Hybrid CORDIC 2.A
Sine/Cosine Generator

20170715

Wilson ROM based Sine/Cosine Generation

[24] Fu & Willson Sine / Cosine Generation

ROM-based

For high resolution, ROM size grows exponentially

quater-wave symmetry

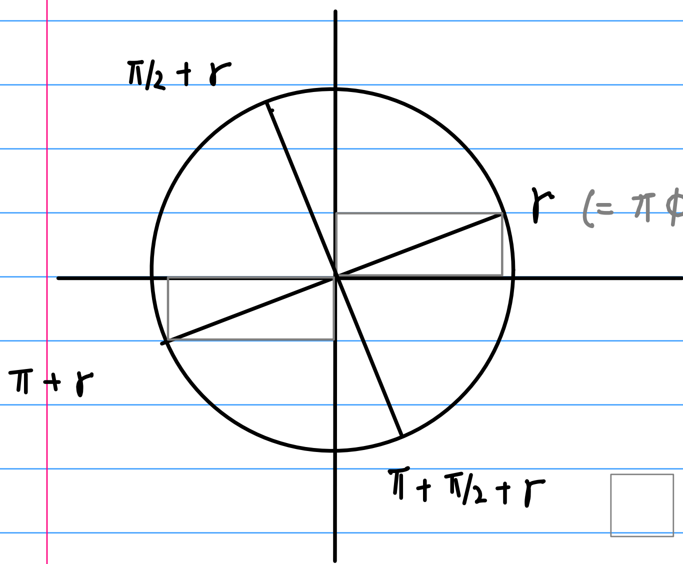
$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

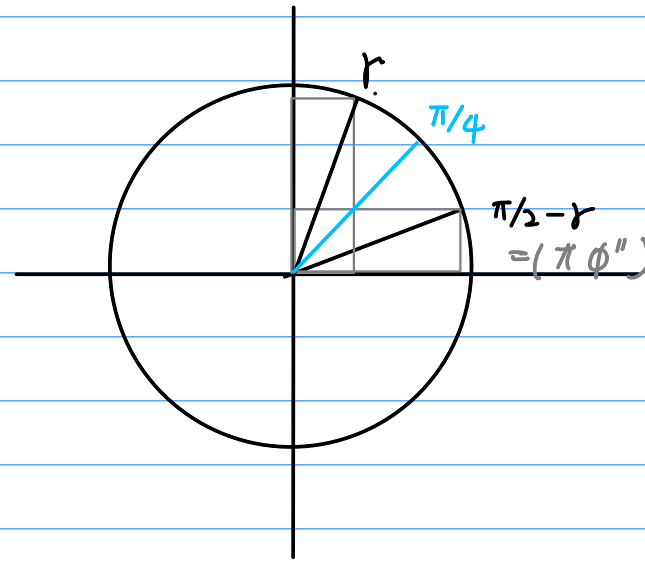
ϕ $[0, 2\pi] \longrightarrow [0, \frac{\pi}{4}]$

conditionally interchanging inputs X_0 & Y_0
Conditionally interchanging and negating outputs X & Y

$$X = X_0 \cos\phi - Y_0 \sin\phi$$
$$Y = Y_0 \cos\phi + X_0 \sin\phi$$

Madisetti VLSI arch





for frequency synthesis

argument: signed normalized by π angle [-1, 1]

binary representation of a radian angle required

$$[-1, 1] \xrightarrow[\phi]{} [0, \pi/4] \xrightarrow[\theta]{} \text{sine/cosine generator}$$

$\pi\phi$ ← (sine/cosine generator)

① a phase accumulator ϕ $[-1, 1]$
② a radian converter $\phi \to \theta$
③ a sine/cosine generator $\sin\theta$, $\cos\theta$
④ an output stage $\sin\theta$, $\cos\theta$ → $\sin\pi\phi$, $\cos\pi\phi$

.

