Unveiling the breakdown of Euclidean geometry in linearized general relativity without Riemannian calculus

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Perhaps the best way to show that spacetime possesses Rienannian curvature is to begin with the assumption of a 'flat' universe. A reasonably simple effort to construct a tensor theory of gravity cast within the familiar formalism of the 'flat' Minkowskian spacetime is shown to violate Euclidean geometry. Careful use of ordinary (non-Reimannian) calculus can ensure that little in the derivation needs to be retracted when this breakdown of Euclidean geometry does emerge.

I. POSTULATES

Our goal is to obtain an equation of motion for a test particle in the presence of a gravitational field generated by an arbitrary energy-momentum stress, $T^{\alpha\beta}$. For a cold fluid this stress tensor is $T^{\alpha\beta} = \rho U^\alpha U^\beta$, where ρ is mass density in the rest frame of a cold fluid moving with 4-velocity $U^\alpha = dX^\alpha/d\tau$. We restrict our discussion to weak fields throughout.[1–3]

Except that 'simple' is used with no rigorous mathematical definition, this paper is also a proof by contradiction that the simplest tensor theory of field motion requires Riemannian curvature of spacetime. We begin with a 'flat' spacetime and adopt the familiar Cartesian-like Minkowskian coordinate system. We soon find ourselves working in a different coordinate system. As occurs with with other non-Cartesian coordinate systems (e.g., spherical), this complicates the vector calculus a bit. Fortunately these complications can be ameliorated: First, most complications can be ignored as higher order terms in this approximate theory. Second, we avoid Riemannian calculus altogether. Riemannian calculus differs from ordinary vector calculus trivially in that the notation is different, and nontrivially in that it involves an intrinsic curvature that we only seem to experience on two-dimensional curved surfaces.[1–3]

It is not easy to justify the introduction of a non-Euclidean calculus into a theory of gravity. Consider for example, an attempt to mimic gravity by subjecting an array of massless observers to acceleration.[1] While these observers will have difficulty constructing a familiar coordinate system, this spacetime always lacks Riemannian curvature. An empty universe is 'flat'.

At this point it is impossible to assume or even speculate that particle motion under the influence of gravity might follow a geodesic (minimum-length) path. Without Riemannian curvature, the geodesic can only represent the uniform motion we commonly associate with constant speed and direction. The following 'postulates' might be better called 'starting points':

Postulate 1. Particles are the limiting case of localized wavepackets of a scalar wave equation.

Postulate 2. The tensor theory of gravity closely resembles the vector-potential form of electrodynamics cast in the Lorentz gauge.[4]

Postulate 3. The motion of a test particle is independent of its mass in that initial position and velocity are sufficient to predict future motion.

Postulate 4. Zero-mass particles experience an intuitively guessed gravitational redshift as they rise against a static gravitational field.[5]

Postulate 5. An apparent contradiction between the first and fourth postulates is resolved by a slight redefinition of spacetime coordinates.

Using theorem by Weinberg, the first postulate informs us that particles obey Hamilton's equations of motion,[6,7]

$$\frac{dx^j}{dt} = \frac{\partial \omega}{\partial k_j}, \qquad \frac{dk_j}{dt} = -\frac{\partial \omega}{\partial x^j}, \qquad \frac{d\omega}{dt} = \frac{\partial \omega}{\partial t}, \tag{1}$$

Conjugate momentum and wavenumber shall be treated as interchangable throughout this paper $\boldsymbol{p} \Leftrightarrow \boldsymbol{k}$, as are the Hamiltonian and frequency, $H(\boldsymbol{p}, \boldsymbol{r}, t) \Leftrightarrow \omega(\boldsymbol{k}, \boldsymbol{r}, t)$. The Hamiltonian shall be referred to as the dispersion relation. In units where $c = 1$, a well-known free particle Hamiltonian is, $\omega^2 = m^2 + k^2$.

