Derivatives (1A)

Young Won Lim
1/28/16

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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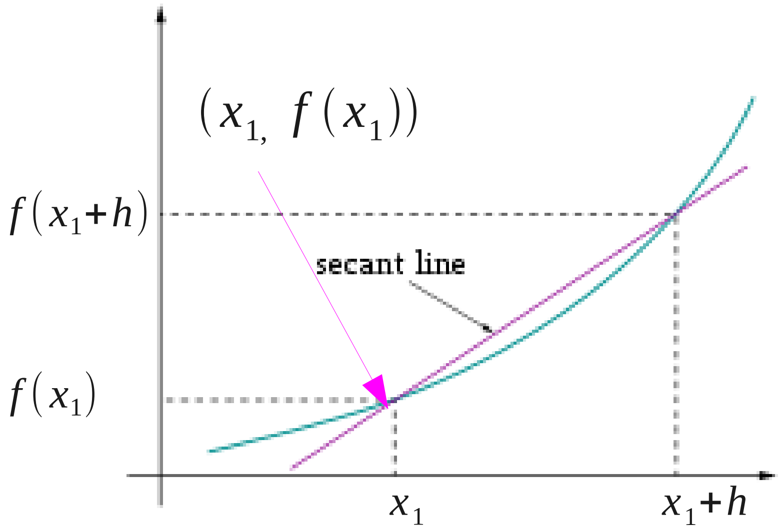
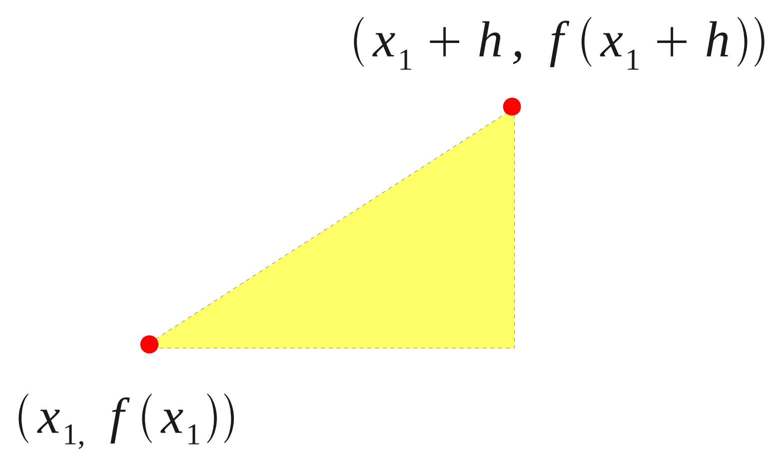
Differentials

A triangle and its slope

$$y = f(x)$$

$$\frac{f(x_1 + h) - f(x_1)}{h}$$





http://en.wikipedia.org/wiki/Derivative

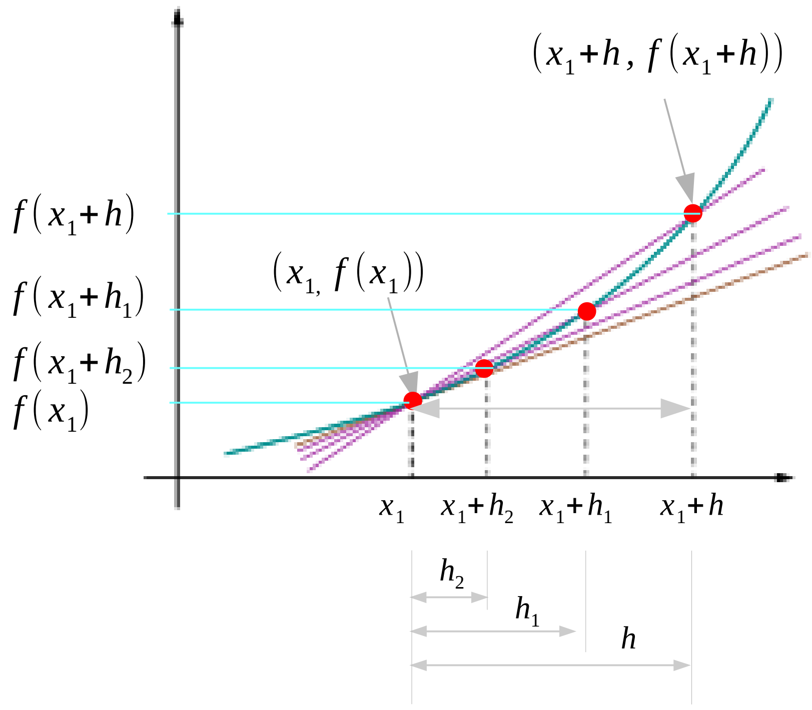
Many smaller triangles and their slopes

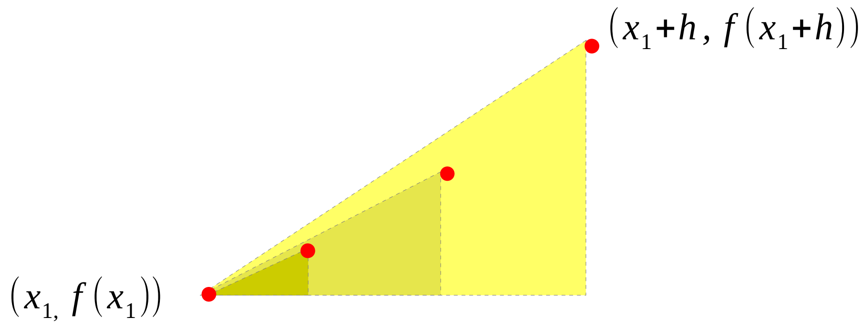
$$\frac{f(x_1 + h) - f(x_1)}{h}$$

$$\frac{f(x_1 + h_1) - f(x_1)}{h_1}$$

$$\frac{f(x_1 + h_2) - f(x_1)}{h_2}$$

$$\lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$





http://en.wikipedia.org/wiki/Derivative

The limit of triangles and their slopes

$$y = f(x)$$

The derivative of the function f at x_1

$$f'(x_1) = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

The derivative function of the function f

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$y' = f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x)$$

5. (*calculus*) The derived function of a function.

   The derivative of $f : f(x) = x^2$ *is* $f' : f'(x) = 2x$

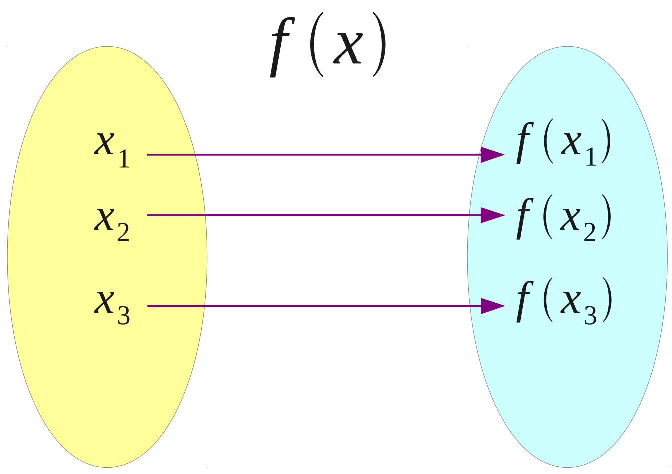
6. (*calculus*) The value of this function for a given value of its independent variable.

   The derivative of $f(x) = x^2$ *at x = 3 is* $f'(3) = 2 * 3 = 6$.

http://en.wiktionary.org/

The derivative as a function

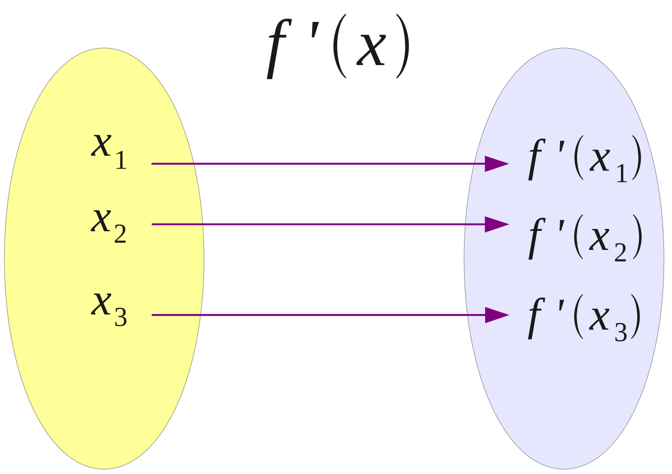
$$y = f(x)$$



Derivative Function

$$y' = f'(x)$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



The notations of derivative functions

Largrange's Notation

$$y' = f'(x)$$

Leibniz's Notation

$$\frac{dy}{dx} = \frac{d}{dx} f(x)$$

not a ratio.