output stage $\quad \sin\theta \rightarrow \sin\pi\phi \qquad [-\pi, +\pi]$

$\cos\theta \rightarrow \cos\pi\phi$

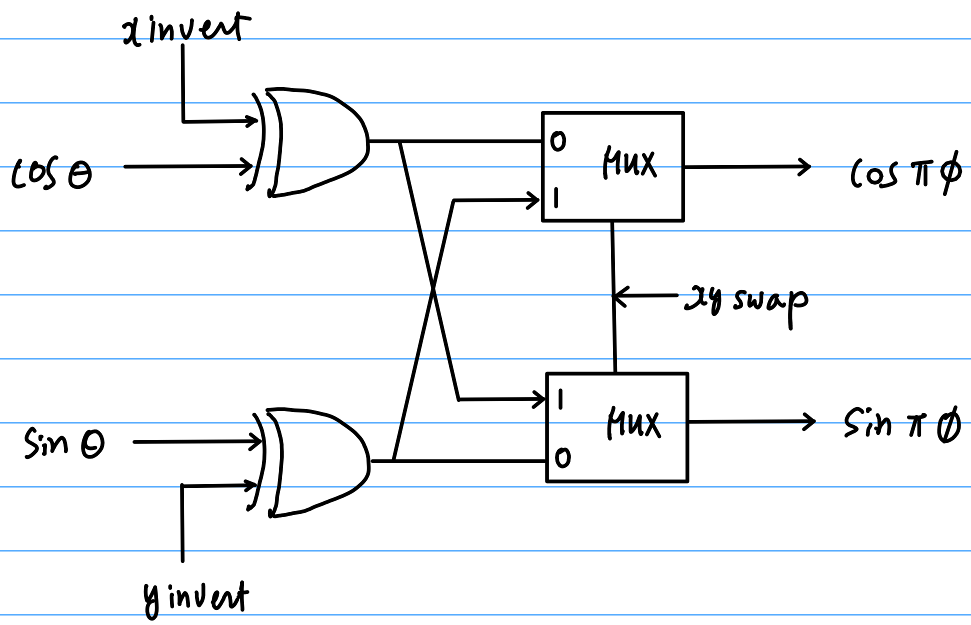
negation / interchange

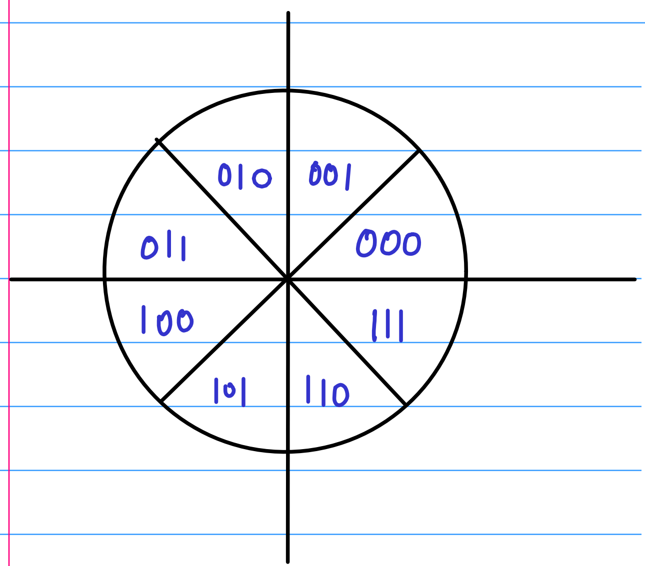
MSB₁, MSB₂, MSB₃ → x invert, y invert, xy swap

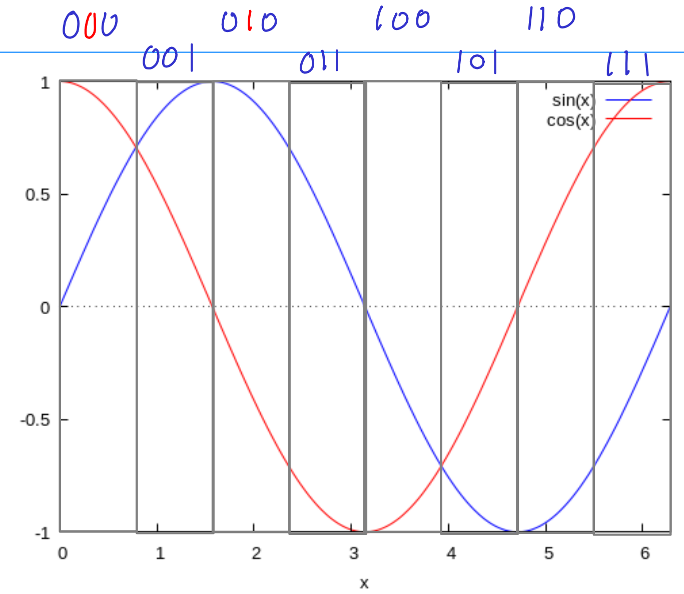
x invert / y invert: the negation of $\cos\theta = X_{N+1}$, $\sin\theta = Y_{N+1}$

