Vector Calculus (H.1) Identities

20160413

Gradient of a vector [edit]

See also: covariant derivative

Since the total derivative of a vector field is a linear mapping from vectors to vectors, it is a tensor quantity.

In rectangular coordinates, the gradient of a vector field **f** = (f_1, f_2, f_3) is defined by

$$\nabla \mathbf{f} = g^{jk}\frac{\partial f^i}{\partial x_j}\mathbf{e}_i\mathbf{e}_k$$

where the Einstein summation notation is used and the product of the vectors $\mathbf{e}_i$, $\mathbf{e}_k$ is a dyadic tensor of type (2,0), or the Jacobian matrix

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)}.$$

In curvilinear coordinates, or more generally on a curved manifold, the gradient involves Christoffel symbols:

$$\nabla \mathbf{f} = g^{jk}\left(\frac{\partial f^i}{\partial x_j} + {\Gamma^i}_{jl} f^l\right)\mathbf{e}_i\mathbf{e}_k$$

where g^{jk} are the components of the metric tensor and the $\mathbf{e}_i$ are the coordinate vectors.

Expressed more invariantly, the gradient of a vector field **f** can be defined by the Levi-Civita connection and metric tensor:[1]

$$\nabla^a \mathbf{f}^b = g^{ac}\nabla_c \mathbf{f}^b$$

where ∇_c is the connection.

Jacobian Matrix

$$\vec{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$J = \frac{d\vec{f}}{d\vec{x}} = \begin{bmatrix} \frac{d\vec{f}}{dx_1} & \frac{d\vec{f}}{dx_2} & \frac{d\vec{f}}{dx_3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \frac{df_1}{dx_3} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \frac{df_2}{dx_3} \\ \frac{df_3}{dx_1} & \frac{df_3}{dx_2} & \frac{df_3}{dx_3} \end{bmatrix}$$

$$J_i^j = \frac{\partial f_i}{\partial x_j} \qquad \begin{array}{cc} & j \\ & \downarrow \\ i \rightarrow & \begin{bmatrix} & & \\ & \square & \\ & & \end{bmatrix} \end{array}$$

$$J_i^j = \frac{\partial f_i}{\partial x_j}$$

$$\begin{array}{cc}
\begin{array}{c} i=1 \rightarrow \\[1.5em] i=2 \rightarrow \\[1.5em] i=3 \rightarrow \end{array} &
\begin{bmatrix}
\frac{d f_1}{d x_1} & \frac{d f_1}{d x_2} & \frac{d f_1}{d x_3} \\[0.5em]
\frac{d f_2}{d x_1} & \frac{d f_2}{d x_2} & \frac{d f_2}{d x_3} \\[0.5em]
\frac{d f_3}{d x_1} & \frac{d f_3}{d x_2} & \frac{d f_3}{d x_3}
\end{bmatrix}
\end{array}$$

$$J_i^j = \frac{\partial f_i}{\partial x_j}$$

$$\begin{array}{c}
\begin{array}{ccc} j=1 & j=2 & j=3 \\ \downarrow & \downarrow & \downarrow \end{array} \\
\begin{bmatrix}
\frac{d f_1}{d x_1} & \frac{d f_1}{d x_2} & \frac{d f_1}{d x_3} \\[0.5em]
\frac{d f_2}{d x_1} & \frac{d f_2}{d x_2} & \frac{d f_2}{d x_3} \\[0.5em]
\frac{d f_3}{d x_1} & \frac{d f_3}{d x_2} & \frac{d f_3}{d x_3}
\end{bmatrix}
\end{array}$$

dyadic (*comparative* **more dyadic**, *superlative* **most dyadic**)

1. Pertaining to the number two; of two parts or elements.
2. Pertaining to the physical sex of a person who is exactly male or female; not intersex.

Etymology [edit]

From New Latin *tensor* ("that which stretches"). Anatomical sense from 1704. In the 1840s introduced by William Rowan Hamilton as an algebraic quantity unrelated to the modern notic of tensor. The contemporary mathematical meaning was introduced (as German *Tensor*) by Woldemar Voigt (1898)[1] and adopted in English from 1915 (in the context of General Relativity), obscuring the earlier Hamiltonian sense. The mathematical object is so named because an early application of tensors was the study of materials stretching under tension.

