

Background – Trigonometry (2A)

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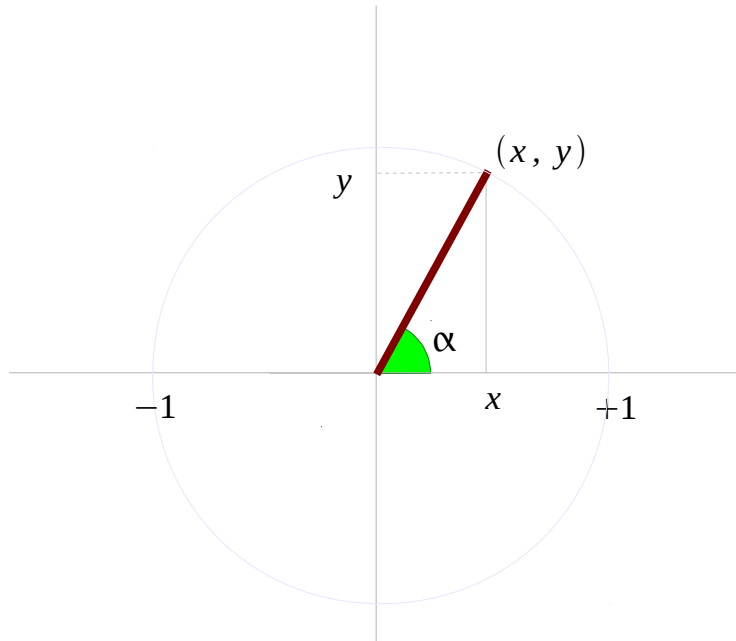
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Complex Numbers

Reference Angle (1)

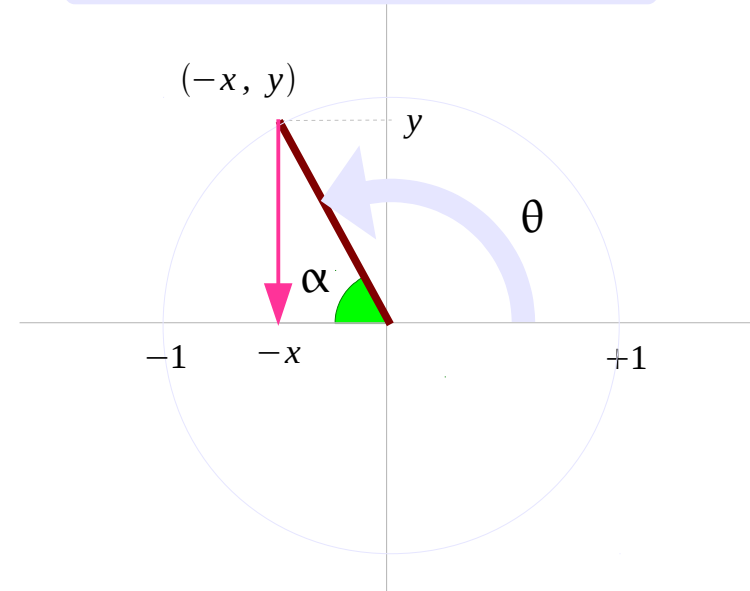
1st Quadrant Angle θ



$$\begin{aligned}\sin \alpha &= y \\ \cos \alpha &= x \\ \tan \alpha &= y/x\end{aligned}$$

