

Power Density Spectrum - Continuous Time

Young W Lim

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Energy and average power in time domain

power density spectrum for continuous time signals

Energy, Average Power – deterministic, time domain

a deterministic signal $x(t)$

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{otherwise} \end{cases}$$

the energy

$$E(T) = \int_{-T}^{+T} x^2(t) dt = \int_{-\infty}^{+\infty} x_T^2(t) dt$$

the average power

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt = \frac{1}{2T} \int_{-\infty}^{+\infty} x_T^2(t) dt$$

Fourier transform

power density spectrum for continuous time signals

Fourier Transform Pair $x(t) \iff X(\omega)$

Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

a deterministic signal $x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Fourier transform of $x_T(t)$

for continuous time signals

bounded duration, bounded variation

for a finite T , $x_T(t)$ is assumed to have bounded variation

$$\int_{-T}^{+T} |x(t)| dt < \infty$$

the Fourier transform of $x_T(t)$

$$\begin{aligned} X_T(\omega) &= \int_{-\infty}^{+\infty} x_T(t) e^{-j\omega t} dt \\ &= \int_{-T}^{+T} x(t) e^{-j\omega t} dt \end{aligned}$$

Fourier transforms of $x_T(t)$ and $X_T(t)$

power density spectrum for continuous time signals

deterministic $X_T(\omega)$ v.s. random $X_T(\omega)$

a **deterministic** sample signal $x_T(t)$

$$x_T(t) \iff X_T(\omega)$$

a **random process** signal $X_T(t)$

$$X_T(t) \iff X_T(\omega)$$

Parseval's theorem (I)

power density spectrum for continuous time signals

for a deterministic $x_T(t)$

a **deterministic** sample signal $x_T(t)$

$$\int_{-\infty}^{+\infty} x_T(\tau)x_T^*(\tau)d\tau = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_T(\omega)X_T^*(\omega)d\omega$$

$$\int_{-\infty}^{+\infty} |x_T(\tau)|^2 d\tau = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

Parseval's theorem (II)

power density spectrum for continuous time signals

for a deterministic $x_T(t)$ v.s. a random $X_T(t)$

- a **deterministic** signal $x_T(t) \iff X_T(\omega)$

$$\int_{-\infty}^{+\infty} |x_T(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

- a **random** signal $X_T(t) \iff X_T(\omega)$

$$\int_{-\infty}^{+\infty} E \left[|X_T(t)|^2 \right] dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E \left[|X_T(\omega)|^2 \right] d\omega$$

Energy and average power in frequency domain

power density spectrum for continuous time signals

Energy, Average Power – Parseval's theorem applied

a deterministic signal $x_T(t)$

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{otherwise} \end{cases} \quad x_T(t) \iff X_T(\omega)$$

the **energy** by Parseval's theorem

$$E(T) = \int_{-T}^{+T} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

the **average power** by Parseval's theorem

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

$E(T)$ and $P(T)$ in frequency domain – deterministic case

power density spectrum for continuous time signals

deterministic $x_T(t) \iff X_T(\omega)$

the **energy** for the **deterministic** $X_T(\omega)$ in $x_T(t) \iff X_T(\omega)$

$$E(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

the **average power** for the **deterministic** $X_T(\omega)$

$$P(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

the **power density spectrum** for the **deterministic** $X_T(\omega)$

$$\lim_{T \rightarrow \infty} \frac{|X_T(\omega)|^2}{2T}$$

$E(T)$ and $P(T)$ in frequency domain – random case

power density spectrum for continuous time signals

random $X_T(t) \iff X_T(\omega)$

the **energy** for the **random** $X_T(\omega)$ in $X_T(t) \iff X_T(\omega)$

$$E(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E[|X_T(\omega)|^2] d\omega$$

the **average power** for the **random** $X_T(\omega)$

$$P(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{E[|X_T(\omega)|^2]}{2T} d\omega$$

the **power density spectrum** for the **random** $X_T(\omega)$

$$\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

Average power $P(T)$ – bounded duration $(-T, +T)$

power density spectrum for continuous time signals

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

- not the average power in a **random process**
only the power in **one** sample function
 - to obtain the **average power** over all possible realizations,
replace $x(t)$ by $X(t)$
take the **expected value** of $x^2(t)$, that is $TE [X^2(t)]$
 - then, the **average power** is a **random variable**
with respect to the **random process** $X(t)$
- not the average power in an **entire** sample function
 - take $T \rightarrow \infty$ to include all power in the **ensemble** member

Average power P_{XX} – unbounded duration $(-\infty, +\infty)$

power density spectrum for continuous time signals

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

- replace $x(t)$ by the **random variable** $X(t)$
- take the **expected value** of $x^2(t)$, that is $E[X^2(t)]$

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt$$

- take $T \rightarrow \infty$ to include all power

$$P_{XX} = \lim_{T \rightarrow \infty} P(T) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt$$

Average power P_{XX} – time average $A[\bullet]$

power density spectrum for continuous time signals

The time average

$$A_T[\bullet] = \frac{1}{2T} \int_{-T}^T [\bullet] dt \qquad A[\bullet] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [\bullet] dt$$

time average and sample average operations

$$\begin{aligned} P_{XX} = \lim_{T \rightarrow \infty} P(T) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt \\ &= \lim_{T \rightarrow \infty} A_T[E[X^2(t)]] \\ &= A[E[X^2(t)]] \end{aligned}$$

Measuring average power

power density spectrum for continuous time signals

for deterministic and random signals

the **average power** $P(T)$ for a deterministic signal $x(t)$

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

the **average power** P_{XX} for a random process $X(t)$

$$\begin{aligned} P_{XX} &= \lim_{T \rightarrow \infty} P(T) \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt \\ &= A[E[X^2(t)]] \end{aligned}$$