The first postulate also explains why the Hamiltonian and congugate momentum variables must transform as a *covariant* (subscripted) 4-vector, $K_\alpha \equiv (-\omega, \boldsymbol{k})$. Since all test particles move in the same way, we are at liberty to invent a Lorentz invariant scalar field ψ that governs the quantum mechanics of massive particles, as well as a fictitious scalar field that resembles light. If the wavepacket's amplitude, $\psi = A(x,t)\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t)$, is invariant, then so is phase, $\boldsymbol{k}\cdot\boldsymbol{r} - \omega t = K_\alpha X^\alpha$. Throughout this paper, $(X^0, X^1, X^2, X^3) = (t, x, y, z) = (t, \boldsymbol{r}) = X^\alpha = (t, x^j)$ all represent variables in the same coordinate system. It will soon emerge that these variables do not exactly represent space and time as measured by clocks and rulers on a conventional cartesian coordinate system.

The second postulate is inspired by the compactness of electrodynamics as stated in potential form with the Lorentz gauge. The field equation in the Lorentz gauge, $\partial_\alpha A^\alpha = 0$, is $\partial^\alpha \partial_\alpha A^\beta = 4\pi J^\beta = 4\pi\rho U^\beta$; and the dispersion relation is $(K_\alpha - qA_\alpha)(K^\alpha - qA^\alpha) + m^2 = 0$.[1,4]

The third postulate is a consequence of Einstein's equivalence principle.[1–3] The fourth postulate is also well known and preceded Einstein's formulation of general relativity.[5] Either conservation of energy or the doppler shift can be used to motivate a plausibility argument that the weak-field gravtiational redshift of light is, $\Delta\omega/\omega \approx -\Delta V$, where the Newtonian potential, V, is defined in units where the gravitational acceleration is $\boldsymbol{a} = -\nabla V$. In the weak field limit, $V << 1$.

The centerpiece of this derivation is an apparent contradiction between the first and fourth postulates. Eq. (1) states that frequency is invariant for a time-invariant dispersion relation $\omega = \omega(\boldsymbol{k}, \boldsymbol{r})$.[7] But the fourth postulate demands that frequency vary for a static gravitational potential $V = V(\boldsymbol{r})$. Rather than attempt to concoct an explicitly time-dependent dispersion relation, $\partial\omega/\partial t \neq 0$, for a particle in static field, $\partial V/\partial t = 0$, we adopt a different approach: Let the variable t not precisely represent measured time. It may be helpful to write down an explicit but mythical transformation between our coordinate variables $X^\alpha = (t, \boldsymbol{r})$ and what might naively be called the 'real' spacetime variables $(t^*, \boldsymbol{r}^*)$ of the (mythical) Minkowskian coordinate system:

$$
\begin{aligned}
t &= t(t^*, \boldsymbol{r}^*) \approx t^* \\
\boldsymbol{r} &= \boldsymbol{r}(t^*, \boldsymbol{r}^*) \approx \boldsymbol{r}^*
\end{aligned}
\tag{2}
$$

Though meaningless, Eq. (2) permits us to avoid Riemannian calculus. It is the only equation that must be retracted when this linearized theory of gravity is completed and shown to violate Euclidean geometry: There are no coordinates (t^*, x^*, y^*, z^*) for which t^* measures simultaneous time for observers on a cartesian grid measured by the three $\boldsymbol{r}^*$ variables .

What does survive the transition to a correct theory of gravity is the concept of locally measured differentials, $(dt^*, d\boldsymbol{r}^*)$. To measure frequency, an observer would count cycles as phase advances:

$$\omega^* dt^* = \omega dt \tag{3}$$

This permits the observed frequency, ω^*, to vary, while the 'Hamiltonian', ω, remains invariant, as one would expect in a static gravitational field.