2nd Quadrant Angle θ

$$90^\circ < \theta < 180^\circ$$



Reference Angle α

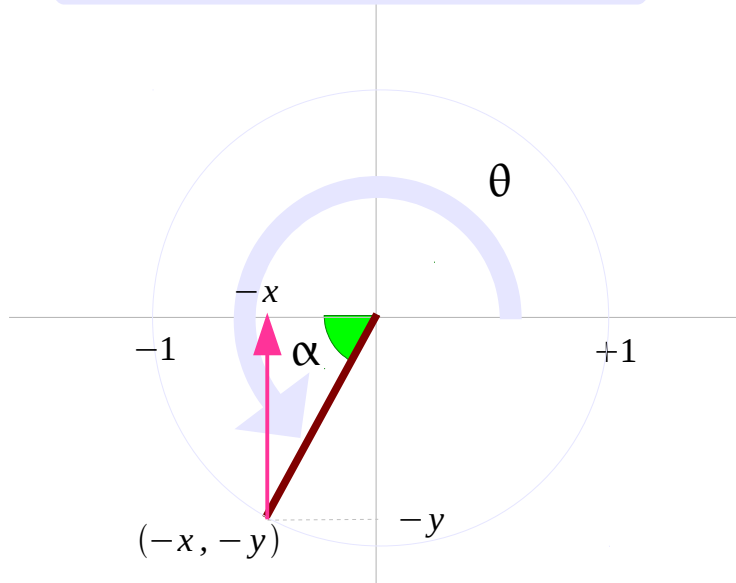
$$\alpha = 180^\circ - \theta$$

$$\begin{aligned}\sin \theta &= + \sin \alpha = (+ y) \\ \cos \theta &= - \cos \alpha = (-x) \\ \tan \theta &= - \tan \alpha = (+ y)/(-x)\end{aligned}$$

Reference Angle (2)

3rd Quadrant Angle θ

$$180^\circ < \theta < 270^\circ$$



Reference Angle α

$$\alpha = \theta - 180^\circ$$

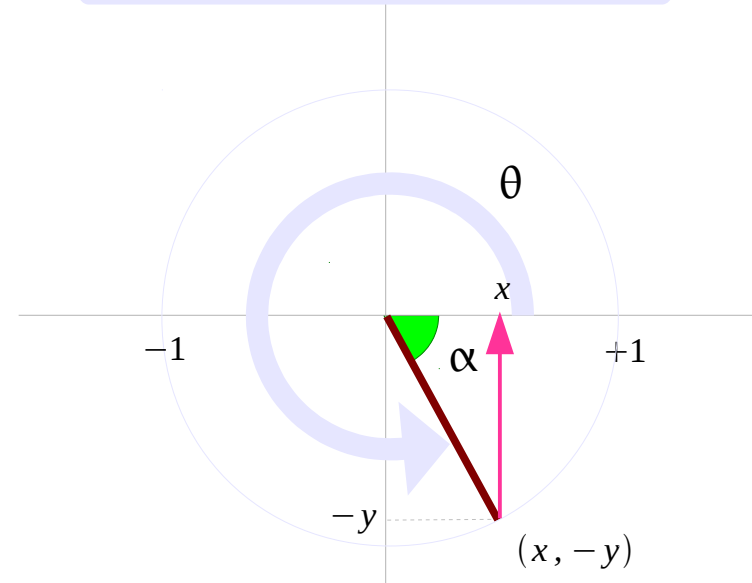
$$\sin \theta = - \sin \alpha = (-y)$$

$$\cos \theta = - \cos \alpha = (-x)$$

$$\tan \theta = + \tan \alpha = (-y)/(-x)$$

4th Quadrant Angle θ

$$270^\circ < \theta < 360^\circ$$



Reference Angle α

$$\alpha = 360^\circ - \theta$$

$$\sin \theta = - \sin \alpha = (-y)$$

$$\cos \theta = + \cos \alpha = (+x)$$

$$\tan \theta = - \tan \alpha = (-y)/(+x)$$

Reference Angle (3)