Wikipedia has an article on:
tensor

Wikipedia has an article on:
Classical Hamiltonian quaternions#Tensor

Pronunciation [edit]

- Hyphenation: ten·sor
- Rhymes: -ɛnsə(ɹ)

Adjective [edit]

tensor (*not comparable*)

1. Of or relating to tensors

Dyadic, outer, and tensor products [edit]

A *dyad* is a tensor of order two and rank two, and is the result of the dyadic product of two vectors (complex vectors in general), whereas a *dyadic* is a general tensor of order two.

There are several equivalent terms and notations for this product:

- the **dyadic product** of two vectors **a** and **b** is denoted by **ab** (no symbol; no multiplication signs, crosses, dots etc.)
- the **outer product** of two column vectors **a** and **b** is denoted and defined as **a** ⊗ **b** or **ab**^{T}, where T means transpose,
- the **tensor product** of two vectors **a** and **b** is denoted **a** ⊗ **b**,

In the dyadic context they all have the same definition and meaning, and are used synonymously, although the **tensor product** is an instance of the more general and abstract use of the term.

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

$$\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$\vec{a} \cdot \vec{b}^T = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix}$$

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

$$\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$\begin{aligned}
\vec{a}\vec{b} = \; & a_1 \vec{i}\, b_1 \vec{i} + a_1 \vec{i}\, b_2 \vec{j} + a_1 \vec{i}\, b_3 \vec{k} \\
& + a_2 \vec{i}\, b_1 \vec{i} + a_2 \vec{i}\, b_2 \vec{j} + a_2 \vec{i}\, b_3 \vec{k} \\
& + a_3 \vec{i}\, b_1 \vec{i} + a_3 \vec{i}\, b_2 \vec{j} + a_3 \vec{i}\, b_3 \vec{k} \\
= \; & a_1 b_1 \vec{i}\vec{i} + a_1 b_2 \vec{i}\vec{j} + a_1 b_3 \vec{i}\vec{k} \\
& + a_2 b_1 \vec{i}\vec{i} + a_2 b_2 \vec{i}\vec{j} + a_2 b_3 \vec{i}\vec{k} \\
& + a_3 b_1 \vec{i}\vec{i} + a_3 b_2 \vec{i}\vec{j} + a_3 b_3 \vec{i}\vec{k}
\end{aligned}$$

$$\mathbf{ab} \equiv \mathbf{a} \otimes \mathbf{b} \equiv \mathbf{ab}^{\mathrm{T}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \begin{pmatrix} b_1 & b_2 & b_3 \end{pmatrix} = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix}.$$

$$\vec{i}\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{i}\vec{j} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{i}\vec{k} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{j}\vec{i} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{j}\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{j}\vec{k} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{j}\vec{i} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\vec{j}\vec{j} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\vec{j}\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{ii} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{ik} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{ji} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{jj} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{jk} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{ki} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \mathbf{kj} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathbf{kk} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

https://en.wikipedia.org/wiki/Vector_calculus_identities

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\begin{aligned}\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \\ &= (\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla)\mathbf{A} - (\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla)\mathbf{B} \\ &= \nabla \cdot (\mathbf{B}\mathbf{A}^{\mathrm{T}}) - \nabla \cdot (\mathbf{A}\mathbf{B}^{\mathrm{T}}) \\ &= \nabla \cdot (\mathbf{B}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{B}^{\mathrm{T}})\end{aligned}$$

.