The metric is defined using a Lorentz invariant:[1–3]

$$-d\tau^2 \equiv dt^{*2} - dx^{*2} - dy^{*2} - dz^{*2} = dX^\alpha \, g_{\alpha\beta} \, dX^\beta \tag{4}$$

The approximation, $t \approx t^*$ and $x^j \approx x^{*j}$, restricts the validity of this derivation to first order in field strength, and also implies that the metric is nearly that of a Minkowskian space:

$$\begin{aligned} g_{\alpha\beta} &= \eta^B_{\alpha\beta} + h_{\alpha\beta} \\ g^{\alpha\beta} &= \eta_T^{\alpha\beta} - h^{\alpha\beta}, \end{aligned} \tag{5}$$

where $g_{\alpha\beta}$ is the *true* metric, and $h_{\alpha\beta}$ is the first order metric perturbation. The Minkowskian metrics,

$$\eta^B_{\alpha\beta} = diag(-1, 1, 1, 1) = \eta_T^{\alpha\beta}. \tag{6}$$

now loses their status as the true metric.

Though Eqs. (5-6) are commonly found in the literature,[1–3] I have temporarily embellished the notation by adding a 'T' for the covariant, and 'B' for the contravariant Minkowskian tensors, respectively. To first order in field strength, $\underline{\underline{\eta}}^B$ and $\underline{\underline{\eta}}^T$ represent different tensors. The contravariant form of η^B is $g^{\alpha\beta}\eta^B_{\beta\gamma}g^{\gamma\delta} \neq \eta_T^{\alpha\delta}$. This explains the peculiar mix of plus and minus signs in Eq. (5). Henceforth, the conventional notation is adopted, $\eta^{\alpha\beta} \equiv \eta_T^{\alpha\beta}$ and $\eta_{\alpha\beta} \equiv \eta^B_{\alpha\beta}$.

II. FIELD EQUATIONS

The second postulate suggests field equations similar to those of electrodynamics. The field, $\Phi^{\mu\nu}$, is generated by the source, $T^{\mu\nu}$, as follows:

$$\partial^\alpha \partial_\alpha \Phi^{\mu\nu} = -16\pi T^{\mu\nu} + \cdots \tag{7}$$

$$\partial_\mu \Phi^{\mu\nu} = 0 + \cdots \tag{8}$$

The factor 16π in Eq. (7) is an arbitrary coupling constant chosen to conform with convention. One advantage of this Lorentz gauge is that an intuitive and simple solution exists: $\Phi^{\mu\nu} = 4\int \{T^{\mu\nu}_{RET}/r\}d^3x$, where $T^{\mu\nu}_{RET}(t) = T^{\mu\nu}(t-r)$ must be evaluated at the 'retarded time'. If the field is due to a stationary mass at the origin, only the 00 term is nonzero: $\Phi^{00} = 4V$ where $V = M/r << 1$ is the Newtonian potential and M is mass.

The higher order terms $(\cdots)$ in Eqns. (7-8) represent the fact that they cannot be exact if the spacetime variables to are not those of an orthonormal Minkowskian coordinate system. Like the 'del' operator ∇ of three-dimensional space, the operator $\partial_\alpha \equiv \partial/\partial x^\alpha$ guarantees symmetry under Lorentz transformations, but only if orthonormal Minkowskian variables are used. Fortunately these higher order terms can be dropped in a linearized theory.

III. DISPERSION RELATION (HAMILTONIAN)

The free-particle dispersion relation, $\omega^2 = m^2 + k^2$, can be written as $0 = m^2 + K_\alpha \eta^{\alpha\beta} K_\beta$. By analogy with electrodynamics, and in a weak-field theory, we seek to add terms that are first order in field strength, where the field is $\Phi^{\alpha\beta}$. Since particle motion is independent of mass, all correction terms must be quadratic in the variables (m, k, ω). To understand this scaling, consider how the dispersion relation changes when the test particle changes from m to χm. A free particle of mass χm will have the same velocity as one of mass m only if k and ω are also changed by a factor of χ. We therefore demand that all terms in the dispersion relation be proportional to χ^2. Assuming that the dispersion relation is simply related to a wave equation, this considerably limits the permissible first order terms that can be added:

$$0 = m^2 + K_\alpha \eta^{\alpha\beta} K_\beta + b_1 \Phi K_\alpha \eta^{\alpha\beta} K_\beta - b_2 K_\alpha \Phi^{\alpha\beta} K_\beta, \tag{9}$$

where the trace may be calculated to first order as $\Phi \equiv g_{\alpha\beta}\Phi^{\alpha\beta} \approx \eta_{\alpha\beta}\Phi^{\alpha\beta}$. There is no need to postulate a term proportional to m^2 because for example, $m^2\Phi$ would be redundant with $K_\alpha \eta^{\alpha\beta} K_\beta \Phi$, at first order.