A Quadrant Angle θ

Reference Angle α

$$\alpha = \pi - \theta$$

$$\sin \theta = + \sin \alpha$$

$$\cos \theta = - \cos \alpha$$

$$\tan \theta = - \tan \alpha$$

$$\sin \theta = - \sin \alpha$$

$$\cos \theta = - \cos \alpha$$

$$\tan \theta = + \tan \alpha$$

$$\alpha = \theta - \pi$$

$$\alpha = \theta$$

$$\sin \theta = \sin \alpha$$

$$\cos \theta = \cos \alpha$$

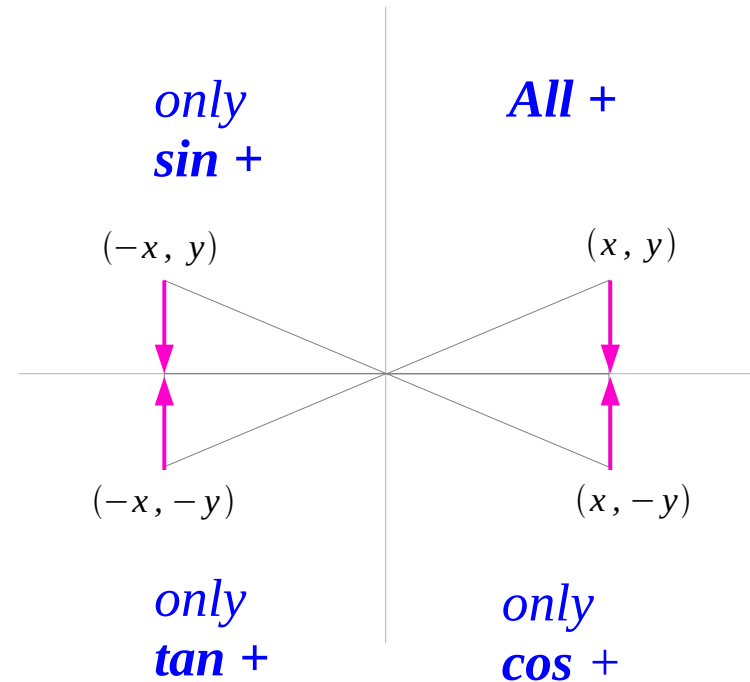
$$\tan \theta = \tan \alpha$$

$$\sin \theta = - \sin \alpha$$

$$\cos \theta = + \cos \alpha$$

$$\tan \theta = - \tan \alpha$$

$$\alpha = 2\pi - \theta$$

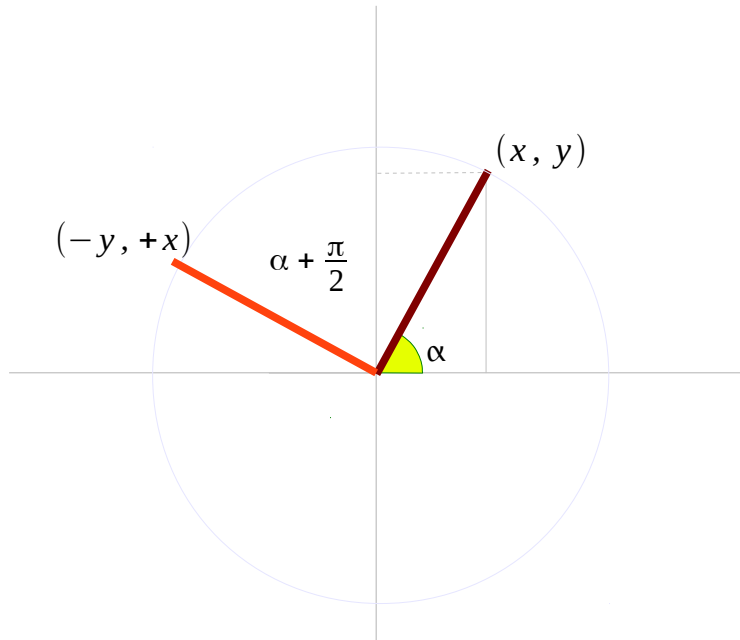


Symmetry

Reflected in $\theta = 0$ ^[4]	Reflected in $\theta = \pi/2$ (co-function identities) ^[5]	Reflected in $\theta = \pi$
$\sin(-\theta) = -\sin \theta$	$\sin(\frac{\pi}{2} - \theta) = +\cos \theta$	$\sin(\pi - \theta) = +\sin \theta$