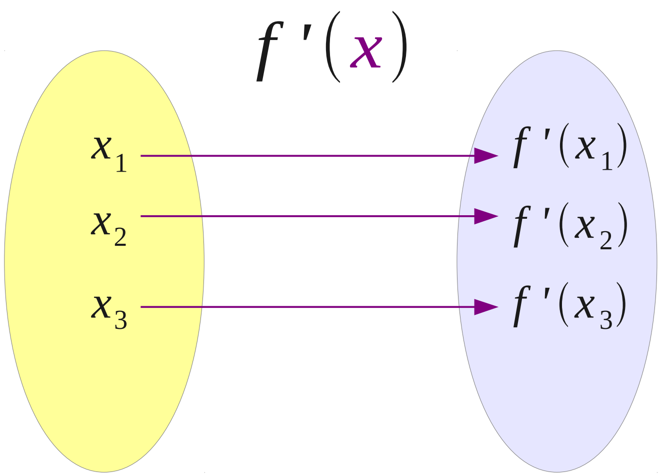
slope of a tangent line

Newton's Notation

$$\dot{y} = \dot{f}(x)$$

Euler's Notation

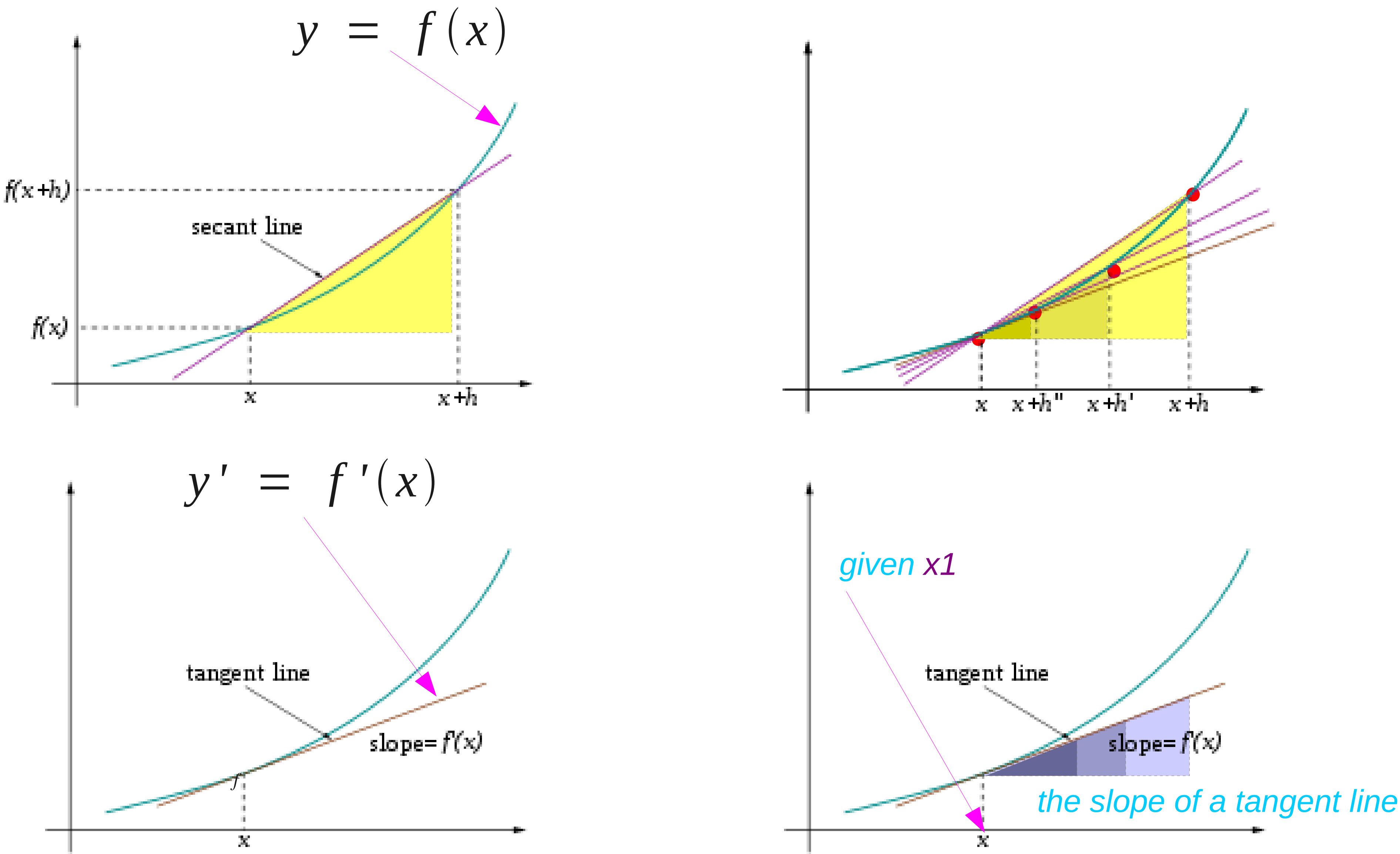
$$D_x y = D_x f(x)$$



$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- derivative with respect to x
- x is an independent variable

Another kind of triangles and their slope

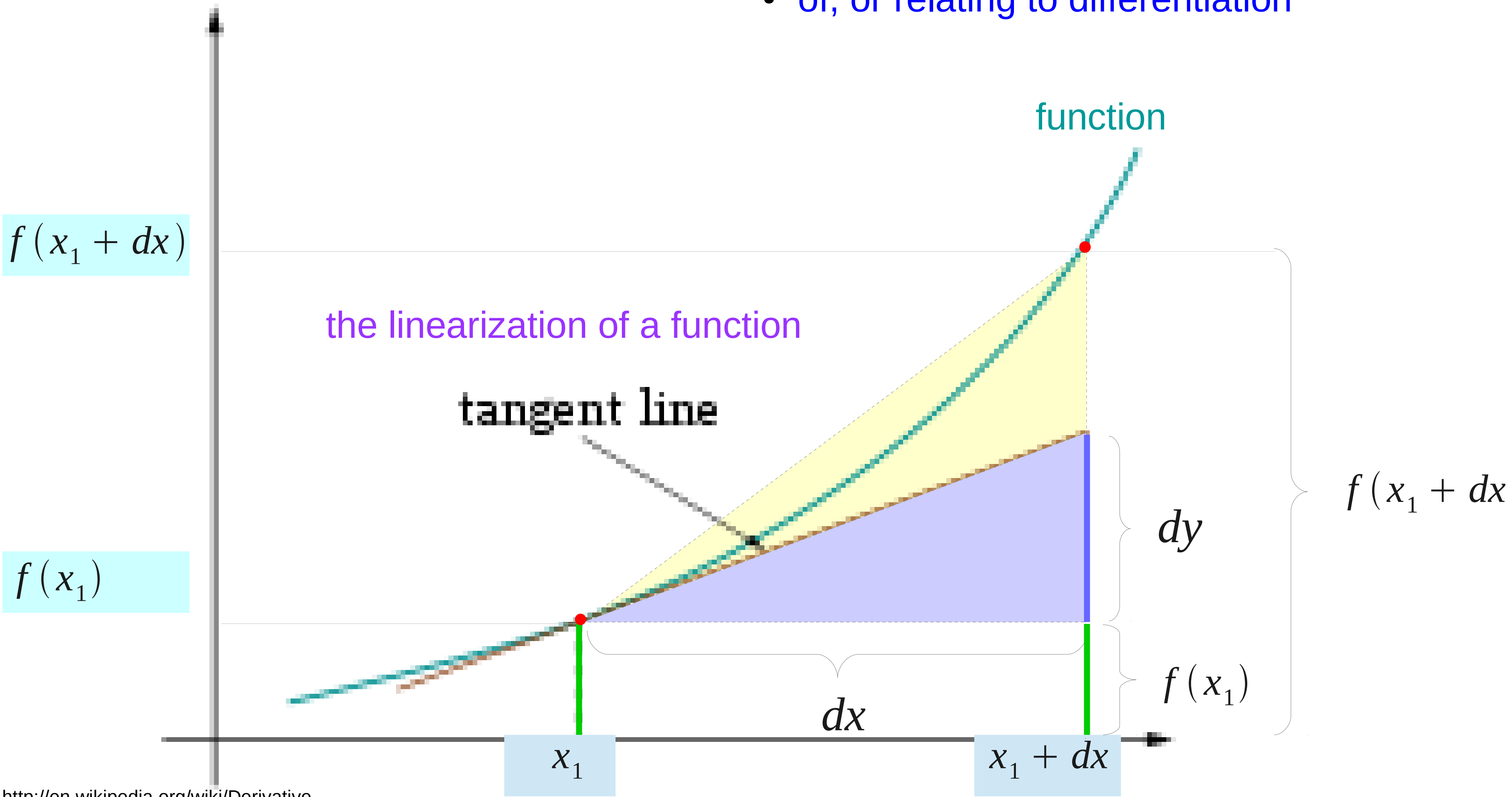


http://en.wikipedia.org/wiki/Derivative

Differential in calculus

Differential: dx, dy, …

- infinitesimals
- a change in the linearization of a function
- of, or relating to differentiation



