Stationarity

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Probability, Random Variables and Random Signal Principles,
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Outline

First Order Stationary

$f_X(x;t)$

if $X(t)$ is to be a **first-order stationary**

$$f_X(x_1;t_1) = f_X(x_1;t_1+\Delta)$$

must be true for any time t_1 and any real number Δ

the first order density function
does not change with a shift in time origin

Consequences of stationarity

$f_X(x;t)$

- $f_X(x, t_1)$ is independent of t_1
  the first order density function
  does not change with a shift in time origin

- the **process mean** value is a **constant**

$$m_X(t) = \overline{X} = constant$$

the process mean value

$$\boxed{m_X(t) = \overline{X} = constant}$$

$$m_X(t_1) = \int_{-\infty}^{\infty} x f_X(x; t_1) dx$$

$$m_X(t_2) = \int_{-\infty}^{\infty} x f_X(x; t_2) dx$$

let $t_2 = t_1 + \Delta$

$$m_X(t_1) = m_X(t_1 + \Delta)$$

Second-Order Stationary Process

$f_X(x_1, x_2; t_1, t_2)$

if $X(t)$ is to be a **second-order stationary**

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

must be true for any time t_1, t_2 and any real number Δ

the second order density function
does not change with a shift in time origin

Second-Order Stationary Process

$f_X(x_1, x_2; t_1, t_2)$

- $f_X(x_1, x_2; t_1, t_2)$ is independent of t_1 and t_2
  the second order density function
  does not change with a shift in time origin
- the **autocorrelation function**

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

N^{th}-order Stationary Processes

$f_X(x_1, \cdots, x_N; t_1, \cdots, t_N)$

if $X(t)$ is to be a **N^{th}-order stationary**

$$f_X(x_1, \cdots, x_N; t_1, \cdots, t_N) = f_X(x_1, \cdots, x_N; t_1 + \Delta, \cdots, t_N + \Delta)$$

must be true for any time $t_1, ..., t_N$
and any real number Δ

the N^{th} order density function
does not change with a shift in time origin

Stationary Process

joint probability distribution

a **stationary process** is a stochastic process whose unconditional joint probability distribution does not change when shifted in time.

Consequently, parameters such as **mean** and **variance** also do not change over time.

https://en.wikipedia.org/wiki/Stationary_process

Stationary Process - nomenclature

nomenclature

- stationary process
- strictly stationary process
- strongly stationary process
- strict sense stationary (SSS) process

https://en.wikipedia.org/wiki/Stationary_process

Strict Sense Stationary Process

for all natural number N

if $X(t)$ is to be a **strict sense stationary (SSS) process**

$$f_X(x_1, \cdots, x_N; t_1, \cdots, t_N) = f_X(x_1, \cdots, x_N; t_1 + \Delta, \cdots, t_N + \Delta)$$

must be true for any time $t_1, ..., t_N$
and any real number Δ
and for all natural number N

- **white noise** is the simplest example of a **strictly stationary process**.

https://en.wikipedia.org/wiki/Stationary_process

Wide Sense Stationary Process

1st and 2nd moments

Wide Sense Stationary (**WSS**) random processes
only require that
1st moment (i.e. the **mean**) and **autocovariance**
do not vary with respect to time and that
the 2nd moment is finite for all times.

- $E[X(t_1)] = E[X(t_2)] = \overline{X} = constant$ for all t_1 and t_2
- $C_{XX}(t_1, t_2) = C_{XX}(t_1 - t_2, 0) \triangleq C_{XX}(\tau)$ for all t_1 and t_2
- $E[|X(t)|^2] < \infty$ for all t

https://en.wikipedia.org/wiki/Stationary_process

Wide Sense Stationary Process - nomenclature

nomenclature

- weak sense stationary (WSS) process
- wide sense stationary (WSS) process

https://en.wikipedia.org/wiki/Stationary_process

WSS - auto-covariance & auto-correlation

mean, auto-covariance, auto-correlation

$$
\begin{aligned}
m_X(t) &= \overline{X} = constant \\
C_{XX}(t_1, t_2) &= E\left[\{X(t_1) - m_x(t_1)\}\{X(t_2) - m_x(t_2)\}\right] \\
&= E\left[\left\{X(t_1) - \overline{X}\right\}\left\{X(t_2) - \overline{X}\right\}\right] \\
&= E\left[X(t_1)X(t_2)\right] - \overline{X}^2 \\
&\triangleq C_{XX}(\tau) \\
&\triangleq R_{XX}(\tau) - \overline{X}^2 \\
R_{XX}(t_1, t_2) &\triangleq R_{XX}(\tau)
\end{aligned}
$$

https://en.wikipedia.org/wiki/Stationary_process

Wide Sense Stationary Process

$m_X(t)$, $R_{XX}(\tau)$

WSS random processes only require that
1st moment (i.e. the **mean**) and **auto**correlation
do not vary with respect to time

$$E[X(t)] = m_X(t) = \overline{X} = constant$$

$$E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

Wide Sense Stationary Process

$m_X(t)$, $R_{XX}(\tau)$

- the 2nd order stationarity is sufficient
  for wide sense stationarity
- if $f_X(x_1; t_1)$ is independent of t_1
  then $E[X(t)] = constant$
- if $f_X(x_1, x_2; t_1, t_2)$ is independent of t_1 and t_2
  then $E[X(t)X(t+\tau)] = R_{XX}(\tau)$

The properties of autocorrelation functions (1)