- The scalar triple product is invariant under a circular shift of its three operands (**a**, **b**, **c**):

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

- Swapping the positions of the operators without re-ordering the operands leaves the triple product unchanged. This follows from the preceding property and the commutative property of the dot product.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$

- Swapping any two of the three operands negates the triple product. This follows from the circular-shift property and the anticommutativity of the cross product.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=-\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b})$$
$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=-\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})$$
$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=-\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a})$$

- The scalar triple product can also be understood as the determinant of the 3 × 3 matrix (thus also its inverse) having the three vectors either as its rows or its columns (a matrix has the same determinant as its transpose):

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\det\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}.$$

https://en.wikipedia.org/wiki/Triple_product

- If the scalar triple product is equal to zero, then the three vectors **a**, **b**, and **c** are coplanar, since the "parallelepiped" defined by them would be flat and have no volume.
- If any two vectors in the triple scalar product are equal, then its value is zero:

$$\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{a}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{b}) = \mathbf{a}\cdot(\mathbf{a}\times\mathbf{a}) = 0$$

- Moreover,

$$[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\mathbf{a} = (\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})$$

- The simple product of two triple products (or the square of a triple product), may be expanded in terms of dot products:[1]

$$((\mathbf{a}\times\mathbf{b})\cdot\mathbf{c})\,((\mathbf{d}\times\mathbf{e})\cdot\mathbf{f}) = \det\left[\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\\ \mathbf{c}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{d} & \mathbf{e} & \mathbf{f}\end{pmatrix}\right] = \det\Bigg[$$

https://en.wikipedia.org/wiki/Triple_product

The **vector triple product** is defined as the cross product of one vector with the cross product of the other two. The following relationship holds:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

This is known as **triple product expansion**, or **Lagrange's formula**,[2][3] although the latter name is also used for several other formulae. Its right hand side can be remembered by using the mnemonic "BAC − CAB", provided one keeps in mind which vectors are dotted together. A proof is provided below.

Since the cross product is anticommutative, this formula may also be written (up to permutation of the letters) as:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -(\mathbf{c} \cdot \mathbf{b})\mathbf{a} + (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

From Lagrange's formula it follows that the vector triple product satisfies:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \underline{\mathbf{c} \times (\mathbf{a} \times \mathbf{b})} = 0$$

which is the Jacobi identity for the cross product. Another useful formula follows:

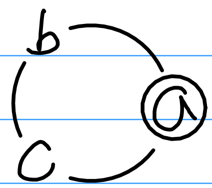
$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \times (\mathbf{a} \times \mathbf{c})$$

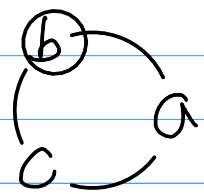
These formulas are very useful in simplifying vector calculations in physics. A related identity regarding gradients and useful in vector calculus is Lagrange's formula of vector cross-product identity:[4]

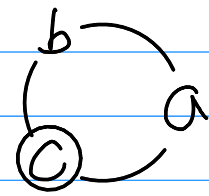
$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{f}) - (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{f}$$

$(a \times b) \times c = a \times (b \times c) - b \times (a \times c)$

https://en.wikipedia.org/wiki/Triple_product

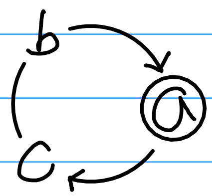


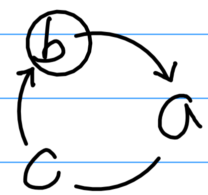


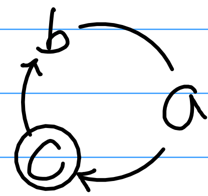


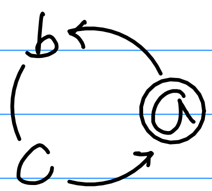
$\vec{a} \times (\vec{b} \times \vec{c})$ $\quad$ $\vec{b} \times (\vec{c} \times \vec{a})$ $\quad$ $\vec{c} \times (\vec{a} \times \vec{b})$

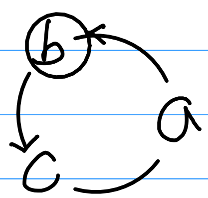
$\mathbf{b}(\mathbf{a} \cdot \mathbf{c})$ $\quad$ $\mathbf{c}(\mathbf{b} \cdot \mathbf{a})$ $\quad$ $\mathbf{a}(\mathbf{c} \cdot \mathbf{b})$