To find the values of the constants (b_1, b_2) we need the need to write Eq. (9) for the special 'static' case in which the gravitating objects are motionless. Here, all tensors in Eq. (9) are diagonal:

$$\Phi^{\alpha\beta} \; = \; diag(-4V, 0, 0, 0)$$

$$\Phi\eta^{\alpha\beta} \;=\; diag(-4V, 4V, 4V, 4V), \tag{10}$$

so that Eq. (9) becomes in the static limit:

$$\omega^2 = m^2 + k^2 + 4(b_2 - b_1)m^2 V + 4b_2 k^2 V + ..., \tag{11}$$

where we have used the zero-order expression, $\omega^2 \approx m^2 + k^2$, to convert and drop terms that are second order in field strength.

Continuing to operate in this static limit, we postulate gravitational dilations in both time and space:

$$\begin{aligned} dt^* &= (1 + b_3 V)dt \\ ds^* &= (1 + b_4 V)ds, \end{aligned} \tag{12}$$

where $ds \equiv (dx^2 + dy^2 + dz^2)$. Although a compelling reason for doing this does not emerge until the appendix is reached, we have also permitted spatial dilation of our coordinates with the (unknown) constant b_4.

The decoupling of space and time variables expressed by Eq. (12) is not usually compatible with special relativity, and is permitted here only because a preferred reference frame exists (that of the gravitating objects). To cast Eq. (12) into Lorentz-invariant form, square terms and subtract to make the Lorentz invariant:

$$-d\tau^2 \equiv dt^{*2} - ds^{*2} = (-1 - 2b_3 V)dt^2 + (1 + 2b_4 V)ds^2. \tag{13}$$

Inspection of Eq. (10) suggests that the metric defined by Eq. (13) can also be written using tensor potentials. It is easily verified that:

$$h_{\alpha\beta} = \frac{1}{2}(b_3 - b_4)\Phi_{\alpha\beta} + \frac{1}{2}b_4\Phi\eta_{\alpha\beta}. \tag{14}$$

Having established Eq. (14) to hold for a static field, we postulate it to hold for time-dependent fields.

IV. PHYSICAL CONSTRAINTS ON (b_1, b_2, b_3, b_4)

Three of the four physical arguments required to find (b_1, b_2, b_3, b_4) are obvious: (1) the recovery of the nonrelativistic dispersion relation, $\omega \approx m + k^2/(2m) + V$ when $\partial\omega/\partial k << 1$, or equivalently when $k << m$; (2) an observed speed of light equal to unity, $\omega^*/k^* = 1$ when $m = 0$; and (3) recovery of the expected gravitational redshift, $\Delta\omega/\omega \approx -\Delta V$. These arguments lead to the constraints: $b_2 - b_1 = \frac{1}{2}$, $b_3 = 2b_2 + b_4$, and $b_3 = 1$, respectively. The fourth constraint is so problematical that we pause to consolidate our unknown parameters (b_1, b_2, b_3, b_4) into one unknown paramater, b, defined so that its 'correct' value is $b = 1$:

$$b_1 = \frac{b}{2}, \quad b_2 = \frac{1+b}{2}, \quad b_3 = 1, \quad b_4 = -b. \tag{15}$$

The appendix shows that $b = 1$ is a consequence of the fact the potential $\Phi^{\alpha\beta}$ is closely related to the coordinate system, a situation that obviously does not exist in electromagnetism. This argument is so difficult that the casual reader is invited to 'cheat' by utilizing experimental evidence concerning the path of light in the presence of a gravitational field.[1–3] As discussed in later sections, this experimental evidence can be used to show $b \approx 1$.