$|R_{XX}(\tau)|, R_{XX}(-\tau), R_{XX}(0)$

$$|R_{XX}(\tau)| \leq R_{XX}(0)$$

$$R_{XX}(-\tau) = R_{XX}(\tau)$$

$$R_{XX}(0) = E\left[X^2(t)\right]$$

$$P\left[|X(t+\tau) - X(t)| > \varepsilon\right] = \frac{2}{\varepsilon^2}\left(R_{XX}(0) - R_{XX}(\tau)\right)$$

The properties of autocorrelation functions (2)

$R_{NN}(\tau)$,$R_{XX}(\tau)$

if $X(t) = \overline{X} + N(t)$
where $N(t)$ is WSS, is **zero-mean**, and
has autocorrelation function $\boldsymbol{R_{NN}}(\tau) \to 0$ as $|\tau| \to \infty$, then

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \overline{X}^2$$

The properties of autocorrelation functions (3)

$R_{NN}(\tau)$, $R_{XX}(\tau)$

if $X(t)$ is **mean square periodic**, i.e, there exists a $T \neq 0$ such that $E\left[\{X(t+T) - X(t)\}^2\right] = 0$ for all t,
then $R_{XX}(t)$ will have a **periodic** component with the same period

The properties of autocorrelation functions (4)

$R_{NN}(\tau)$,$R_{XX}(\tau)$

$R_{XX}(\tau)$ cannot have an arbitrary shape

Crosscorrelation functions (1)

$R_{XY}(t_1, t_2), R_{XY}(t, t+\tau)$

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = R_{XY}(\tau)$$

if

$$R_{XY}(t, t+\tau) = 0$$

then $X(t)$ and $Y(t)$ are called **orthogonal processes**

Crosscorrelation functions (2)

$R_{XY}(t, t+\tau), R_{XY}(\tau)$

if $X(t)$ and $Y(t)$ are **statistically independent**

$$R_{XY}(t,t+\tau) = E[X(t)Y(t+\tau)] = m_X(t)m_Y(t+\tau)$$

if $X(t)$ and $Y(t)$ are **stistically independent** and are at least **WSS**,

$$R_{XY}(\tau) = \overline{XY}$$

which is constant

The properties of crosscorrelation functions (1)

$R_{XY}(\tau), |R_{XY}(\tau)|$

$$R_{XY}(\tau) = R_{XY}(-\tau)$$

$$|R_{XY}(\tau)| = \sqrt{R_{XX}(0)R_{YY}(0)}$$

$$|R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$$

The properties of crosscorrelation functions (2)

$R_{YX}(-\tau)$

$$R_{YX}(-\tau) = E[Y(t)X(t-\tau)] = E[Y(s+\tau)X(s)] = R_{XY}(\tau)$$

$$E\left[\{Y(t+\tau)+\alpha X(t)\}^2\right] \geq 0$$

the **geometric mean** of two positive numbers
cannot exceed their **arithmetic mean**

The properties of crosscorrelation functions (3)

$|R_{XY}(\tau)|$

$$|R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$$

$$\sqrt{R_{XX}(0)R_{YX}(0)} \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$$

Covariance Functions

$C_{XX}(t, t+\tau), C_{XY}(t, t+\tau)$

$$C_{XX}(t, t+\tau) = E[\{X(t) - m_X(t)\}\{X(t+\tau) - m_X(t+\tau)\}]$$
$$C_{XY}(t, t+\tau) = E[\{X(t) - m_X(t)\}\{Y(t+\tau) - m_Y(t+\tau)\}]$$
$$C_{XX}(t, t+\tau) = R_{XX}(t, t+\tau) - m_X(t)m_X(t+\tau)$$
$$C_{XY}(t, t+\tau) = R_{XY}(t, t+\tau) - m_X(t)m_Y(t+\tau)$$

at least jointly WSS

$$C_{XX}(\tau) = R_{XX}(\tau) - \overline{X}^2$$

$$C_{XY}(\tau) = R_{XY}(\tau) - \overline{XY}$$

The properties of covariance functions

$C_{XX}(0)$

For a **WSS process**, variance does not depend on time and if $\tau = 0$

$$C_{XX}(0) = R_{XX}(0) - \overline{X}^2$$

$$\sigma_X^2 = E\left[\{X(t) - E[X(t)]\}^2\right] = C_{XX}(0)$$

it the two random processes **uncorrelated**

$$C_{XY}(t, t+\tau) = R_{XY}(t, t+\tau) - m_X(t) m_Y(t+\tau) = 0$$

$$R_{XY}(t, t+\tau) = m_X(t) m_Y(t+\tau)$$

Discrete-Time Processes and Sequences (1)

$R_{XX}[n, n+k], R_{YY}[n, n+k], C_{XX}[n, n+k], C_{YY}[n, n+k]$

$m_X[n] = \overline{X}$, $m_Y[n] = \overline{Y}$

$$R_{XX}[n, n+k] = R_{XX}[k]$$

$$R_{YY}[n, n+k] = R_{YY}[k]$$

$$C_{XX}[n, n+k] = R_{XX}[k] - \overline{X}^2$$

$$C_{YY}[n, n+k] = R_{YY}[k] - \overline{Y}^2$$

Discrete-Time Processes and Sequences (2)

$R_{XY}[n, n+k], R_{YX}[n, n+k], C_{XY}[n, n+k], C_{YX}[n, n+k]$

$m_X[n] = \overline{X}$, $m_Y[n] = \overline{Y}$

$$R_{XY}[n, n+k] = R_{XY}[k]$$

$$R_{YX}[n, n+k] = R_{YX}[k]$$

$$C_{XY}[n, n+k] = R_{XY}[k] - \overline{XY}$$

$$C_{YX}[n, n+k] = R_{YX}[k] - \overline{YX}$$