Power Spectral Density - Continuous Time

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February 11, 2021

Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Energy, Average Power – deterministic, time domain

a deterministic signal $x(t)$

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & otherwise \end{cases}$$

the **energy**

$$E(T) = \int_{-T}^{+T} x^2(t)dt = \int_{-\infty}^{+\infty} x_T^2(t)dt$$

the **average power**

$$P(T) = \frac{1}{2T}\int_{-T}^{+T} x^2(t)dt = \frac{1}{2T}\int_{-\infty}^{+\infty} x_T^2(t)dt$$

Fourier Transform Pair $x(t) \Longleftrightarrow X(\omega)$

Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

a deterministic signal $x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega$$

Fourier transform of $x_T(t)$
for continuous time signals

bounded duration, bounded variation

for a finite T, $x_T(t)$ is assumed to have bounded variation

$$\int_{-T}^{+T} |x(t)| dt < \infty$$

the Fourier transform of $x_T(t)$

$$X_T(\omega) = \int_{-\infty}^{+\infty} x_T(t) e^{-j\omega t} dt$$

$$= \int_{-T}^{+T} x(t) e^{-j\omega t} dt$$

Fourier transforms of $x_T(t)$ and $X_T(t)$

power spectral density for continuous time signals

deterministic $X_T(\omega)$ v.s. random $X_T(\omega)$

a **deterministic** sample signal $x_T(t)$

$$x_T(t) \Longleftrightarrow X_T(\omega)$$

a **random process** signal $X_T(t)$

$$X_T(t) \Longleftrightarrow X_T(\omega)$$

for a deterministic $x_T(t)$

a **deterministic** sample signal $x_T(t)$

$$\int_{-\infty}^{+\infty} x_T(\tau) x_T^*(\tau) d\tau = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_T(\omega) X_T^*(\omega) d\omega$$

$$\int_{-\infty}^{+\infty} |x_T(\tau)|^2 \, d\tau = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 \, d\omega$$

for a deterministic $x_T(t)$ v.s. a random $X_T(t)$

- a **deterministic** signal $x_T(t) \iff X_T(\omega)$

$$\int_{-\infty}^{+\infty} |x_T(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

- a **random** signal $X_T(t) \iff X_T(\omega)$

$$\int_{-\infty}^{+\infty} E\left[|X_T(t)|^2 \right] dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E\left[|X_T(\omega)|^2 \right] d\omega$$

Energy, Average Power – Parseval's theorem applied

a deterministic signal $x_T(t)$

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & otherwise \end{cases} \qquad x_T(t) \Longleftrightarrow X_T(\omega)$$

the **energy** by Parseval's theorem

$$E(T) = \int_{-T}^{+T} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

the **average power** by Parseval's theorem

$$P(T) = \frac{1}{2T}\int_{-T}^{+T} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

$E(T)$ and $P(T)$ in frequency domain – deterministic case

power spectral density for continuous time signals

deterministic $x_T(t) \Longleftrightarrow X_T(\omega)$

the **energy** for the **deterministic** $X_T(\omega)$ in $x_T(t) \Longleftrightarrow X_T(\omega)$

$$E(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

the **average power** for the **deterministic** $X_T(\omega)$

$$P(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

the **power spectral density** for the **deterministic** $X_T(\omega)$

$$\lim_{T\to\infty} \frac{|X_T(\omega)|^2}{2T}$$

$E(T)$ and $P(T)$ in frequency domain – random case

power spectral density for continuous time signals

random $X_T(t) \Longleftrightarrow X_T(\omega)$

the **energy** for the **random** $X_T(\omega)$ in $X_T(t) \Longleftrightarrow X_T(\omega)$

$$E(T) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E\left[|X_T(\omega)|^2\right] d\omega$$

the **average power** for the **random** $X_T(\omega)$

$$P(T) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T} d\omega$$

the **power spectral density** for the **random** $X_T(\omega)$

$$\lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

Average power $P(T)$ – bounded duraton $(-T, +T)$

power spectral density for continuous time signals

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

- not the average power in a **random process**
  only the power in one **sample function**
  - to obtain the **average power** over all possible **realizations**,
    replace $x(t)$ by $X(t)$
    take the **expected value** of $x^2(t)$, that is $TE\left[X^2(t)\right]$
  - then, the **average power** is a **random variable**
    with respect to the **random process** $X(t)$