V. EQUATIONS OF MOTION: STATIC CASE

Apply Hamilton's equations of motion (1) to the static-case dispersion relation, continuing to treat the fourth parameter ($b = 1$) as unknown,[7]

$$\omega = \left\{m^2 + k^2 + 2m^2 V + (2+2b)k^2 V\right\}^{\frac{1}{2}} \tag{16}$$

Define $\boldsymbol{v} = d\boldsymbol{r}/dt$. To zero order in field strength, $\boldsymbol{k} \approx \omega\boldsymbol{v} \approx \gamma m\boldsymbol{v}$, where $\gamma = (1 - v^2)^{-1/2}$. Using, $dV/dt = \boldsymbol{v} \cdot \boldsymbol{\nabla} V$, the acceleration, $\boldsymbol{a} = d\boldsymbol{v}/dt$, is to first order:

$$\boldsymbol{a} = -(1+b^2)\nabla V + (2+2b)\boldsymbol{v}\,\boldsymbol{v} \cdot \boldsymbol{\nabla} V \tag{17}$$

The reader is advised not view the relativistic correction term with $\boldsymbol{v}\,\boldsymbol{v} \cdot \boldsymbol{\nabla} V$ as the generalization of $\boldsymbol{v} \times \boldsymbol{B}$ in the magnetic force. To understand why, let both gravity and the motion of light to be aligned along the y-axis. Eq. (17) seems to suggest that the photon accelerates at a rate of $(2 + b)\partial V/\partial y$. In fact, the photon is not accelerating at all – the extra term simply ensures that the observed velocity dy^*/dt^* remains constant. To put it another way, the so-called 'acceleration' of light is entirely an artifact of how we chose to label spacetime.

The experimentally observed deflection of starlight by the Sun's gravity is consistent with a downward acceleration of $-2\partial V/\partial y$ for light directed in the x direction. Hence an alternative to the proof in the appendix that $b = 1$, is the experimentally observed deflection of light by gravity. The 'anomalous' factor of 2 in $-2\partial V/\partial y$ is what made the confirmation of Einstein's theory in 1919 so spectacular.[1] As shown in the next section, this 'anomalous' deflection is also an artifact of how we choose to label spacetime; photons directed perpendicular to gravity 'fall' at just the rate intuition says they should.

VI. VIOLATION OF EUCLIDEAN GEOMETRY

Though we have relabelled spacetime, nothing so far has forced us to abandon the concept of a 'flat' universe. For that reason, we now construct a simple geometry that is actually forbidden by the (nonlinear) general theory of relativity. Consider a planet consisting of an infinite sheet of ultra-dense material that occupies the plane $y = 0$. Take $V = 0$ on this plane and suppose one side is populated by people who experience Earth's gravitational acceleration, $a_G \approx 9.8m/s^2$. Solve Eq. (7) to yield $V = a_G y$ for $y > 0$. The weak-field approximation ($V << 1$) demands that we remain close to the surface of the planet: $y << 1/a_G \equiv c^2/a_G \approx 1-$light year.

If $b = 1$, then squares of length L in the xy plane will not be measured as squares. Eq. (12) implies that top side is be shorter than the bottom by a factor of approximately $1 - a_G L$. As these 'squares' fill space,[8] careful surveyors will measure the planet to be bowel shaped with a radius of curvature, $1/a_G$. The breakdown of Euclidean geometry would evident to anybody who managed to drill a hole through the planet and discover that part of the universe is 'missing'.