xy swap: Interchange

negate before swap

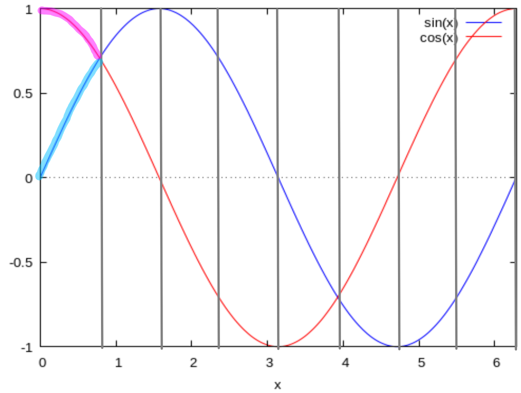




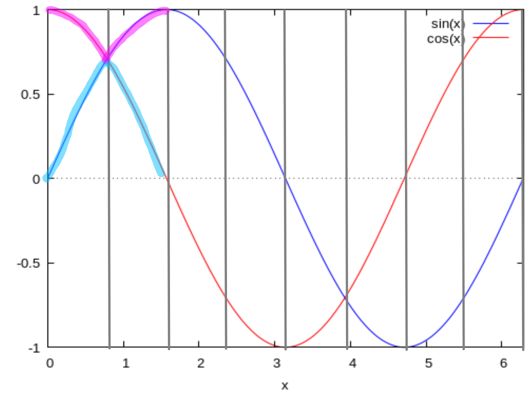


	cos	sin			
	xinv	yinv	swap	$\cos\pi\theta$	$\sin\pi\phi$
000	0	0	0	$\cos\theta$	$\sin\theta$
001	0	0	1	$\sin\theta$	$\cos\theta$
010	0	1	1	$-\sin\theta$	$\cos\theta$
011	1	0	0	$-\cos\theta$	$\sin\theta$
100	1	1	0	$-\cos\theta$	$-\sin\theta$
101	1	1	1	$-\sin\theta$	$-\cos\theta$
110	1	0	1	$\sin\theta$	$-\cos\theta$
111	0	1	0	$\cos\theta$	$-\sin\theta$