$\cos(-\theta) = +\cos \theta$	$\cos(\frac{\pi}{2} - \theta) = +\sin \theta$	$\cos(\pi - \theta) = -\cos \theta$
$\tan(-\theta) = -\tan \theta$	$\tan(\frac{\pi}{2} - \theta) = +\cot \theta$	$\tan(\pi - \theta) = -\tan \theta$
$\csc(-\theta) = -\csc \theta$	$\csc(\frac{\pi}{2} - \theta) = +\sec \theta$	$\csc(\pi - \theta) = +\csc \theta$
$\sec(-\theta) = +\sec \theta$	$\sec(\frac{\pi}{2} - \theta) = +\csc \theta$	$\sec(\pi - \theta) = -\sec \theta$
$\cot(-\theta) = -\cot \theta$	$\cot(\frac{\pi}{2} - \theta) = +\tan \theta$	$\cot(\pi - \theta) = -\cot \theta$

<http://en.wikipedia.org/wiki/Derivative>

Shift by $+\pi/2$



$$\cos\left(\alpha + \frac{\pi}{2}\right) = -\sin \alpha$$

$$\sin\left(\alpha + \frac{\pi}{2}\right) = +\cos \alpha$$

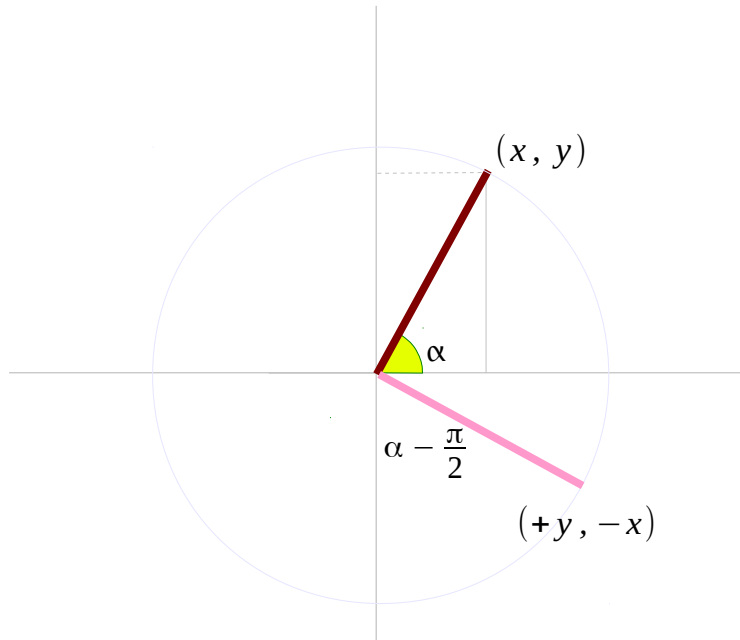
$$\tan\left(\alpha + \frac{\pi}{2}\right) = -\cot \alpha$$

