B) \land C \equiv A \land (B \land C), \quad (A \lor B) \lor C \equiv A \lor (B \lor C)$
Distributivity Law	$A \land (B \lor C) \equiv (A \land B) \lor (A \land C), \quad A \lor (B \land C) \equiv (A \lor B) \land (A \lor C)$
De Morgan's Law	$\neg(A \land B) \equiv \neg A \lor \neg B, \quad \neg(A \lor B) \equiv \neg A \land \neg B$
Implication Elimination	$A \Rightarrow B \equiv \neg A \lor B$
If and Only If Elimination	$A \Leftrightarrow B \equiv A \Rightarrow B \land B \Rightarrow A$
Contraposition Law	$A \Rightarrow B \equiv \neg B \Rightarrow \neg A$
Double Negation	$\neg\neg A \equiv A$

Material & Logical Implications (1)

Material implication

a binary connective that creates a new sentence

A→**B** is a compound sentence using the symbol →.

Sometimes it refers to the truth function of this connective.

Logical implication

a relation between *two sentences* **A** and **B**

between compound propositions

any model that makes **A** true also makes **B** true

Entailment **A** ⊨ **B**, or **A** ⇛ **B**

Or sometimes, confusingly, as **A** ⇒ **B**,

although some people use ⇒ for material implication.

http://math.stackexchange.com/questions/68932/whats-the-difference-between-material-implication-and-logical-implication

Material & Logical Implications (2)

Material implication

- a symbol at *the object level* *between atomic propositions*
- a function of the truth value of two sentences in one fixed model

logical implication

- a relation at *the meta level* *between compound propositions*
- **not** directly about the truth values of sentences in a particular model
- about the relation between the truth values of the sentences when all models are considered.

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

A	B	$A \wedge B$	$A \wedge B \Rightarrow A$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

http://math.stackexchange.com/questions/68932/whats-the-difference-between-material-implication-and-logical-implication

Notations

Material implication	$\phi \rightarrow \psi$	$\phi \Rightarrow \psi$
Logical implication	$\phi \Rightarrow \psi$	$\phi \Rrightarrow \psi$

http://math.stackexchange.com/questions/68932/whats-the-difference-between-material-implication-and-logical-implication

Propositional formula

a propositional **formula** is
a type of **syntactic formula**
which is **well formed** and
has **a truth value**.

If the values of *all variables*
in a propositional **formula** are given,
it determines a **unique truth value**.

A propositional **formula** may also be called
a **propositional expression**,
a **sentence**, or
a **sentential formula**.

https://en.wikipedia.org/wiki/Propositional_formula

Propositional formula denotes a proposition

In mathematics, a propositional **formula** is often more briefly referred to as a "**proposition**",

but, more precisely, a propositional **formula** is
not a **proposition**
but a **formal expression** that denotes a **proposition**,
a **formal objec**t under discussion,

just like an expression such as "x + y"
is not a value,
but denotes a value.

https://en.wikipedia.org/wiki/Propositional_formula

Valid and Satisfiable Propositional Statements

A **formula** is **valid**
if it is **true** for *all values* of its **terms**.

Satisfiability refers to
the existence of **a** combination of values
to make the expression **true**.

A proposition is **satisfiable**
if there is at least one **true** result in its truth table

A proposition is **valid**
if all values it returns in the truth table are **true**.

http://math.stackexchange.com/questions/258602/what-is-validity-and-satisfiability-in-a-propositional-statement

Valid and Satisfiable Propositional Statements

Satisfiability -the other way of interpretation

A propositional statement is **satisfiable**
if and only if, its truth table is **not contradiction**.

Not contradiction means,
it could be a **tautology** also.

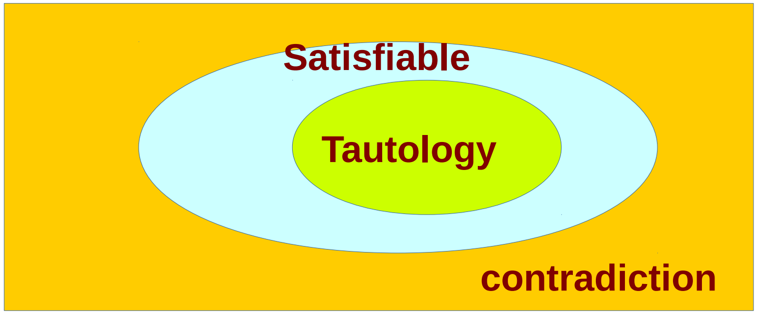
Tautology → **Satisfiable** (O)
Satisfiability → **Tautology**. (X)

if a propositional statement is **Tautology**, then its always **valid**.

Tautology implies (**Satisfiability** + **Validity**).

http://math.stackexchange.com/questions/258602/what-is-validity-and-satisfiability-in-a-propositional-statement

Valid and Satisfiable Propositional Statements



¬, ∧,
∨

contradiction

				F
				F
				F
				F
				F
				F

Satisfiable proposition

				T

Tautology

				T
				T
				T
				T
				T
				T

Valid proposition

http://math.stackexchange.com/questions/258602/what-is-validity-and-satisfiability-in-a-propositional-statement

Terms and Formulas

In analogy to natural language,
where a **noun** phrase refers to an **object** and
a whole **sentence** refers to a **fact**,

in mathematical logic,
a **term** denotes a mathematical **object** and
a **formula** denotes a mathematical **fact.**

In particular, terms appear as **components** of a formula.

https://en.wikipedia.org/wiki/Term_(logic)

Expression

A **first-order term** is recursively constructed from
- **constant** symbols,
- **variables** and
- **function** symbols.

An **expression** formed
by applying a **predicate** symbol
to an appropriate number of **terms** is called an **atomic formula**,
which evaluates to **true** or **false** in bivalent logics,
given an **interpretation**.

https://en.wikipedia.org/wiki/Term_(logic)

Terms and Formula Examples

For example, (x + 1) $*$ (x + 1) is a **term** built from
the **constant** 1,
the **variable** x, and
the binary **function** symbols + and $*$;

it is part of the **atomic formula** (x + 1) $*$ (x + 1) ≥ 0
which evaluates to **true** for each real-numbered value of x.

https://en.wikipedia.org/wiki/Term_(logic)

Logical Equivalences

¬, ∧ ,
∨

$\wedge \vee \neg \urcorner \Rightarrow$
$\Leftrightarrow \equiv$

$\Rightarrow$
$\Leftrightarrow$
$\equiv$

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