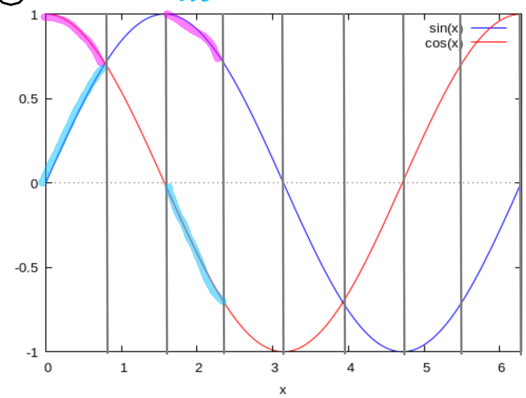
(0) cos θ / sin θ



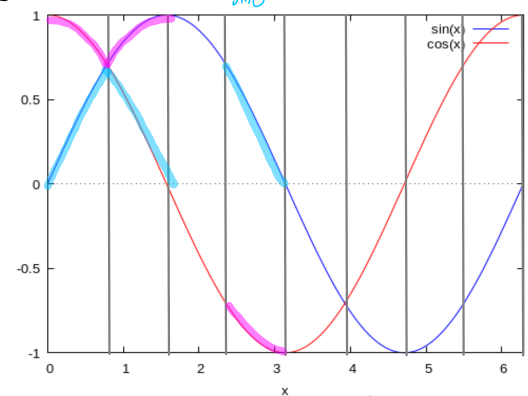
(1) sin θ / cos θ



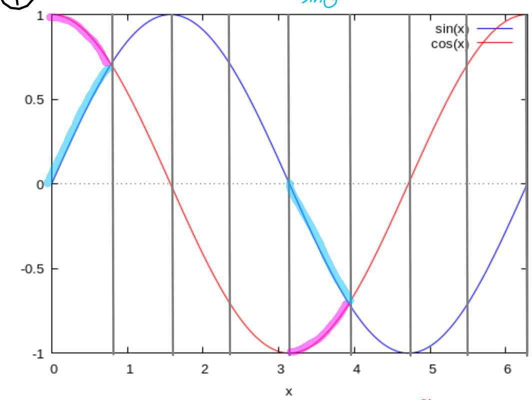
(2) −sin θ / cos θ



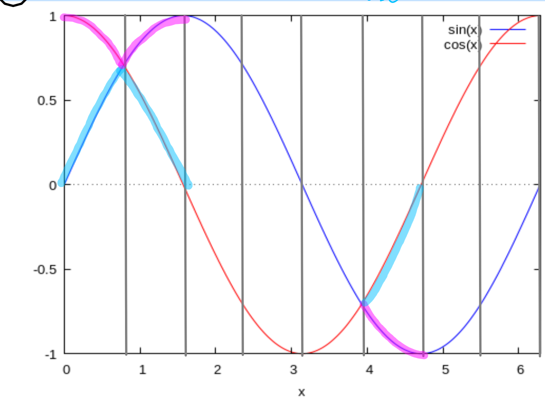
(3) −cos θ / sin θ



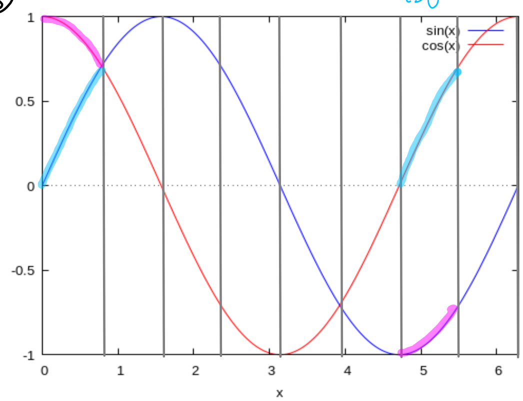
(4) −cos θ / −sin θ



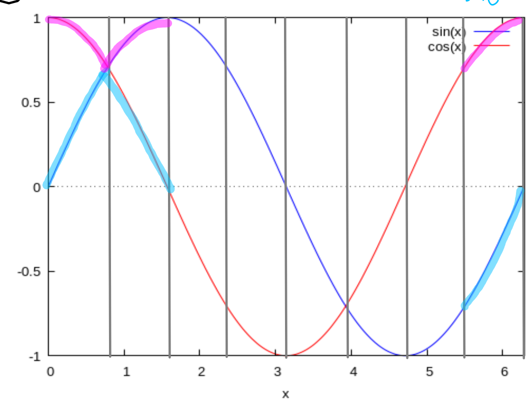