$$\cot\left(\alpha + \frac{\pi}{2}\right) = -\tan \alpha$$

$$\csc\left(\alpha + \frac{\pi}{2}\right) = +\sec \alpha$$

$$\sec\left(\alpha + \frac{\pi}{2}\right) = -\csc \alpha$$

Shift by $-\pi/2$



$$\cos\left(\alpha - \frac{\pi}{2}\right) = +\sin \alpha$$

$$\sin\left(\alpha - \frac{\pi}{2}\right) = -\cos \alpha$$

$$\tan\left(\alpha - \frac{\pi}{2}\right) = -\cot \alpha$$

$$\cot\left(\alpha - \frac{\pi}{2}\right) = -\tan \alpha$$

$$\csc\left(\alpha - \frac{\pi}{2}\right) = -\sec \alpha$$

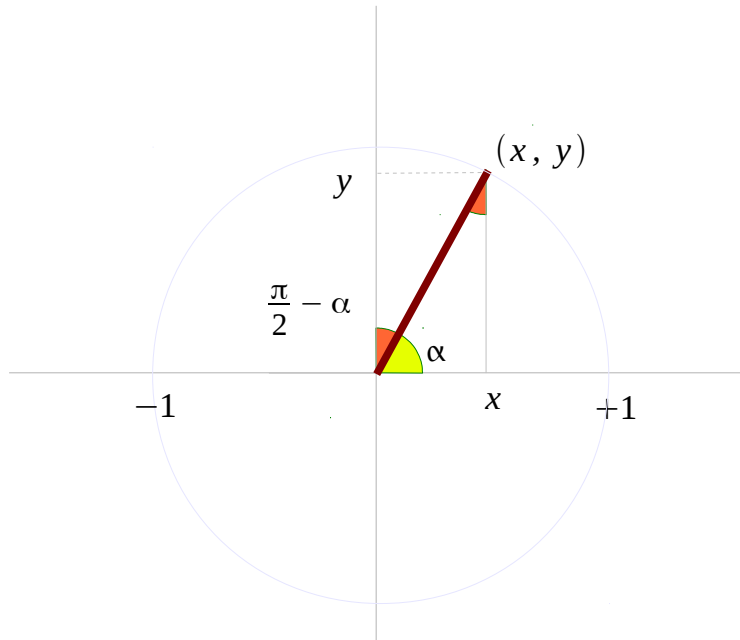
$$\sec\left(\alpha - \frac{\pi}{2}\right) = +\csc \alpha$$

Shifts and periodicity

Shift by $\pi/2$	Shift by π Period for tan and cot ^[6]	Shift by 2π Period for sin, cos, csc and sec ^[7]
$\sin(\theta + \frac{\pi}{2}) = +\cos\theta$	$\sin(\theta + \pi) = -\sin\theta$	$\sin(\theta + 2\pi) = +\sin\theta$
$\cos(\theta + \frac{\pi}{2}) = -\sin\theta$	$\cos(\theta + \pi) = -\cos\theta$	$\cos(\theta + 2\pi) = +\cos\theta$
$\tan(\theta + \frac{\pi}{2}) = -\cot\theta$	$\tan(\theta + \pi) = +\tan\theta$	$\tan(\theta + 2\pi) = +\tan\theta$
$\csc(\theta + \frac{\pi}{2}) = +\sec\theta$	$\csc(\theta + \pi) = -\csc\theta$	$\csc(\theta + 2\pi) = +\csc\theta$
$\sec(\theta + \frac{\pi}{2}) = -\csc\theta$	$\sec(\theta + \pi) = -\sec\theta$	$\sec(\theta + 2\pi) = +\sec\theta$
$\cot(\theta + \frac{\pi}{2}) = -\tan\theta$	$\cot(\theta + \pi) = +\cot\theta$	$\cot(\theta + 2\pi) = +\cot\theta$