http://en.wikipedia.org/wiki/Derivative

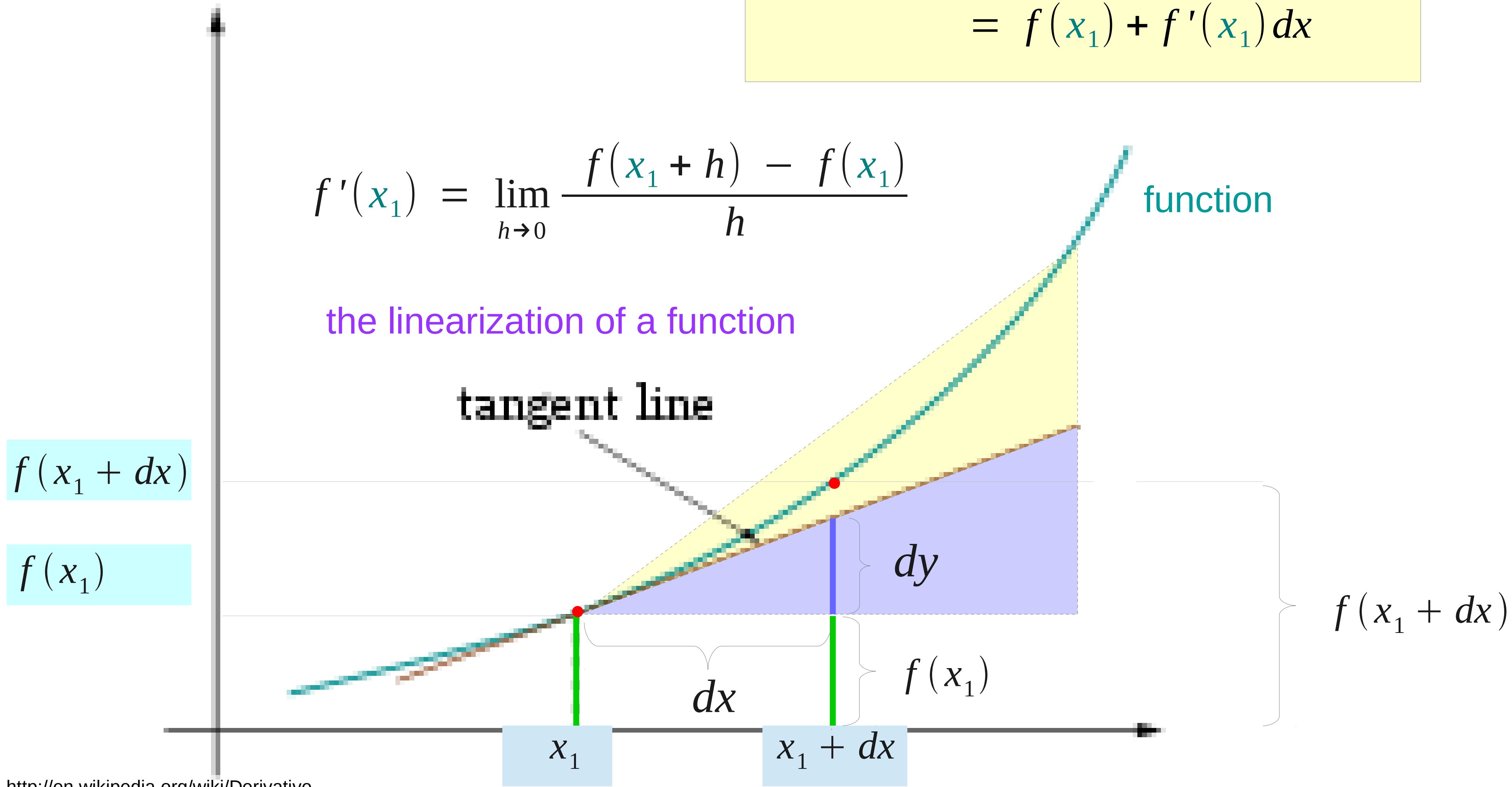
Approximation

Differential: dx, dy, …

$$f(x_1 + dx) \approx f(x_1) + dy$$
$$= f(x_1) + f'(x_1)\,dx$$

$$f'(x_1) = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

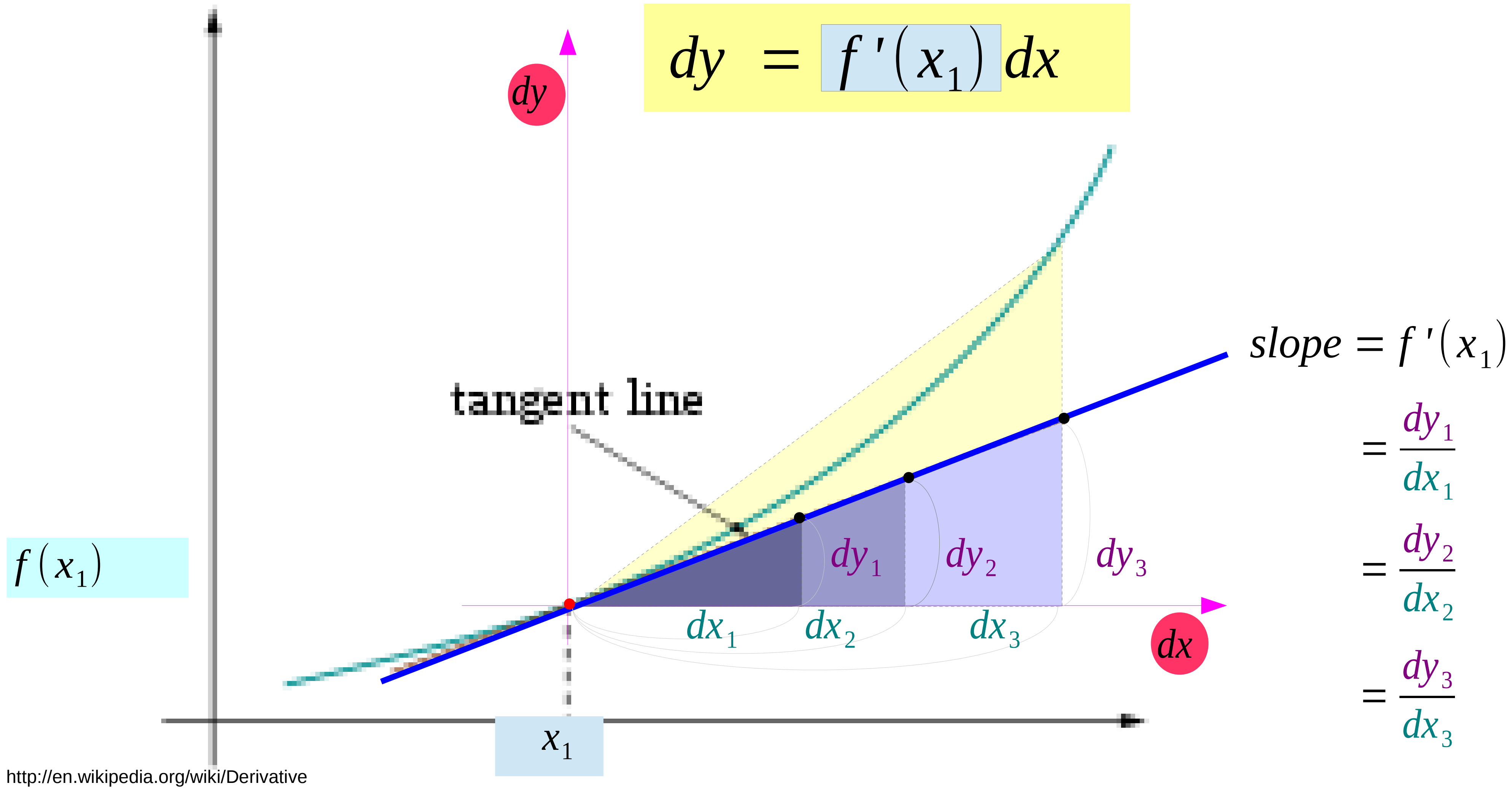
the linearization of a function



http://en.wikipedia.org/wiki/Derivative

Line equation in the new coordinate.

$$dy = f'(x_1)\,dx$$

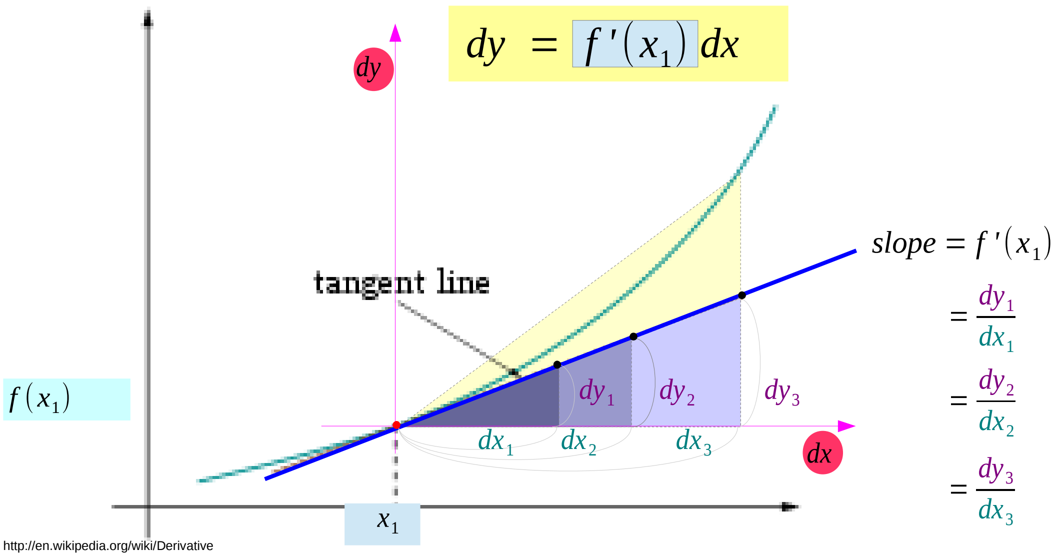


http://en.wikipedia.org/wiki/Derivative

Differentials and Derivatives (1)

$$dy = f'(x)\, dx$$

$$dy = \frac{df}{dx}\, dx$$



differentials ***derivative***

$$\frac{dy}{dx} = f'(x)$$

ratio ***not a ratio***

Differentials and Derivatives (2)