- not the average power in an entire **sample function**
  - take $T \to \infty$ to include all power in the **ensemble** member

Average power P_{XX} – unbounded duraton $(-\infty, +\infty)$

power spectral density for continuous time signals

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

- replace $x(t)$ by the **random variable** $X(t)$
- take the **expected value** of $x^2(t)$, that is $E\left[X^2(t)\right]$

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} E\left[X^2(t)\right] dt$$

- take $T \to \infty$ to include all power

$$\boxed{P_{XX} = \lim_{T\to\infty} P(T)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} E\left[X^2(t)\right] dt$$

The time average

$$A_T[\bullet] = \frac{1}{2T}\int_{-T}^{T} [\bullet]dt \qquad A[\bullet] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [\bullet]dt$$

time average and sample average operations

$$\boxed{P_{XX} = \lim_{T\to\infty} P(T)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} E\left[X^2(t)\right] dt$$
$$= \lim_{T\to\infty} A_T\left[E\left[X^2(t)\right]\right]$$
$$= A\left[E\left[X^2(t)\right]\right]$$

for deterministic and random signals

the **average power** $P(T)$ for a **deterministic** signal $x(t)$

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

the **average power** P_{XX} for a **random process** $X(t)$

$$P_{XX} = \lim_{T \to \infty} P(T)$$
$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} E\left[X^2(t)\right] dt$$
$$= A\left[E\left[X^2(t)\right]\right]$$

the average power via power density

the average power P_{XX} for the **random process** $X_T(\omega)$

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \boxed{\lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}} d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \boxed{S_{XX}(\omega)} d\omega$$

the **Power Spectral Density (PSD)** $S_{XX}(\omega)$

$$\boxed{S_{XX}(\omega)} = \boxed{\lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}}$$

- $S_{XX}(\omega) \geq 0$
- $S_{XX}(-\omega) = S_{XX}(\omega)$ $\quad X(t)$ real
- $S_{XX}(\omega)$ real
- $\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = A\left[E\left[X^2(t)\right]\right]$
- $S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$
- $\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A[R_{XX}(t, t+\tau)]$
- $S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t+\tau)]\, e^{-j\omega\tau} d\tau$

Equations involving $S_{XX}(\omega)$

power spectral density for continuous time signals

the average power P_{xx} and the inverse Fourier transform of $S_{XX}(\omega)$

the **average power** related equation

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega = A\left[E\left[X^2(t)\right]\right]$$

the **autocorrelation** related equation

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{j\omega t}d\omega = A[R_{XX}(t, t+\tau)]$$

the average power P_{xx}

the average power related equation

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega = A\left[E\left[X^2(t)\right]\right]$$

- a **random process** $X(t)$ in time domain
- a **random process** $X(\omega)$ in frequency domain

$$X(t) = \lim_{T\to\infty} X_T(t) \qquad X(\omega) = \lim_{T\to\infty} X_T(\omega)$$

- Parseval's theorem over $X_T(t) \iff X_T(\omega)$

Average power P_{XX} in time / frequency domain

power spectral density for continuous time signals

Average power P_{XX} using $X_T(t)$ and $X_T(\omega)$

- Using a **random process** $X_T(t)$ in time domain

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} E\left[X^2(t)\right] dt$$
$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{+\infty} E\left[X_T^2(t)\right] dt$$
$$= \lim_{T\to\infty} A_T\left[E\left[X^2(t)\right]\right] = \boxed{A\left[E\left[X^2(t)\right]\right]}$$

- Using a **random process** $X_T(\omega)$ in frequency domain

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{\lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}} d\omega$$
$$= \boxed{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{S_{XX}(\omega)} d\omega}$$

the Inverse Fourier transform of $S_{XX}(\omega)$

the average power related equation

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{j\omega t}d\omega = A[R_{XX}(t, t+\tau)]$$