<http://en.wikipedia.org/wiki/Derivative>

Co-function Identities



$$\sin \alpha = y \Rightarrow \cos\left(\frac{\pi}{2} - \alpha\right)$$

$$\cos \alpha = x \Rightarrow \sin\left(\frac{\pi}{2} - \alpha\right)$$

$$\tan \alpha = y/x \Rightarrow \cot\left(\frac{\pi}{2} - \alpha\right)$$

$$\cos\left(\frac{\pi}{2} - \alpha\right) = \sin \alpha$$

$$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha$$

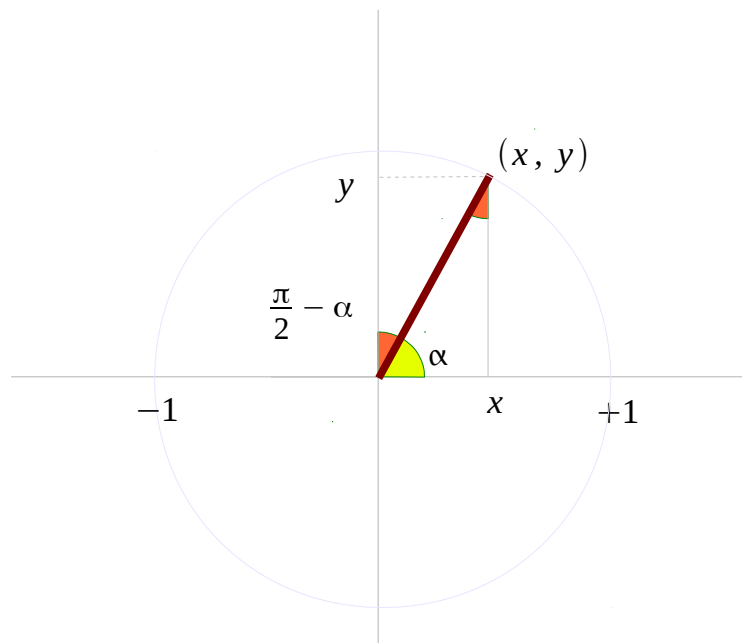
$$\tan\left(\frac{\pi}{2} - \alpha\right) = \cot \alpha$$

$$\cot\left(\frac{\pi}{2} - \alpha\right) = \tan \alpha$$

$$\csc\left(\frac{\pi}{2} - \alpha\right) = \sec \alpha$$

$$\sec\left(\frac{\pi}{2} - \alpha\right) = \csc \alpha$$

Angle Sum and Difference Identities (1)



$$\sin(\alpha + \beta) = \sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta$$

$$\sin(60^\circ + 30^\circ) = 1$$

+	$\sin(60^\circ) = \frac{\sqrt{3}}{2}$	X	$\sin(30^\circ) = \frac{1}{2}$
+	$\cos(60^\circ) = \frac{1}{2}$	X	$\cos(30^\circ) = \frac{\sqrt{3}}{2}$

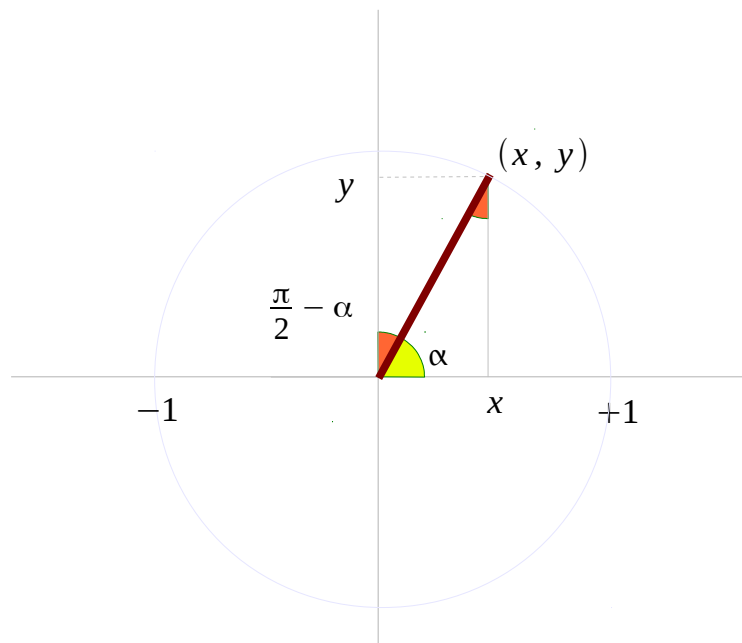
$$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = 1$$

$$\sin(60^\circ - 30^\circ) = \frac{1}{2}$$

+	$\sin(60^\circ) = \frac{\sqrt{3}}{2}$	X	$\sin(30^\circ) = \frac{1}{2}$
-	$\cos(60^\circ) = \frac{1}{2}$	X	$\cos(30^\circ) = \frac{\sqrt{3}}{2}$

$$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

Angle Sum and Difference Identities (2)