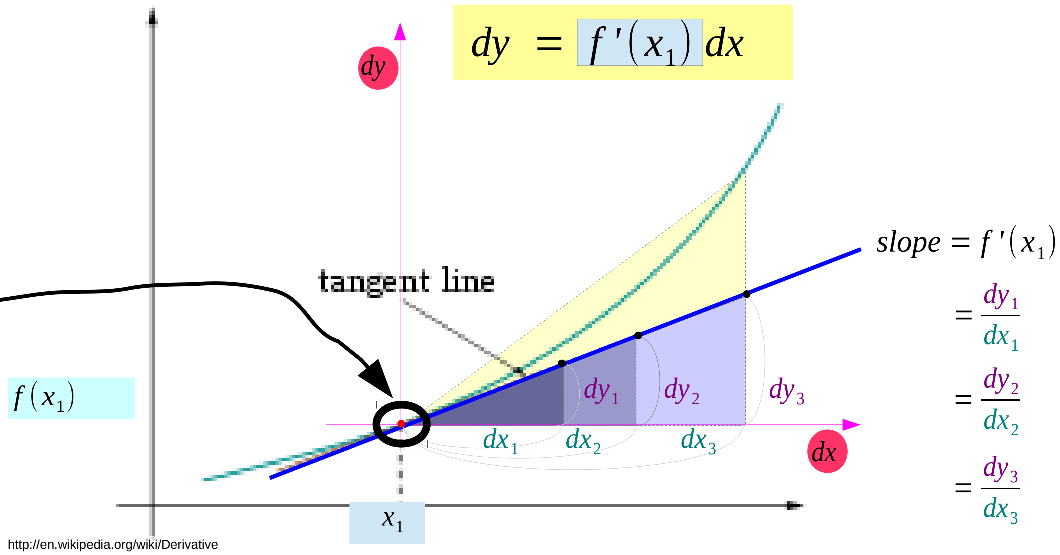
$$f(x_1 + dx) \approx f(x_1) + dy$$
$$= f(x_1) + f'(x_1)\,dx$$

for small enough dx

$\lim_{dx \to 0}$

$$f(x_1 + dx) = f(x_1) + dy$$
$$= f(x_1) + f'(x_1)\,dx$$

$$\lim_{dx \to 0} \frac{f(x_1 + dx) - f(x_1)}{dx} = f'(x_1)$$



Differentials and Derivatives (3)

$$dy = f'(x)\, dx \quad \Rightarrow \quad \int dy = \int f'(x)\, dx$$

$$dy = \frac{df}{dx}\, dx \quad \Rightarrow \quad \int dy = \int \frac{df}{dx}\, dx$$

$$y = f(x)$$

$$dy = \dot{f}\, dx$$

$$dy = D_x f\, dx$$

$$\int dy = \int 1\, dy = y$$

Integration Constant C

place a constant

place another constant

$$\int dy = \int f'(x)\, dx$$

$$\int dy = \int \frac{df}{dx}\, dx$$

$$y + C_1 = f(x) + C_2$$

$$y = f(x) + C$$

differs by a constant

place only one constant from the beginning

$$\int dy = \int f'(x)\, dx + C$$

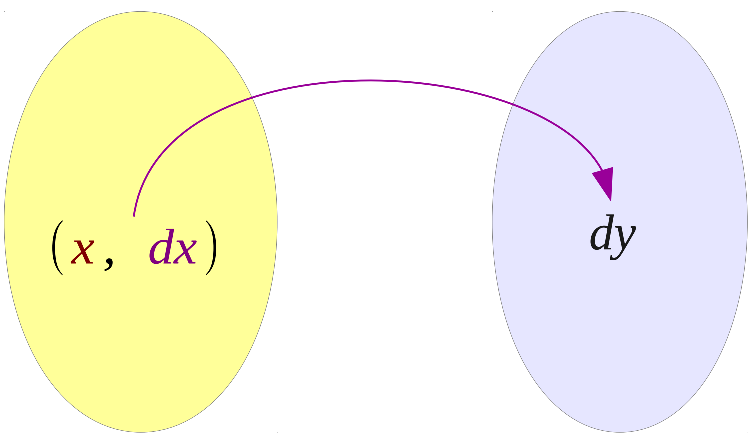
$$\int dy = \int \frac{df}{dx}\, dx + C$$

$$y = f(x) + C$$

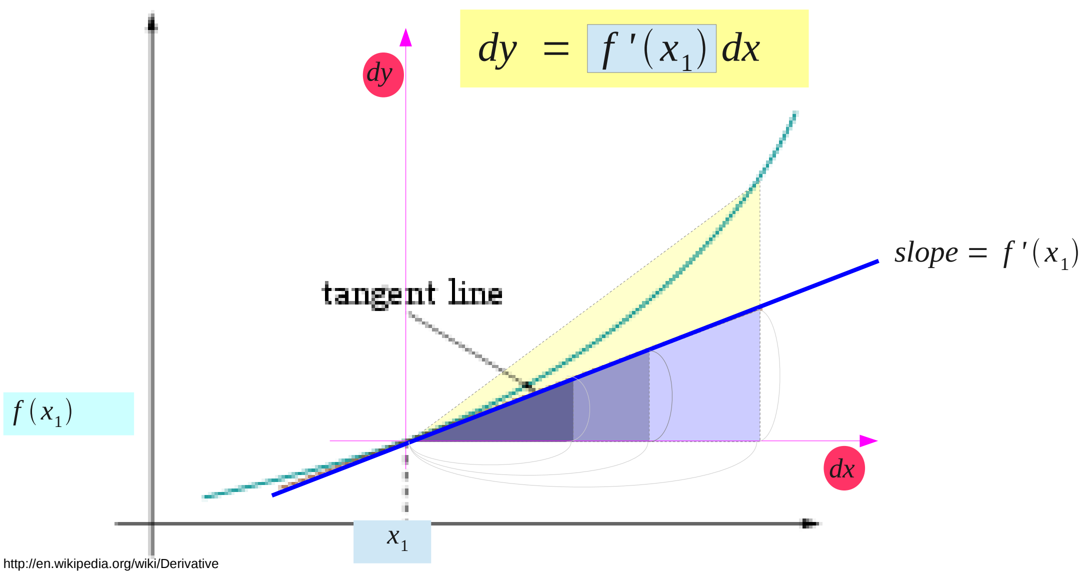
Differential as a function

*The **differential** of a function **f(x)** of a single real variable **x** is the function of two independent real variables **x** and **dx** given by*

$$dy = f'(x)\,dx$$



Line equation in the new coordinate.



Applications of Differentials (1)

Substitution Rule

$$\int f(\,g(x)\,)\cdot g'(x)\,dx \quad = \quad \int f(\,u\,)\,du$$

(I) $u = g(x)$ $\qquad du = g'(x)dx$ $\qquad du = \dfrac{dg}{dx}dx$

(II) $\displaystyle\int f(g)\frac{dg}{dx}\,dx \quad = \quad \int f(\,g\,)\,dg$

Applications of Differentials (2)

Integration by parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$u = f(x) \qquad du = f'(x)\,dx \qquad du = \frac{df}{dx}\,dx$$

$$v = g(x) \qquad dv = g'(x)\,dx \qquad dv = \frac{dg}{dx}\,dx$$

$$\int f(x)\underline{g'(x)\,dx} = f(x)g(x) - \int \underline{f'(x)}g(x)\,\underline{dx}$$