Power density spectrum $S_{XX}(\omega)$

power density spectrum for continuous time signals

the average power via power density

the average power P_{XX} for the random process $X_T(\omega)$

$$\begin{aligned} P_{XX} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{S_{XX}(\omega)} d\omega \end{aligned}$$

the power density spectrum $S_{XX}(\omega)$

$$\boxed{S_{XX}(\omega)} = \boxed{\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}}$$

Properties of Power Spectrum

power density spectrum for continuous time signals

- $S_{XX}(\omega) \geq 0$
- $S_{XX}(-\omega) = S_{XX}(\omega)$ $X(t)$ real
- $S_{XX}(\omega)$ real
- $\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = A [E [X^2(t)]]$
- $S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$
- $\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A [R_{XX}(t, t + \tau)]$
- $S_{XX}(\omega) = \int_{-\infty}^{+\infty} A [R_{XX}(t, t + \tau)] e^{-j\omega \tau} d\tau$

Equations involving $S_{XX}(\omega)$

power density spectrum for continuous time signals

the average power P_{XX} and the inverse Fourier transform of $S_{XX}(\omega)$

the **average power** related equation

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = A[E[X^2(t)]]$$

the **autocorrelation** related equation

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A[R_{XX}(t, t + \tau)]$$

Average power related equation

power density spectrum for continuous time signals

the average power P_{xx}

the average power related equation

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = A [E [X^2(t)]]$$

- a random process $X(t)$ in time domain
- a random process $X(\omega)$ in frequency domain

$$X(t) = \lim_{T \rightarrow \infty} X_T(t) \quad X(\omega) = \lim_{T \rightarrow \infty} X_T(\omega)$$

- Parseval's theorem over $X_T(t) \iff X_T(\omega)$

Average power P_{XX} in time / frequency domain

power density spectrum for continuous time signals

Average power P_{XX} using $X_T(t)$ and $X_T(\omega)$

- Using a random process $X_T(t)$ in time domain

$$\begin{aligned}P_{XX} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt \\&= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-\infty}^{+\infty} E[X_T^2(t)] dt \\&= \lim_{T \rightarrow \infty} A_T [E[X^2(t)]] = \boxed{A[E[X^2(t)]]}\end{aligned}$$

- Using a random process $X_T(\omega)$ in frequency domain

$$\begin{aligned}P_{XX} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}} d\omega \\&= \boxed{\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}\end{aligned}$$

Autocorrelation related equation

power density spectrum for continuous time signals

the Inverse Fourier transform of $S_{XX}(\omega)$

the average power related equation

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A[R_{XX}(t, t + \tau)]$$

- auto-correlation function

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] \quad \Rightarrow R_{XX}(\tau)$$

- a random process $X(t)$ in time domain
- a random process $X(\omega)$ in frequency domain

Power Density Spectrum and Auto-correlation

power density spectrum for continuous time signals

Fourier transform pairs

- $A[R_{XX}(t, t + \tau)] \iff S_{XX}(\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t + \tau)] e^{-j\omega\tau} d\tau$$

$$A[R_{XX}(t, t + \tau)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega$$

- $R_{XX}(\tau) \iff S_{XX}(\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$$

for a WSS $X(t)$, $A[R_{XX}(t, t + \tau)] = R_{XX}(\tau)$

Power Spectrum and Auto-Correlation Functions

power density spectrum for continuous time signals

Definition

the power spectrum

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

the auto-correlation function

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$

Fourier transforms of autocorrelation functions

power density spectrum for continuous time signals

Definition

Fourier transform of an autocorrelation functions

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$S_{\dot{X}\dot{X}}(\omega) = \int_{-\infty}^{+\infty} R_{\dot{X}\dot{X}}(\tau) e^{-j\omega\tau} d\tau$$

- auto-correlation function

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] \Rightarrow R_{XX}(\tau)$$

$$R_{\dot{X}\dot{X}}(t, t + \tau) = E[\dot{X}(t)\dot{X}(t + \tau)] \Rightarrow R_{\dot{X}\dot{X}}(\tau)$$

- a random process $X(t)$ in time domain
- $\dot{X}(t) = \frac{d}{dt}X(t)$: the derivative of $X(t)$

Fourier transform of a derivative function

power density spectrum for continuous time signals

Definition

Fourier transform of an autocorrelation functions

$$x(t) \iff X(\omega)$$

$$\frac{d^n}{dt^n} x(t) \iff (j\omega)^n X(\omega)$$

Power Density Spectrum and Auto-correlation

power density spectrum for continuous time signals

Definition

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t + \tau)] e^{-j\omega\tau} d\tau$$
$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A[R_{XX}(t, t + \tau)]$$

for a WSS $X(t)$, $A[R_{XX}(t, t + \tau)] = R_{XX}(\tau)$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$
$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$

Power Spectrum and Auto-Correlation Functions

power density spectrum for continuous time signals

Definition

the power spectrum

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

the auto-correlation function

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$

RMS Bandwidth

power density spectrum for continuous time signals

Definition

the standard deviation is

a measure of the spread in a density function.

the analogous quantity for the normalized power spectrum is

a measure of its spread that we call the rms bandwidth
(root-mean-square)

$$W_{rms}^2 = \frac{\int_{-\infty}^{+\infty} \omega^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$

RMS Bandwidth and Mean Frequency

power density spectrum for continuous time signals

Definition

the mean frequency $\bar{\omega}_0$

$$\bar{\omega}_0 = \frac{\int_{-\infty}^{+\infty} \omega S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$

the rms bandwidth

$$W_{rms}^2 = \frac{4 \int_{-\infty}^{+\infty} (\omega - \bar{\omega}_0)^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$