$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$$

$$\cos(30^\circ + 60^\circ) = 0$$

$$\begin{array}{l} \color{red}{-} \sin(60^\circ) = \frac{\sqrt{3}}{2} \quad \color{blue}{-} \sin(30^\circ) = \frac{1}{2} \\ \color{blue}{+} \cos(60^\circ) = \frac{1}{2} \quad \color{blue}{-} \cos(30^\circ) = \frac{\sqrt{3}}{2} \end{array}$$

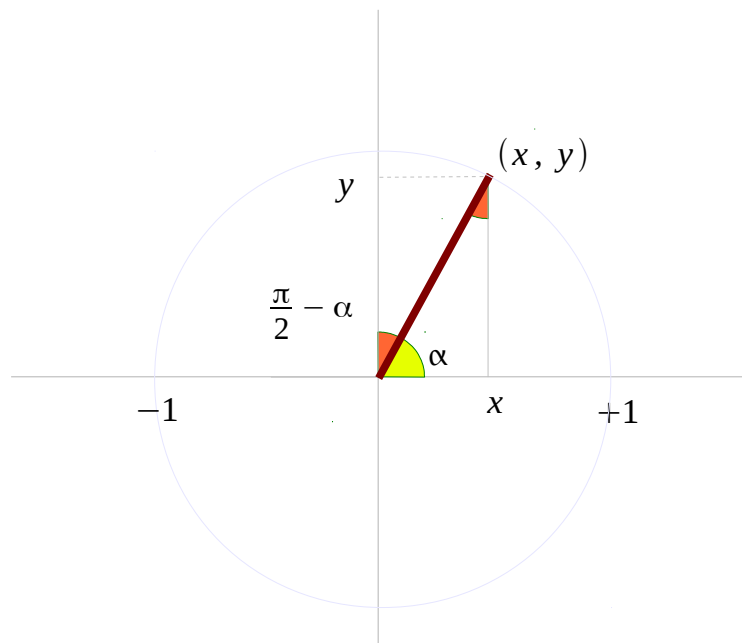
$$\frac{\sqrt{3}}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 0$$

$$\cos(30^\circ - 60^\circ) = \frac{\sqrt{3}}{2}$$

$$\begin{array}{l} \color{blue}{+} \sin(60^\circ) = \frac{\sqrt{3}}{2} \quad \color{blue}{-} \sin(30^\circ) = \frac{1}{2} \\ \color{blue}{+} \cos(60^\circ) = \frac{1}{2} \quad \color{blue}{-} \cos(30^\circ) = \frac{\sqrt{3}}{2} \end{array}$$

$$\frac{\sqrt{3}}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

Angle Sum and Difference Identities (3)



$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

$$\tan(30^\circ + 60^\circ) = +\infty$$

$$\tan(30^\circ) = \frac{1}{\sqrt{3}} \quad \text{—————} \quad \tan(60^\circ) = \sqrt{3}$$

$$\tan(60^\circ) = \sqrt{3} \quad \text{—————} \quad \tan(30^\circ) = \frac{1}{\sqrt{3}}$$

$$\frac{\frac{1}{\sqrt{3}} + \sqrt{3}}{1 - \frac{1}{\sqrt{3}} \cdot \sqrt{3}} = +\infty$$

$$\tan(30^\circ - 60^\circ) = -\frac{1}{\sqrt{3}}$$

$$\tan(30^\circ) = \frac{1}{\sqrt{3}} \quad \text{—————} \quad \tan(60^\circ) = \sqrt{3}$$

$$\tan(60^\circ) = \sqrt{3} \quad \text{—————} \quad \tan(30^\circ) = \frac{1}{\sqrt{3}}$$

$$\frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + \frac{1}{\sqrt{3}} \cdot \sqrt{3}} = -\frac{1}{\sqrt{3}}$$