An optical illusion would seem to give the planet twice this curvature because light directed nearly parallel to the planet's surface 'falls' at a rate of a_G (not $2a_G$). This is shown in Fig. 1 where the path of light is depicted as path a. This path obeys $\ddot{y} = -2a_G$ (assuming it is directed nearly perpendicular to the surface). If such a photon obeyed $\ddot{y} = 0$, observers would conclude that gravity repelled the photon, causing it to curve upward with the planet's surface along a contour of constant y. If gravity had no influence on the photon's path it would obey $\ddot{y} = -a_G$. This is depicted as path b in Fig. 1. The so-called 'anomalous' deflection of light by gravity is not so anomalous after all.[8]

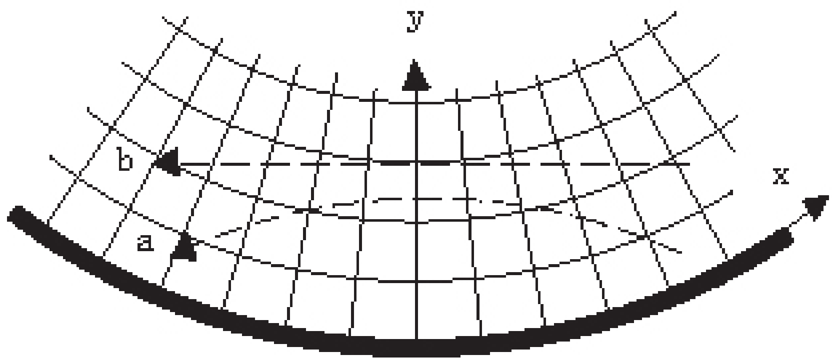

FIG. 1: A planet with Earthlike gravity is formed from a sheet of gravitating matter fills the plane $y = 0$. Observers on both sides measure the planet to be bowl shaped. The photon follows path a.

VII. CORRESPONDENCE WITH EINSTEIN'S THEORY

Taking $b = 1$ we now use Eq. (15) to substitute the corrected metric into the dispersion relation. Eq. (14) permits us to convert the first order terms containing $\Phi_{\alpha\beta}$ into terms involving the metric correction $h^{\alpha\beta} \approx \eta^{\alpha\gamma} h_{\gamma\delta} \eta^{\delta\beta}$. All first order the correction terms in Eq. (9) are remarkably absorbed into the free-particle dispersion relation as we replace the approximate metric $\eta^{\alpha\beta}$ by the corrected metric $g^{\alpha\beta}$:

$$0 = m^2 + K_\alpha \left\{ \eta^{\alpha\beta} - h^{\alpha\beta} \right\} K_\beta = m^2 + K_\alpha g^{\alpha\beta} K_\beta. \tag{18}$$

Provided one understands that only the covariant form K_α contains the Hamiltonian and conjugate momentum variables, this can be written as $0 = m^2 + K_\alpha K^\alpha$. Such an elegant dispersion relation can only be associated with geodesic motion! This has been verified and shown to hold in the full nonlinear theory as well.[9] Recall from classical mechanics[7] that one purpose of the canonical equations of motion (Eq. (1)) is to express the conjugate momentum variables in terms of generalized velocity $(dx/dt, dy/dt, dz/dt) = dX^j/dt = dX^j/dX^0$. This relationship is also remarkably elegant in both the linearized and nonlinear theory: $K_\alpha = mU_\alpha$.

Eq. (7) is a well known version of linearized Relativity in the Lorentz gauge.[1–3] In this gauge the linearized Riemannian curvature tensor is:[3]

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}(\partial_\beta \partial_\gamma h_{\alpha\delta} + \partial_\alpha \partial_\delta h_{\beta\gamma} - \partial_\beta \partial_\delta h_{\alpha\gamma} - \partial_\alpha \partial_\gamma h_{\beta\delta}) \tag{19}$$

APPENDIX A: PROOF THAT $b = 1$

Consider the uniqueness of solutions to the field equations of Eqs. (7 - 8). Let the par of tensors $\{\Phi_1, T_1\}$ represent one solution and let $\{\Phi_2, T_2\}$ represent another solution. Subtract to obtain,

$$\partial_\alpha \partial^\alpha (\Phi_2^{\mu\nu} - \Phi_1^{\mu\nu}) = -16\pi (T_2^{\mu\nu} - T_1^{\mu\nu}). \tag{A1}$$

This implies that a uniqueness theorem prevents two different stress-energy tensors from generating the same field: If $\Phi_1 = \Phi_2$ within any small neighborhood, then $T_1 = T_2$. One has to be careful because two different tensors can represent the same situation but in two different metrics. On the other hand, the physical significance of stress-energy $T^{\mu\nu}$ cannot vanish in one coordinate system and be non-zero in another coordinate system. No 'change of variables' can convert a region with no stress-energy into a region that contains a nonzero stress-energy. This condition will lead to the correct value of $b = 1$.