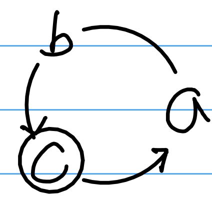








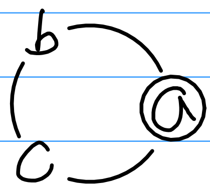


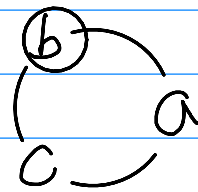


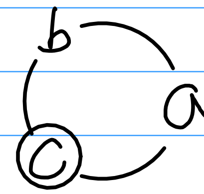
$-\mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ $\quad$ $-\mathbf{a}(\mathbf{b} \cdot \mathbf{c})$ $\quad$ $-\mathbf{b}(\mathbf{c} \cdot \mathbf{a})$

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$$

$$\begin{array}{ccc} \| & \| & \| \\ \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) & \mathbf{c}(\mathbf{b} \cdot \mathbf{a}) & \mathbf{a}(\mathbf{c} \cdot \mathbf{b}) \\ -\mathbf{c}(\mathbf{a} \cdot \mathbf{b}) & -\mathbf{a}(\mathbf{b} \cdot \mathbf{c}) & -\mathbf{b}(\mathbf{c} \cdot \mathbf{a}) \end{array}$$







$\vec{a} \times (\vec{b} \times \vec{c})$ $\quad$ $\vec{b} \times (\vec{c} \times \vec{a})$ $\quad$ $\vec{c} \times (\vec{a} \times \vec{b})$

a b c $\quad$ b c a $\quad$ c a b

a b c $\quad$ b c a $\quad$ c a b

$b(a \cdot c)$ $\quad$ $c(b \cdot a)$ $\quad$ $a(c \cdot b)$

$-c(a \cdot b)$ $\quad$ $-a(b \cdot c)$ $\quad$ $-b(c \cdot a)$

$$\boxed{\vec{a}\times(\vec{b}\times\vec{c})} + \boxed{\vec{b}\times(\vec{c}\times\vec{a})} + \boxed{\vec{c}\times(\vec{a}\times\vec{b})} = 0$$

$$\begin{array}{ccc} \| & \| & \| \\ \boxed{\begin{array}{c} b\,(a\cdot c) \\ -c\,(a\cdot b) \end{array}} & \boxed{\begin{array}{c} c\,(b\cdot a) \\ -a\,(b\cdot c) \end{array}} & \boxed{\begin{array}{c} a\,(c\cdot b) \\ -b\,(c\cdot a) \end{array}} \end{array}$$

$$-\boxed{\vec{a}\times(\vec{b}\times\vec{c})} - \boxed{\vec{b}\times(\vec{c}\times\vec{a})} = \boxed{\vec{c}\times(\vec{a}\times\vec{b})}$$

$$\begin{array}{ccccc} & \| & & \| & & \| \\ - & \boxed{\begin{array}{c} b\,(a\cdot c) \\ -c\,(a\cdot b) \end{array}} & - & \boxed{\begin{array}{c} c\,(b\cdot a) \\ -a\,(b\cdot c) \end{array}} & & \boxed{\begin{array}{c} a\,(c\cdot b) \\ -b\,(c\cdot a) \end{array}} \end{array}$$

$$\begin{aligned}
\nabla\times(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \\
&= (\nabla\cdot\mathbf{B} + \mathbf{B}\cdot\nabla)\mathbf{A} - (\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla)\mathbf{B} \\
&= \nabla\cdot(\mathbf{B}\mathbf{A}^{\mathrm{T}}) - \nabla\cdot(\mathbf{A}\mathbf{B}^{\mathrm{T}}) \\
&= \nabla\cdot(\mathbf{B}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{B}^{\mathrm{T}})
\end{aligned}$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) - \mathbf{b}\times(\mathbf{a}\times\mathbf{c})$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = -\mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = -(\mathbf{c}\cdot\mathbf{b})\mathbf{a} + (\mathbf{c}\cdot\mathbf{a})\mathbf{b}$$