$$\int u\,dv = uv - \int v\,du$$

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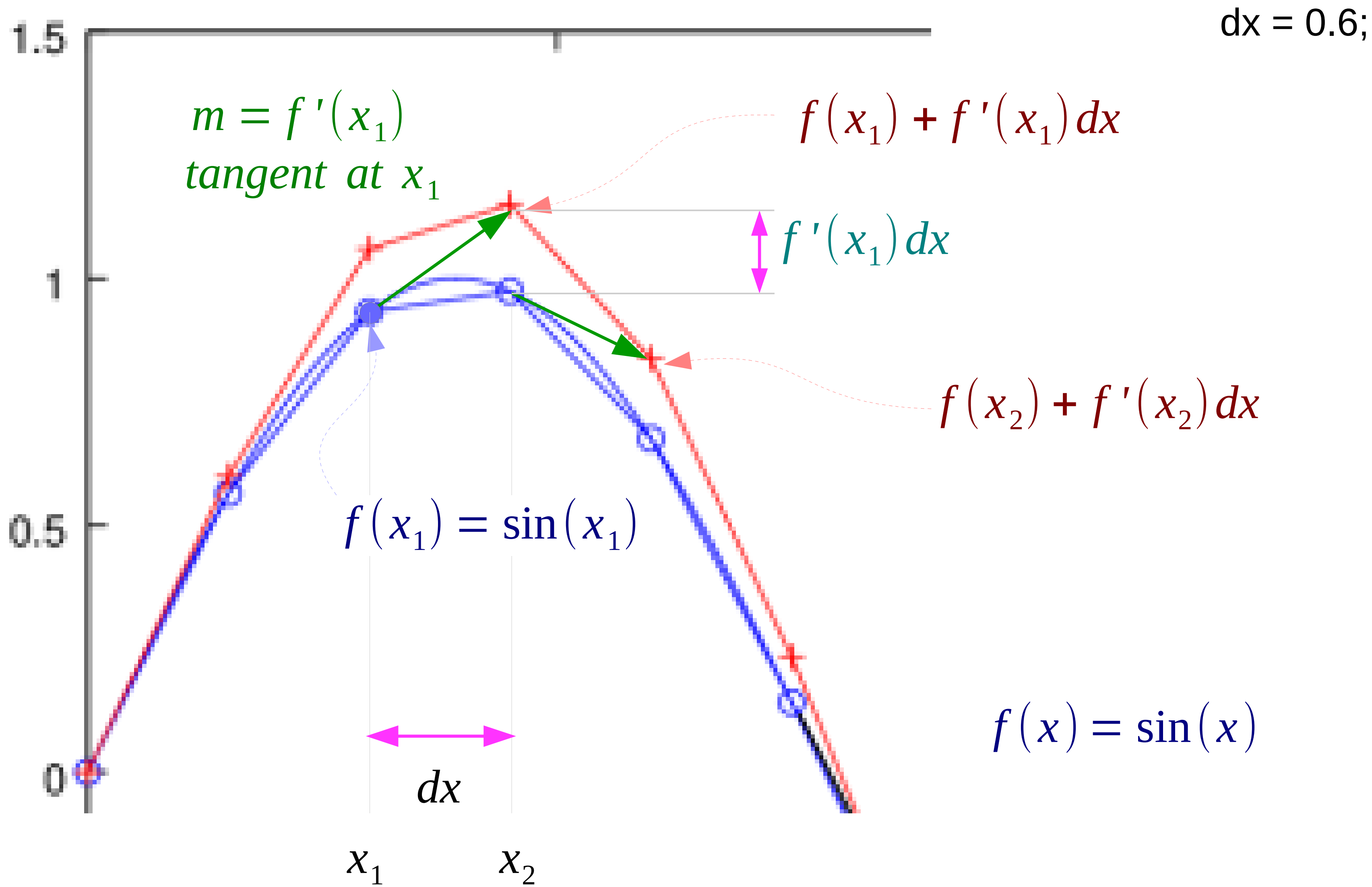
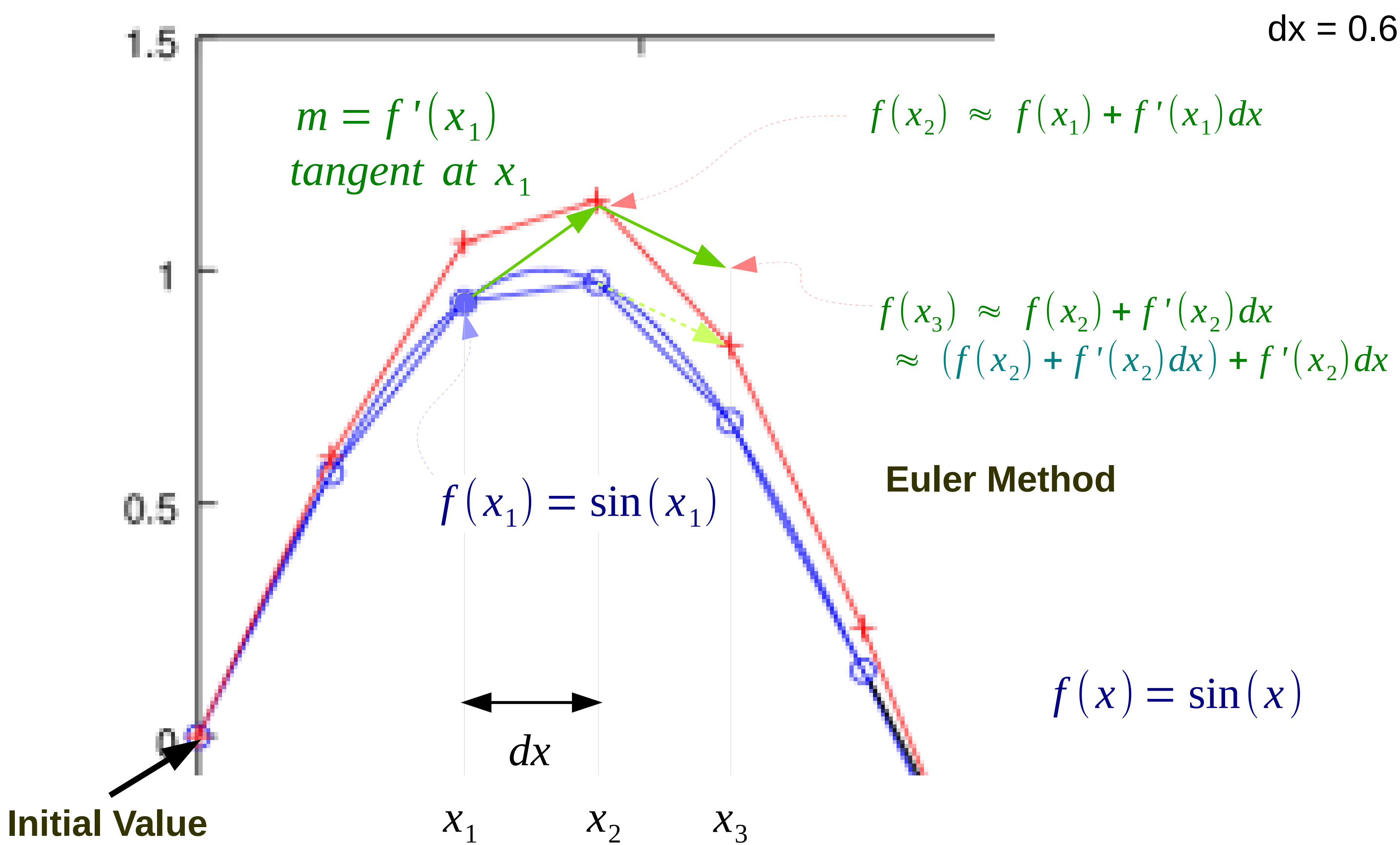
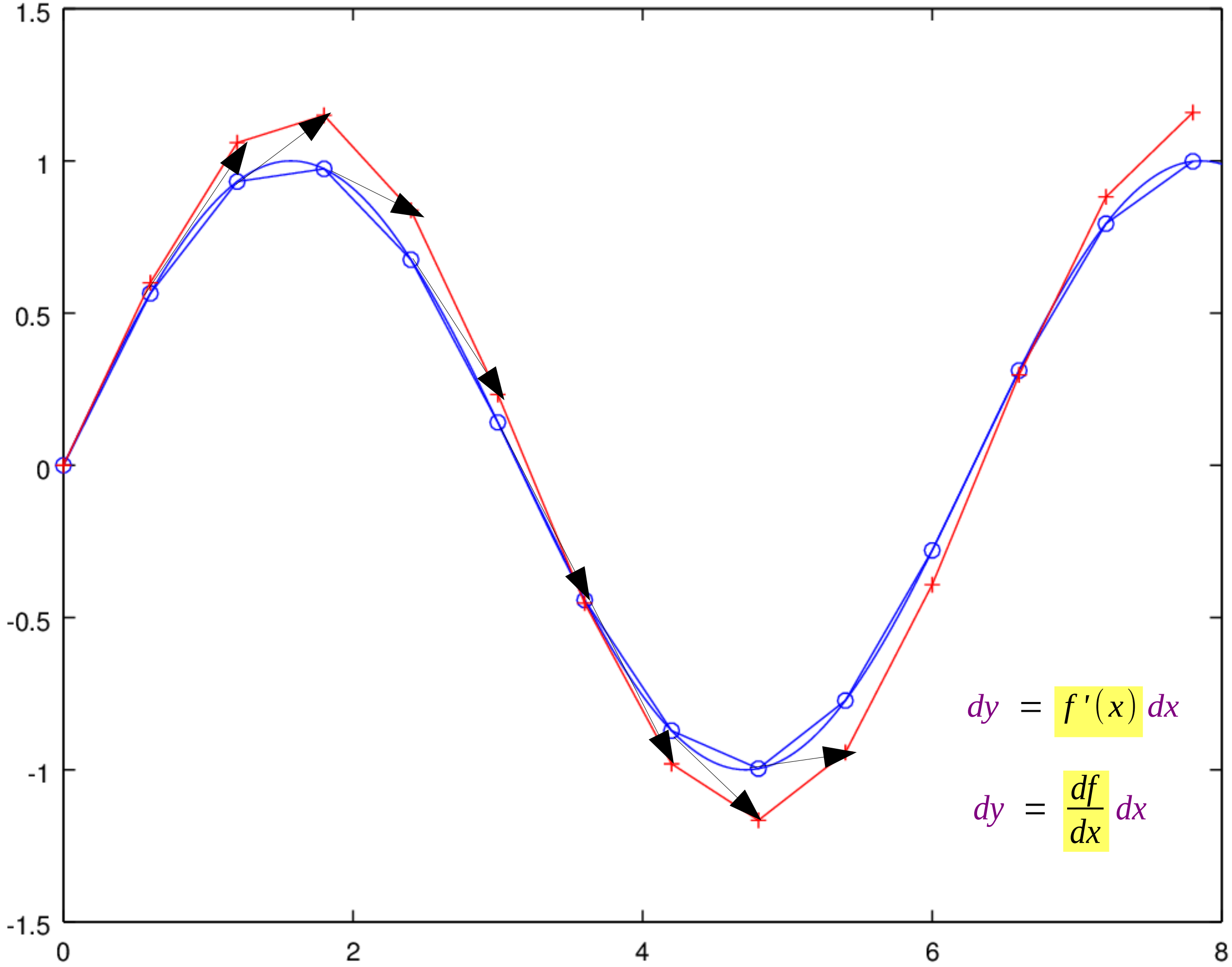
Derivatives and Differentials (large dx)



Euler Method of Approximation (large *dx*)