Consider a universe completely devoid of energy-momentum and adopt the Cartesian-like variables $X^{\alpha *} = (t^*, x^*, y^*, z^*)$ that are governed by Minkowskian metric $\eta_{\alpha\beta}$. Change variables to new coordinates $X^\alpha = (X^0, x^1, X^2, X^3)$ using the transformation,

$$X^{\alpha *} = X^\alpha + \xi^\alpha \tag{A2}$$

where $\xi = \xi(X^0, X^1, X^2, X^3)$ is a differentiable vector field that is sufficiently small near the origin that the coordinates $X^{\alpha *}$ and X^α are nearly identical. The new coordinates may possess a metric that differs from $\eta_{\alpha\beta}$. Use the smallness of ξ^α to Taylor expand $d\xi^\alpha \approx \partial_\mu \xi^\alpha dX^\mu$. Use $d\tau = (-dZ^{*\alpha} \eta_{\alpha\beta} dZ^{*\beta})^{1/2}$ to find the metric for the new variables, defined as $g_{\alpha\beta}$:

$$-d\tau^2 = (dX^\alpha + dX^\mu \partial_\mu \xi^\alpha) \eta_{\alpha\beta} (dX^\beta + dX^\nu \partial_\nu \xi^\beta) = dX^\alpha g_{\alpha\beta} dX^\beta. \tag{A3}$$

If $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, then the metric perturbation is,

$$h = \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha \tag{A4}$$

Take the trace of Eq. (14) after simplifying the constants using Eq. (15), and use the fact that the trace of $\eta_{\alpha\beta} = 4$. The field equations (7-8) can now be expressed in terms of $h_{\alpha\beta}$:

$$\frac{2}{b+1} \partial_\alpha \partial^\alpha \left(h_{\mu\nu} + \frac{b}{1-3b} h \eta_{\mu\nu} \right) = -16\pi T_{\mu\nu} = 0, \tag{A5}$$

$$\partial^\alpha\left(h_{\mu\nu}+\frac{b}{1-3b}h\eta_{\mu\nu}\right)=0. \tag{A6}$$

Eq. (A4) implies that the trace $h=2\partial_\alpha\xi^\alpha$, and permits us to rewrite Eq. (A6) as,

$$\partial_\alpha\partial^\alpha\xi_\mu=\frac{1-b}{3b-1}\partial_\mu\partial_\beta\xi^\beta. \tag{A7}$$

Substitute Eq. (A4) into Eq. (A5) and use Eq. (A7) to obtain:

$$\frac{1-b}{3b-1}\frac{4}{b+1}\left\{\partial_\mu\partial_\nu+\frac{b\eta_{\mu\nu}}{1-3b}\partial_\alpha\partial_\beta\right\}(\partial_\beta\xi^\beta)=0. \tag{A8}$$

Note the factor $(1-b)$. To force $b=1$ we seek a vector field for which the left-side of Eq. (A8) would be non-zero if $b\neq 1$. One such field is,

$$\xi^\alpha=\varepsilon\left\{xyt,\,\frac{1}{2}\frac{b-1}{3b-1}t^2y,\,\frac{1}{2}\frac{b-1}{3b-1}t^2x,\,0\right\}, \tag{A9}$$

where ε is a sufficiently small that the linear approximation is valid. The metric generated by Eq. (A9) diverges far from the origin, thus violating the assumptions of this first order theory. This does not seem to be a problem because both the uniqueness theorem, as well as our demand that transformations not ‘create’ energy apply in any small neighborhood.

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