(5) −sin θ / −cos θ



(6) sin θ / −cos θ

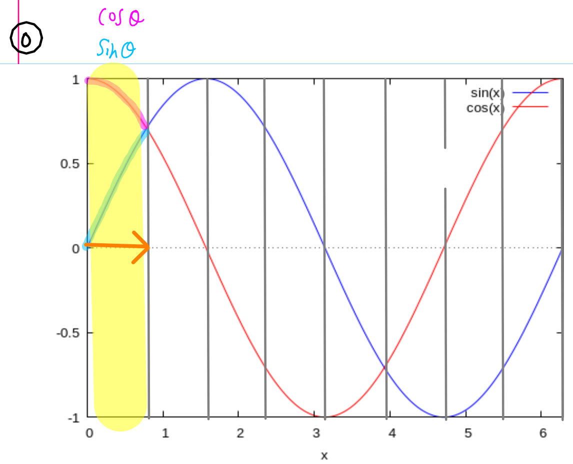


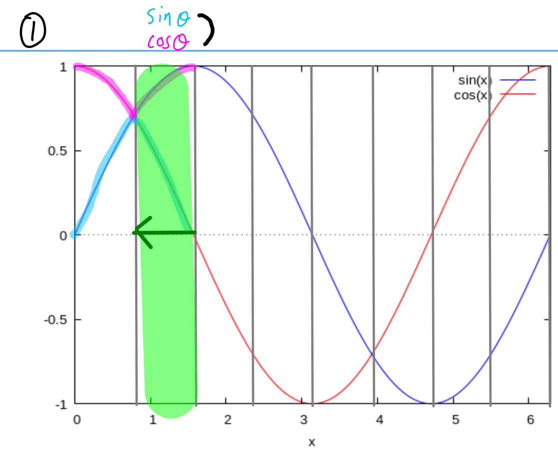
(7) cos θ / −sin θ

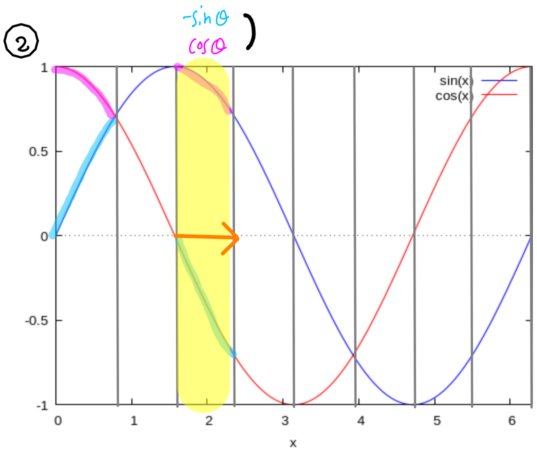


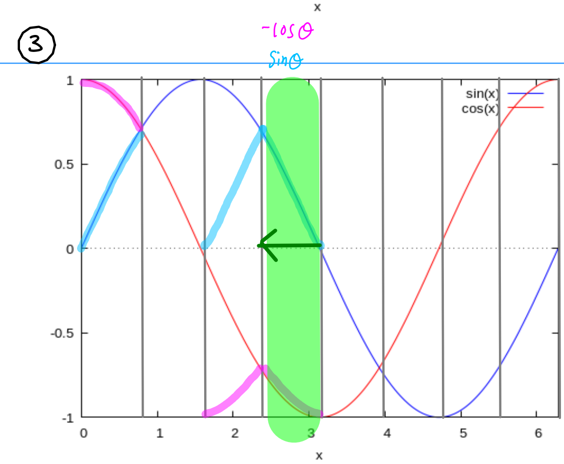
cos φ
sin φ

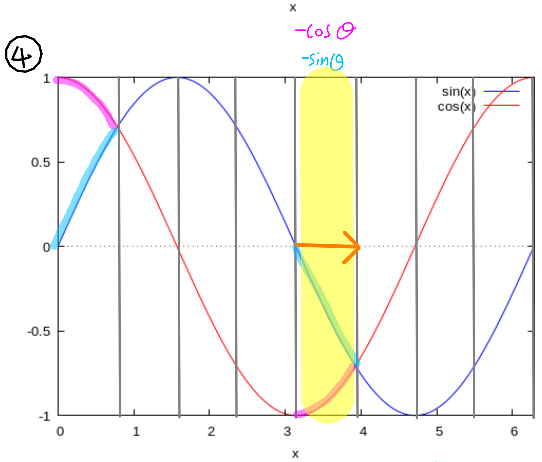
Madisetti & Willson, DDS Freq synthesizer