Derivatives and Differentials (large *dx = 0.6*)



dx = 0.6;

$$dy = f'(x)\,dx \quad \Rightarrow \quad \int dy = \int f'(x)\,dx$$

$$dy = \frac{df}{dx}\,dx \quad \Rightarrow \quad \int dy = \int \frac{df}{dx}\,dx$$

$$y = f(x)$$

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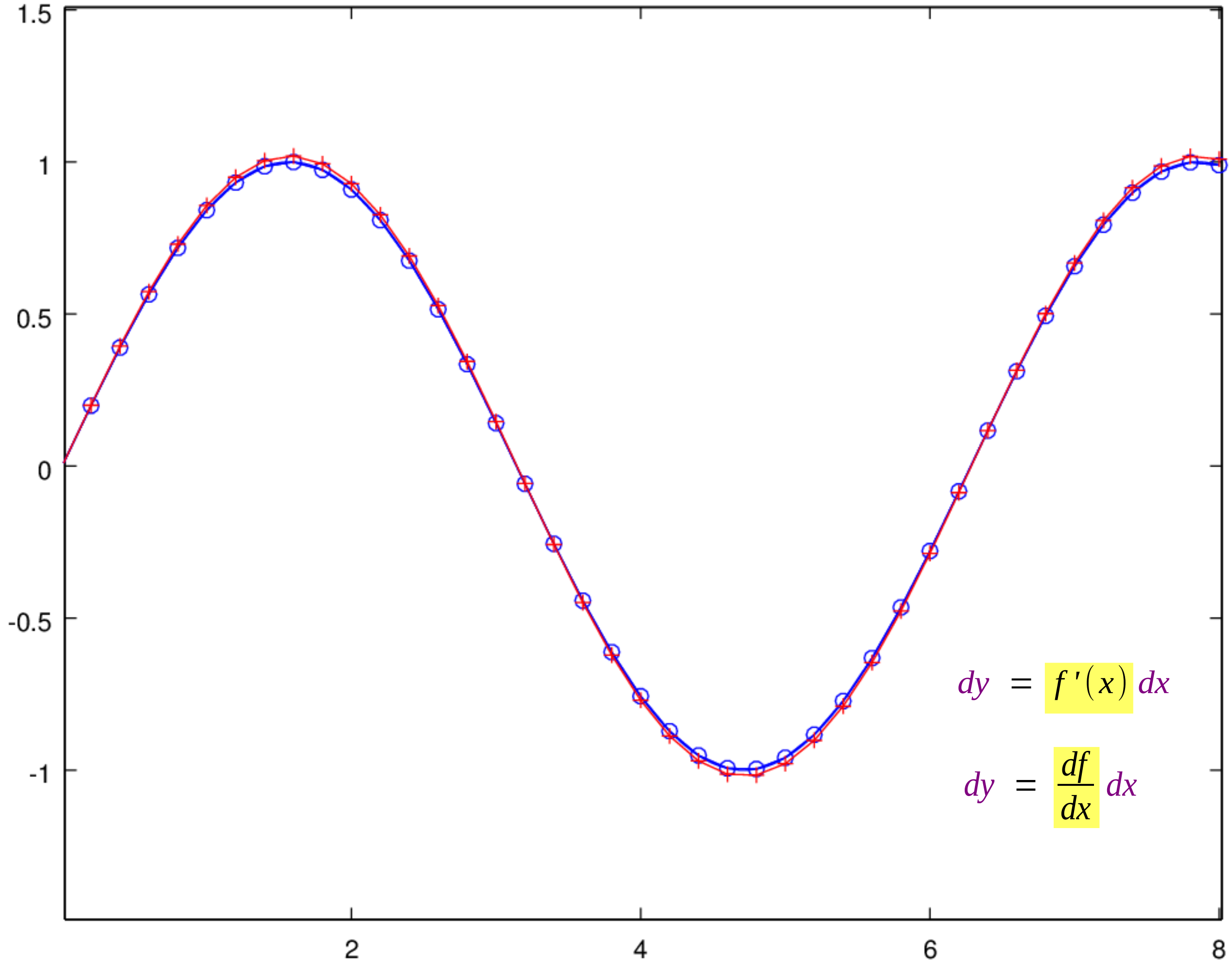
Derivatives and Differentials (small *dx = 0.2*)



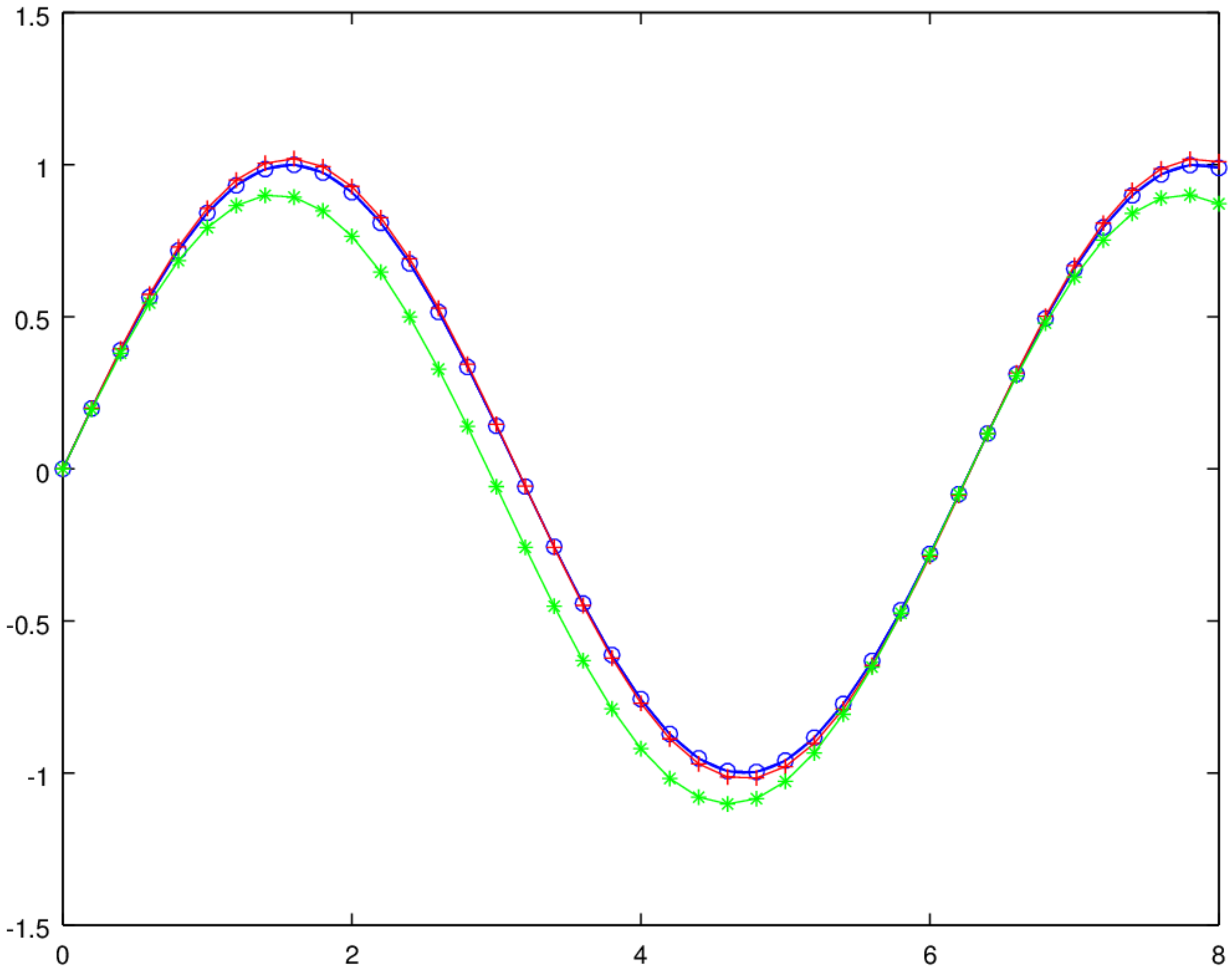
dx = 0.2;

$$dy = f'(x)\,dx \quad \Rightarrow \quad \int dy = \int f'(x)\,dx$$

$$dy = \frac{df}{dx}\,dx \quad \Rightarrow \quad \int dy = \int \frac{df}{dx}\,dx$$

$$y = f(x)$$

Euler's Method of Approximation



Octave Code

```
clf; hold off;
dx = 0.2;

x = 0 : dx : 8;
y = sin(x);
plot(x, y);
t = sin(x) + cos(x)*dx ;
y1 = [y(1), t(1:length(y)-1)];

y2 = [0];
y2(1) = y(1);
for i=1:length(y)-1
  y2(i+1) = y2(i) + cos((i)*dx)*dx;
endfor

hold on
t = 0:0.01:8;
plot(t, sin(t), "color", "blue");
plot(x, y,   "color", 'blue',  "marker", 'o');
plot(x, y1,  "color", 'red',   "marker", '+');
plot(x, y2,  "color", 'green',  "marker", '*');
```

Prerequisite to First Order ODEs

Partial Derivatives

Function of one variable $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Function of two variable $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

treating y as a constant

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

treating x as a constant

Partial Derivatives Notations

Function of one variable $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Function of two variables $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = z_x = f_x$$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

treating y as a constant

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = z_y = f_y$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

treating x as a constant

Higher-Order & Mixed Partial Derivatives

Second-order Partial Derivatives

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Third-order Partial Derivatives

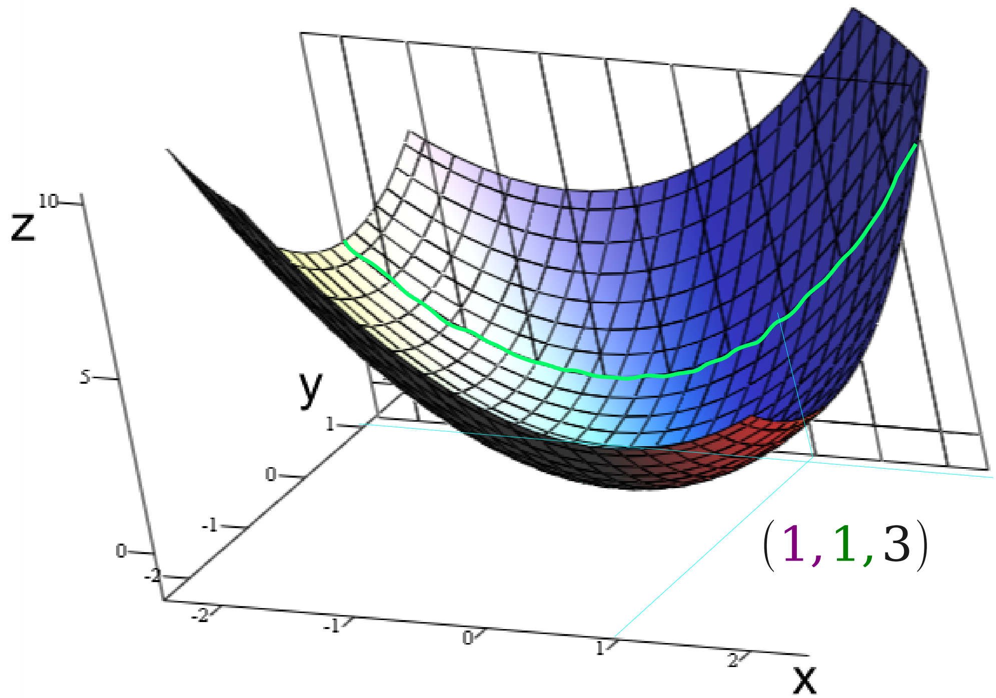
$$\frac{\partial^3 z}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) \qquad \frac{\partial^3 z}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y^2}\right)$$