Angle Sum and Difference Identities (4)

$$\sin(\alpha + \beta) = \sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta$$

$$\begin{array}{cc} \sin \alpha & \times & \sin \beta \\ \cos \alpha & \times & \cos \beta \end{array}$$

$$\cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta$$

$$\begin{array}{cc} - \sin \alpha & \text{---} & \sin \beta \\ + \cos \alpha & \text{---} & \cos \beta \end{array}$$

$$\frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta}{\cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta}$$

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

$$\sin(\alpha - \beta) = \sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta$$

$$\begin{array}{cc} + \sin \alpha & \times & \sin \beta \\ - \cos \alpha & \times & \cos \beta \end{array}$$

$$\cos(\alpha - \beta) = \cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta$$

$$\begin{array}{cc} + \sin \alpha & \text{---} & \sin \beta \\ + \cos \alpha & \text{---} & \cos \beta \end{array}$$

$$\frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = \frac{\sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta}{\cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta}$$

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

Product to Sum (1)

$$+ \sin(\alpha + \beta) = \sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta$$

$$+ \sin(\alpha - \beta) = \sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta$$

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cdot \cos \beta$$

$$\sin \alpha \cdot \cos \beta = \frac{1}{2} \{ \sin(\alpha + \beta) + \sin(\alpha - \beta) \}$$

$$+ \cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta$$

$$+ \cos(\alpha - \beta) = \cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta$$

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cdot \cos \beta$$

$$\cos \alpha \cdot \cos \beta = \frac{1}{2} \{ + \cos(\alpha + \beta) + \cos(\alpha - \beta) \}$$

$$+ \sin(\alpha + \beta) = \sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta$$

$$- \sin(\alpha - \beta) = \sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta$$

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2 \cos \alpha \sin \beta$$

$$\cos \alpha \cdot \sin \beta = \frac{1}{2} \{ \sin(\alpha + \beta) - \sin(\alpha - \beta) \}$$

$$- \cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta$$

$$+ \cos(\alpha - \beta) = \cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta$$

$$- \cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \sin \alpha \sin \beta$$

$$\sin \alpha \cdot \sin \beta = \frac{1}{2} \{ - \cos(\alpha + \beta) + \cos(\alpha - \beta) \}$$

Product to Sum (2)

$$\sin(\alpha \pm \beta) = \sin \alpha \cdot \cos \beta \pm \cos \alpha \cdot \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cdot \cos \beta \mp \sin \alpha \cdot \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta$$

$$\sin \alpha \cdot \cos \beta = \frac{1}{2} \{ \sin(\alpha + \beta) + \sin(\alpha - \beta) \}$$

$$\sin \alpha \cdot \cos \beta = \frac{1}{2} \{ + \sin(\alpha + \beta) + \sin(\alpha - \beta) \}$$

$$\cos \alpha \cdot \sin \beta = \frac{1}{2} \{ + \sin(\alpha + \beta) - \sin(\alpha - \beta) \}$$

$$\cos \alpha \cdot \cos \beta = \frac{1}{2} \{ + \cos(\alpha + \beta) + \cos(\alpha - \beta) \}$$

$$\cos \alpha \cdot \cos \beta = \frac{1}{2} \{ + \cos(\alpha + \beta) + \cos(\alpha - \beta) \}$$

$$\sin \alpha \cdot \sin \beta = \frac{1}{2} \{ -\cos(\alpha + \beta) + \cos(\alpha - \beta) \}$$

References

- [1] <http://en.wikipedia.org/>
- [2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [3] E. Kreyszig, “Advanced Engineering Mathematics”
- [4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”
- [5] www.chem.arizona.edu/~salzmanr/480a