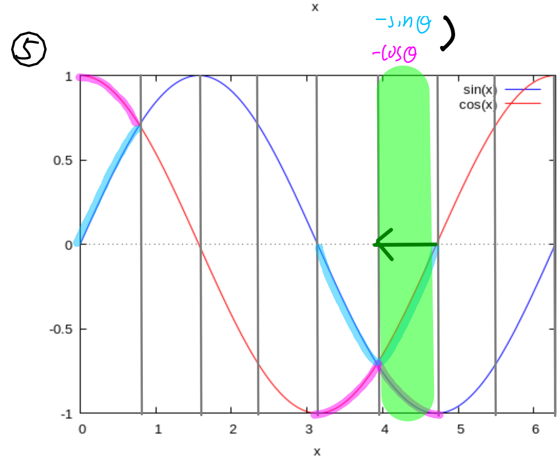


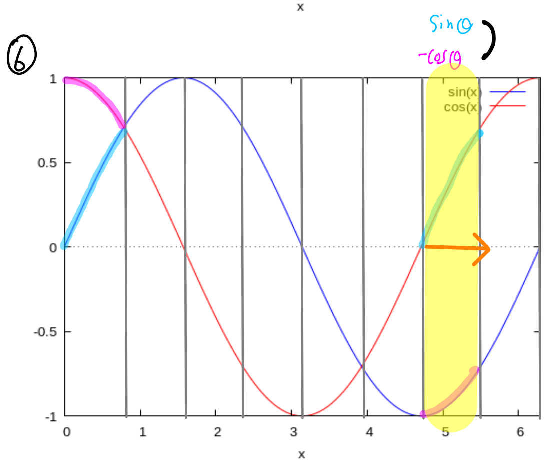


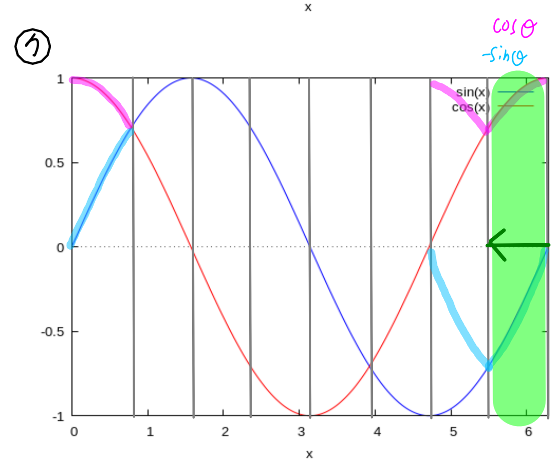




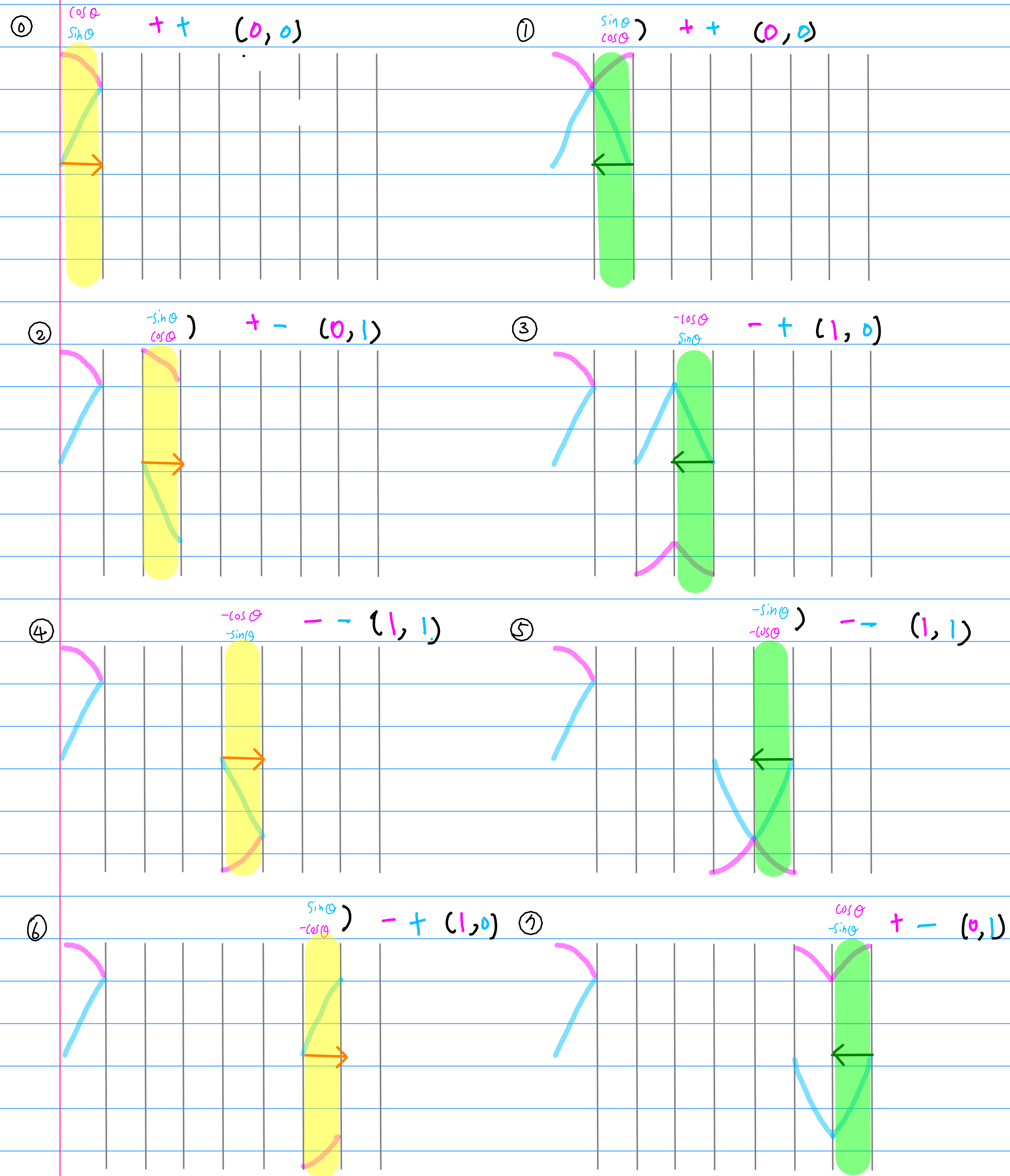








sin φ



⓪ cos θ sin θ + + (0, 0)
① sin θ cos θ) + + (0, 0)
② -sin θ cos θ) + − (0, 1)
③ -cos θ sin θ − + (1, 0)
④ -cos θ -sin θ − − (1, 1)
⑤ -sin θ -cos θ) − − (1, 1)
⑥ sin θ -cos θ) − + (1, 0)
⑦ cos θ -sin θ + − (0, 1)

	Xinv	Yinv	swap	$\cos\pi\phi$	$\sin\pi\phi$
0 0 0	0	0	0	$\cos\theta$	$\sin\theta$
0 0 1	0	0	1	$\sin\theta$	$\cos\theta$
0 1 0	0	1	1	$-\sin\theta$	$\cos\theta$
0 1 1	1	0	0	$-\cos\theta$	$\sin\theta$
1 0 0	1	1	0	$-\cos\theta$	$-\sin\theta$
1 0 1	1	1	1	$-\sin\theta$	$-\cos\theta$
1 1 0	1	0	1	$\sin\theta$	$-\cos\theta$
1 1 1	0	1	0	$\cos\theta$	$-\sin\theta$