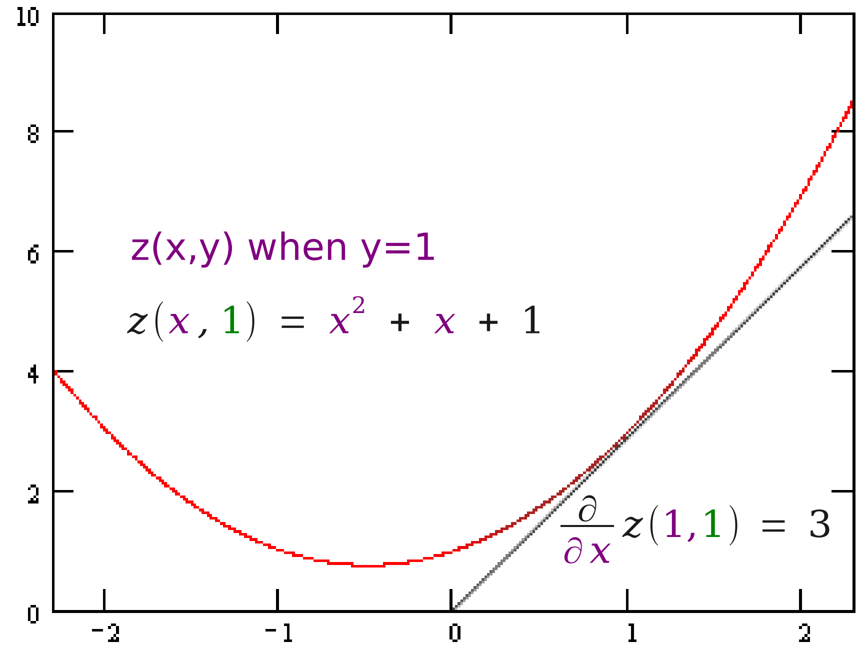
Mixed Partial Derivatives

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \overset{?}{=} \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \quad \Longleftrightarrow \quad \frac{\partial z}{\partial x},\ \frac{\partial z}{\partial y},\ \frac{\partial^2 z}{\partial x \partial y},\ \frac{\partial^2 z}{\partial y \partial x}$$ all defined and continuous

Partial Derivative Examples (1)





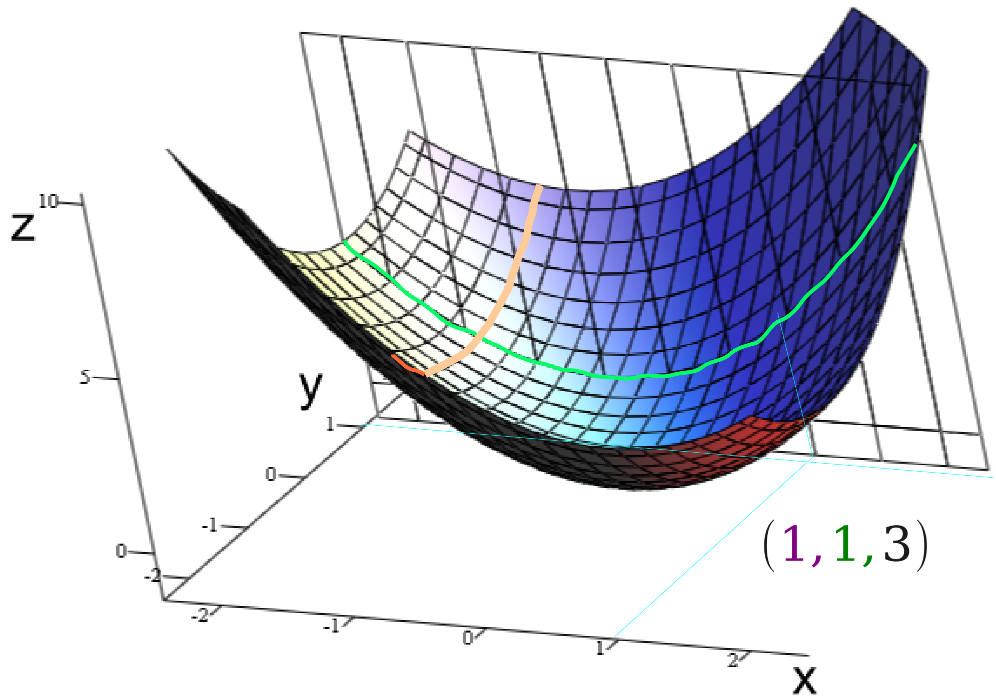
$$z = x^2 + xy + y^2 \Rightarrow \frac{\partial z}{\partial x} = 2x + y$$

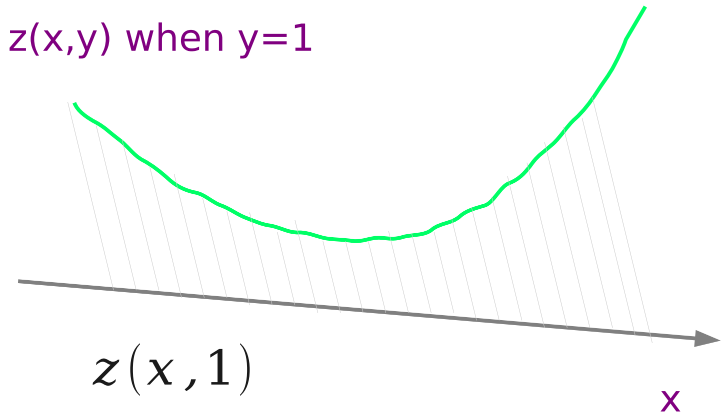
$$z = x^2 + xy + y^2 \Rightarrow \frac{\partial z}{\partial y} = x + 2y$$

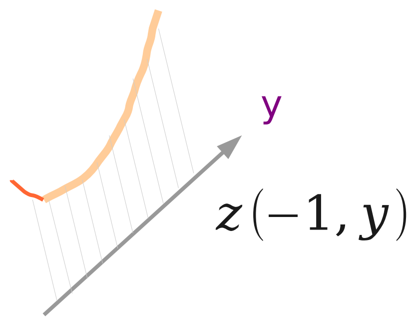
tangent at x=1 of the function z(x,1)

http://en.wikipedia.org/wiki/Partial_derivative

Partial Derivative Examples (2)





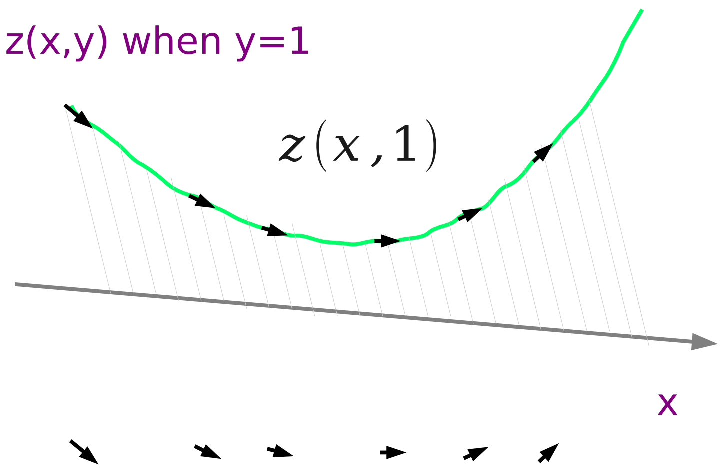


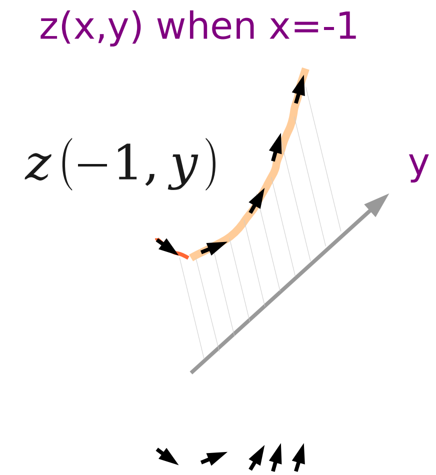
$$z = x^2 + xy + y^2 \Rightarrow \frac{\partial z}{\partial x} = 2x + y$$

$$z = x^2 + xy + y^2 \Rightarrow \frac{\partial z}{\partial y} = x + 2y$$

http://en.wikipedia.org/wiki/Partial_derivative

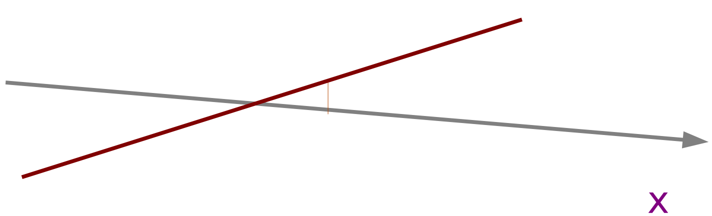
Partial Derivative Examples (3)