- auto-correlation function

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] \qquad \Rightarrow R_{XX}(\tau)$$

- a **random process** $X(t)$ in time domain
- a **random process** $X(\omega)$ in frequency domain

Fourier transform pairs

- $A[R_{XX}(t, t+\tau)] \Longleftrightarrow S_{XX}(\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t+\tau)]\, e^{-j\omega\tau} d\tau$$

$$A[R_{XX}(t, t+\tau)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega$$

- $R_{XX}(\tau) \Longleftrightarrow S_{XX}(\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$

for a WSS $X(t)$, $A[R_{XX}(t, t+\tau)] = R_{XX}(\tau)$

$S_{XX}(\omega)$ and $R_{XX}(\tau)$

the power spectral density

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

the auto-correlation function

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$

Fourier transform of $\frac{d^n}{dt^n}x(t)$

$$x(t) \Longleftrightarrow X(\omega)$$

$$\frac{d^n}{dt^n}x(t) \Longleftrightarrow (j\omega)^n X(\omega)$$

$$X_T(t) \Longleftrightarrow X_T(\omega)$$

$$Y(t) = \frac{d}{dt}X_T(t) \Longleftrightarrow (j\omega)X_T(\omega) = Y(\omega)$$

$S_{\dot{X}\dot{X}}(\omega)$ and $S_{XX}(\omega)$

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T} \qquad X_T(t) \Longleftrightarrow X_T(\omega)$$

$$S_{YY}(\omega) = \lim_{T\to\infty} \frac{E\left[|Y_T(\omega)|^2\right]}{2T} \qquad Y_T(t) \Longleftrightarrow Y_T(\omega)$$

$$= \lim_{T\to\infty} \frac{\omega^2 E\left[|X_T(\omega)|^2\right]}{2T} \qquad Y_T(t) = \dot{X}_T(t)$$

$$= \omega^2 S_{XX}(\omega) \qquad Y_T(\omega) = \omega^2 X_T(\omega)$$

$E\left[X_T(\omega)\right]$ and $E\left[\dot{X}_T(\omega)\right]$

$$Y(t) = \dot{X}_T(t) = \frac{d}{dt} X_T(t)$$

$$\begin{aligned} |Y(\omega)|^2 &= Y_T(\omega) Y_T^*(\omega) \\ &= (j\omega) X_T(\omega) (-j\omega) X_T(\omega) \\ &= \omega^2 |X(\omega)|^2 \end{aligned}$$

$$S_{YY}(\omega) = \lim_{T\to\infty} \frac{E\left[|Y(\omega)|^2\right]}{2T}$$

$$\begin{aligned} S_{\dot{X}\dot{X}}(\omega) &= \lim_{T\to\infty} \frac{\omega^2 E\left[|X(\omega)|^2\right]}{2T} \\ &= \omega^2 S_{XX}(\omega) \end{aligned}$$

Fourier transform of an autocorrelation functions

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$S_{\dot{X}\dot{X}}(\omega) = \int_{-\infty}^{+\infty} R_{\dot{X}\dot{X}}(\tau) e^{-j\omega\tau} d\tau$$

$$\omega^2 S_{XX}(\omega) = \int_{-\infty}^{+\infty} \omega^2 R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

Definition

the standard deviation is
a measure of the spread in a density function.
the analogous quantity for the normalized power spectral density is
a measure of its spread that we call the rms bandwidth
(root-mean-square)

$$W_{rms}^2 = \frac{\int_{-\infty}^{+\infty} \omega^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$

RMS Bandwidth and Mean Frequency
power spectral density for continuous time signals

Definition

the mean frequence $\bar{\omega}_0$

$$\bar{\omega}_0 = \frac{\int_{-\infty}^{+\infty} \omega S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$

the rms bandwidth

$$W_{rms}^2 = \frac{4 \int_{-\infty}^{+\infty} (\omega - \bar{\omega}_0)^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$