$$\begin{aligned}\nabla\times(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}\\ &= (\nabla\cdot\mathbf{B} + \mathbf{B}\cdot\nabla)\mathbf{A} - (\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla)\mathbf{B}\\ &= \nabla\cdot(\mathbf{B}\mathbf{A}^{\mathrm{T}}) - \nabla\cdot(\mathbf{A}\mathbf{B}^{\mathrm{T}})\\ &= \nabla\cdot(\mathbf{B}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{B}^{\mathrm{T}})\end{aligned}$$

$$\nabla\times(A\times B) = \nabla\times(A\times B) + \nabla\times(A\times B)$$

$$\begin{aligned}\nabla\times(A\times B) &= A(\nabla\cdot B) - B(\nabla\cdot A)\\ &= (B\cdot\nabla)A - B(\nabla\cdot A)\end{aligned}$$

$$\begin{aligned}\nabla\times(A\times B) &= A(\nabla\cdot B) - B(\nabla\cdot A)\\ &\ (B\cdot\nabla)A - (A\cdot\nabla)B\end{aligned}$$

$$\begin{aligned}\nabla\times(A\times B) &= A(\nabla\cdot B) - B(\nabla\cdot A)\\ &\quad + (B\cdot\nabla)A - (A\cdot\nabla)B\end{aligned}$$

http://math.stackexchange.com/questions/836041/curl-of-cross-product-of-two-vectors

Directional Derivatives

$$\mathbf{B} \cdot \nabla = \langle B_x, B_y, B_z \rangle \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

$$= B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}$$

$$\cancel{=} \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

$$(\mathbf{B} \cdot \nabla) f = \langle B_x, B_y, B_z \rangle \cdot \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle f$$

$$= B_x \frac{\partial f}{\partial x} + B_y \frac{\partial f}{\partial y} + B_z \frac{\partial f}{\partial z}$$

$$= \mathbf{B} \cdot (\underbrace{\nabla f}_{\text{grad}}) \qquad f_B$$

$$\vec{u} = \langle a, b, c \rangle$$

$$D_{\vec{u}} f(x,y,z) = a \cdot f_x(x,y,z) + b \cdot f_y(x,y,z) + c \cdot f_z(x,y,z)$$

$$= \langle f_x, f_y, f_z \rangle \cdot \langle a, b, c \rangle$$

$$\mathbf{B}\cdot\nabla = B_x,\ B_y,\ B_z$$

$$= B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}$$

$$\cancel{=}\ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

$$\nabla\square = \frac{\partial\square}{\partial x}\vec{i} + \frac{\partial\square}{\partial y}\vec{j} + \frac{\partial\square}{\partial z}\vec{k}$$

Op

$$\mathbf{B}\cdot\nabla = \left(B_x\vec{i} + B_y\vec{j} + B_z\vec{k}\right)\cdot\left(\frac{\partial\square}{\partial x}\vec{i} + \frac{\partial\square}{\partial y}\vec{j} + \frac{\partial\square}{\partial z}\vec{k}\right)$$

$$= B_x\frac{\partial\square}{\partial x} + B_y\frac{\partial\square}{\partial y} + B_z\frac{\partial\square}{\partial z}$$

Scalar

$$\nabla\cdot\mathbf{B} = \left(\frac{\partial\square}{\partial x}\vec{i} + \frac{\partial\square}{\partial y}\vec{j} + \frac{\partial\square}{\partial z}\vec{k}\right)\cdot\left(B_x\vec{i} + B_y\vec{j} + B_z\vec{k}\right)$$

$$= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

Dot Del Operator $\dot{\nabla}$

Op $\quad \mathbf{B} \cdot \nabla = B_x \frac{\partial \square}{\partial x} + B_y \frac{\partial \square}{\partial y} + B_z \frac{\partial \square}{\partial z}$

Scalar $\quad \nabla \cdot \mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$

$$\mathbf{B} \cdot \nabla \neq \nabla \cdot \mathbf{B}$$

$$\mathbf{B} \cdot \nabla = \dot{\nabla} \cdot \mathbf{B}$$

dot del