0 0	0 0
0 1	1 0
1 1	1 1
1 0	0 1

0	0	0	0
0	1	1	0
1	1	1	1
1	0	0	1

Madisetti & Willson, DDS Freq synthesizer

$$\theta = \sum_{k=1}^{N} b_k \theta_k$$

b_k sign + N bit — (N+1) bit fractional b

$b_k \in \{0, 1\}$

$\theta_k = 2^{-k}$

θ is constrained to be positive $\boxed{b_0 = 0}$

$$\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

$r_k \in \{-1, +1\}$ signed digits

ϕ_0 constant

(+) subrotation by 2^{-k}

2 equal (+) half rotations by 2^{-k-1}

(0) subrotation

2 equal opposite half rotations by $\pm 2^{-k-1}$

Binary Representation

$b_k = 1$: rotation by 2^{-k}

$b_k = 0$: zero rotation

k-th rotation

- fixed rotation by 2^{-k-1}
- { pos rotation ← $b_k = 1$
  neg rotation ← $b_k = 0$

Combining all the fixed rotations

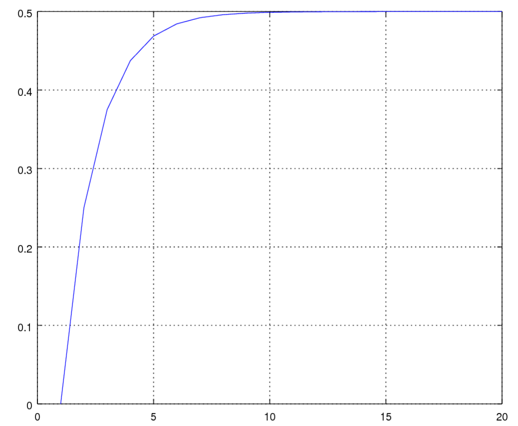
→ initial fixed rotation

	b_1	b_2	b_3		b_N
	2^{-1}	2^{-2}	2^{-3}		2^{-N}
fixed ⇒	$+2^{-2}$	$+2^{-3}$	$+2^{-4}$		$+2^{-N-1}$
	$(b_1=1)$ $+2^{-2}$	$(b_2=1)$ $+2^{-3}$	$(b_3=1)$ $+2^{-4}$		$(b_N=1)$ $+2^{-N-1}$
	$(b_1=0)$ -2^{-2}	$(b_2=0)$ -2^{-3}	$(b_3=0)$ -2^{-4}		$(b_N=0)$ -2^{-N-1}

initial fixed rotation

$$\phi_0 = \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{N+1}}$$

$$= \frac{\frac{1}{2^2}\left(1-\frac{1}{2^N}\right)}{\left(1-\frac{1}{2}\right)} = \frac{1}{2}\left(1-\frac{1}{2^N}\right) = \frac{1}{2} - \frac{1}{2^{N+1}}$$