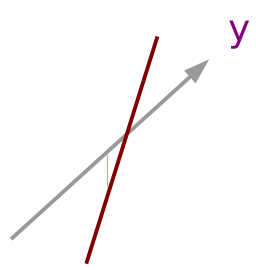




$$\frac{\partial z}{\partial x} = 2x + y \;\Rightarrow\; 2x + 1$$

$$\frac{\partial z}{\partial y} = -1 + 2y \;\Rightarrow\; x + 2y$$





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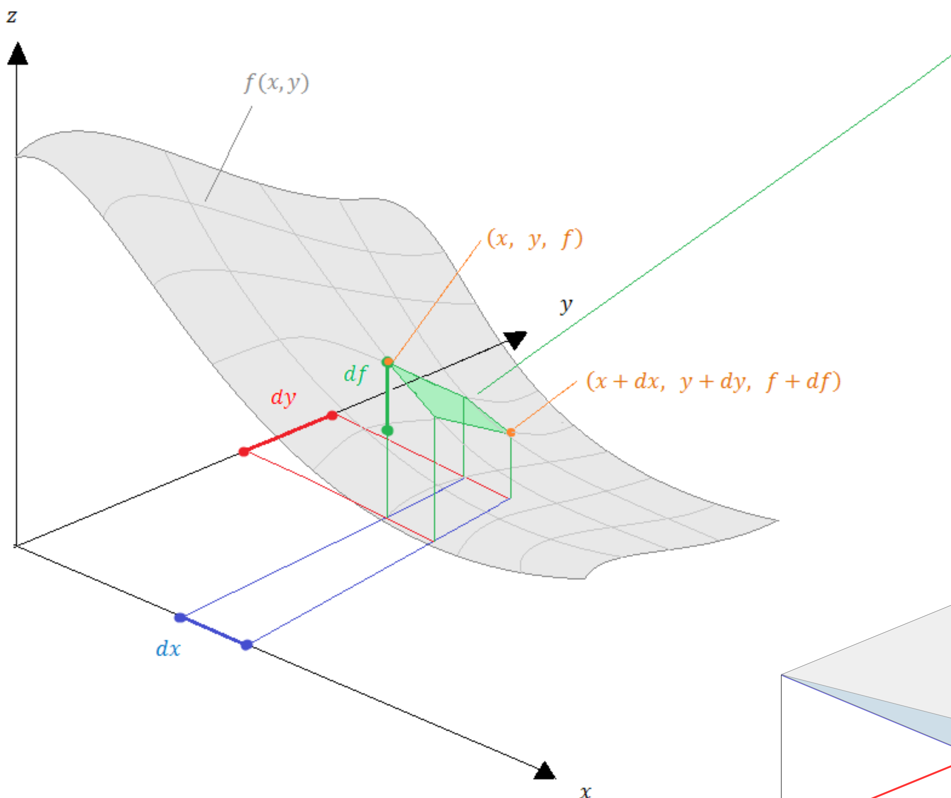
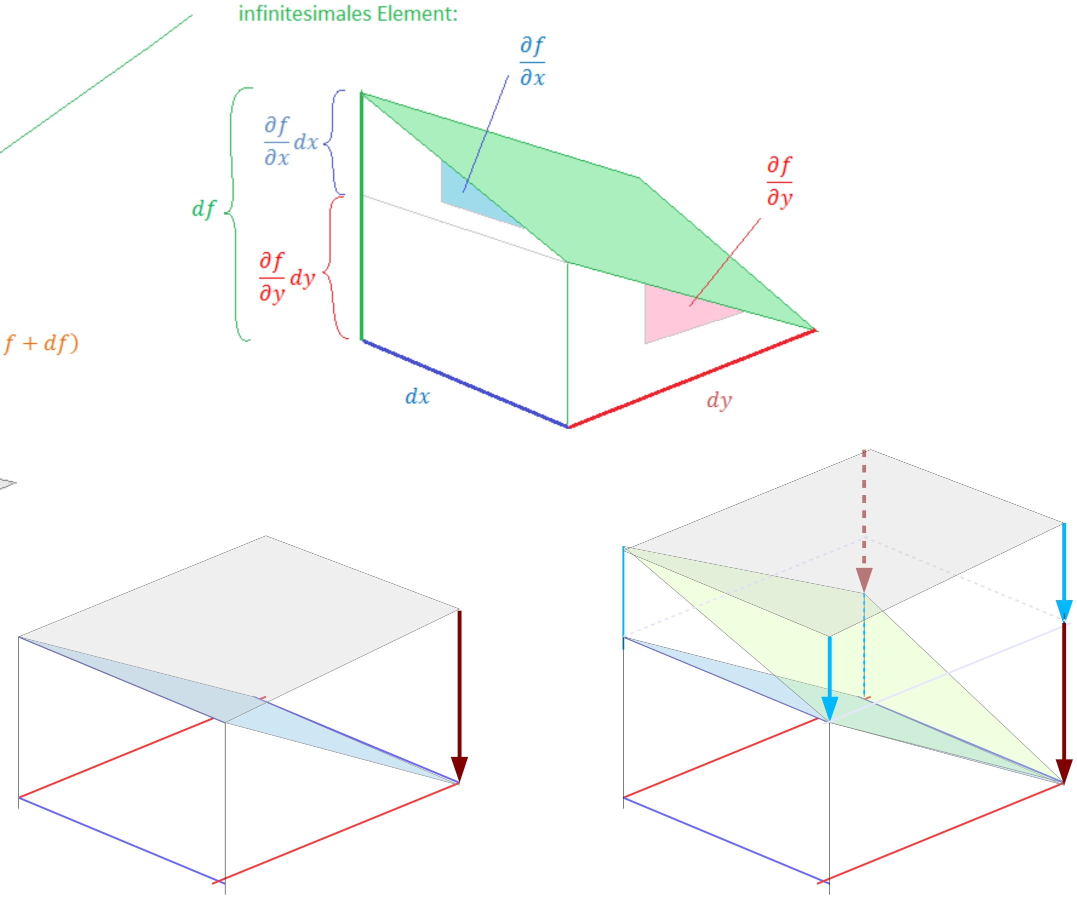
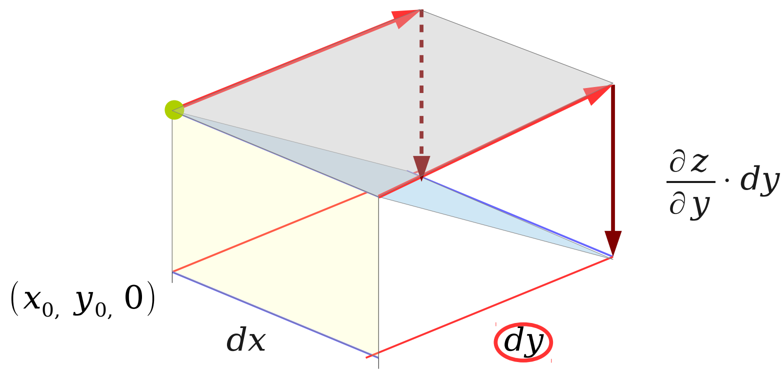
Total Differential





http://de.wikipedia.org/wiki/Totales_Differential
Muhammet Cakir

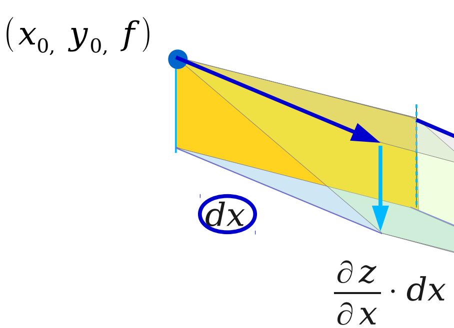
Total Differential

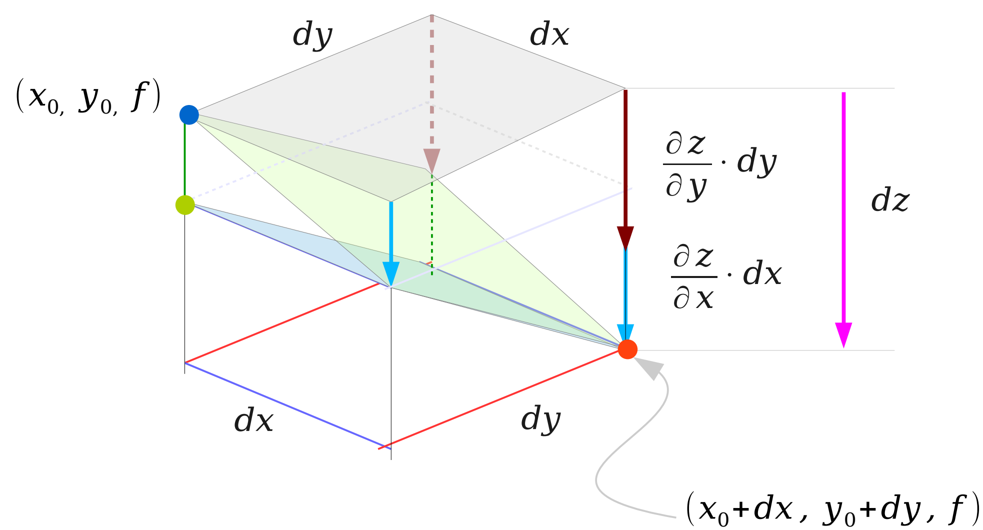


$$z = f(x, y)$$

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

differential, or
total differential





References

[1] http://en.wikipedia.org/
[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[3] E. Kreyszig, “Advanced Engineering Mathematics”
[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”