Signed Digit Recoding

the rotation after recoding

— a fixed initial rotation ϕ_0

a sequence of ⊕/⊖ rotations

$b_k = 1$ $+ 2^{-k-1}$ rotation

$b_k = 0$ $- 2^{-k-1}$ rotation

$r_k = (2b_{k-1} - 1)$

$2 \cdot 1 - 1 = +1$ $b_{k-1} = 1 \longrightarrow r_k = +1$

$2 \cdot 0 - 1 = -1$ $b_{k-1} = 0 \longrightarrow r_k = -1$

The recoding need not be explicitly performed

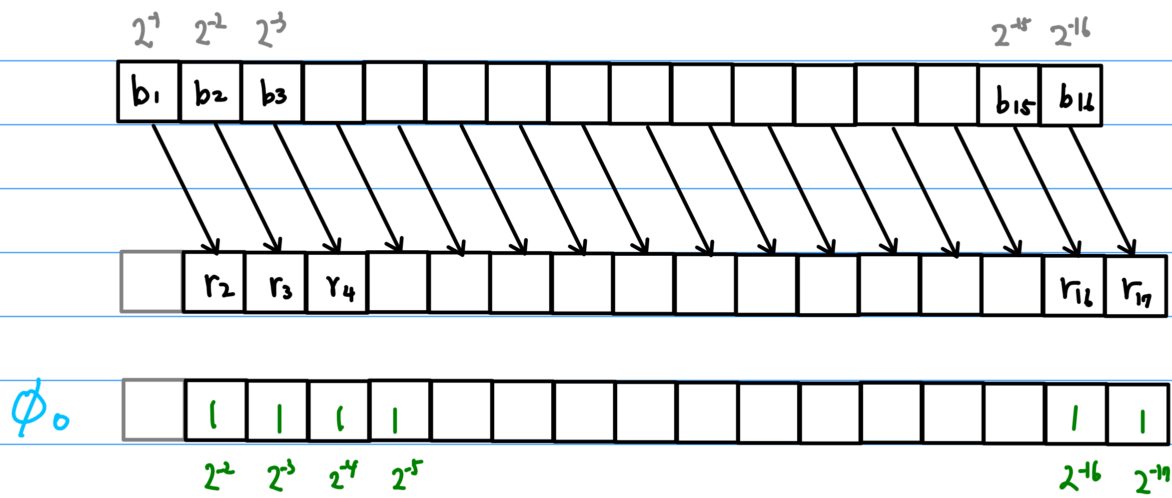
Simply replacing $b_k = 0$ with -1

This recoding maintains

a constant scaling factor K

$$\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

Binary Representation $\{b_k\}$



Signed Digit Recoding $\{r_k\}$

The Scaling K.
The initial rotation ϕ_0

rotation starting point
$$(X_0, Y_0) = (K\cos\phi_0, K\sin\phi)$$

- fixed
- no error buildup

- rotation direction
  immediately obtained from the binary representation
  → no need for comparison

the subangles $\theta_k = 2^{-k}$ used in recoding
the subangles $\theta_k = \tan^{-1}(2^{-k})$ used in CORDIC

$\tan\theta_k$ multipliers used
in the first few subrotation stages
cannot be implemented
as a simple shift-and-add operations

→ ROM implementation

reduced chip area
higher operating speed.

Architecture

1. phase accumulator $\phi \in [-1, +1]$
2. radian converter $\phi \to \theta \in [0, \frac{\pi}{4}]$
3. sine/cosine generator $\sin(\theta)$ $\cos(\theta)$
4. output stage $\sin(\pi\phi)$ $\cos(\pi\phi)$

Overflowing 2's complement accumulator

normalized by π angle ϕ

need radian angle $\theta \in [0, \frac{\pi}{4}]$

$0 < \theta < 1$ rad

N-bit binary representation of θ

controls the direction of subrotation

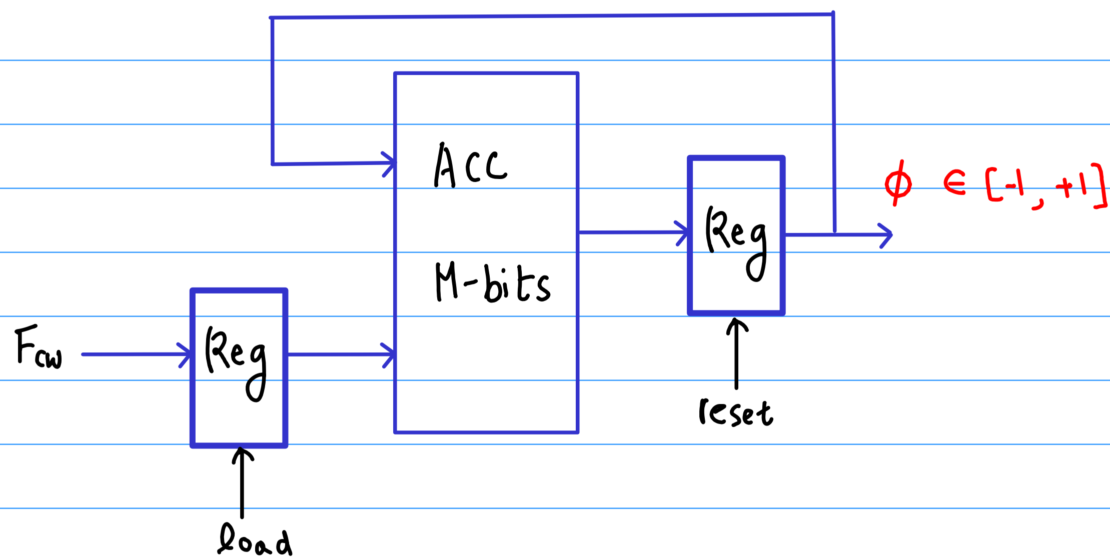
N-bit precision of $\cos\theta$ & $\sin\theta$

Output stage $\theta \rightarrow \pi\phi$

$\sin\theta \rightarrow \sin\pi\phi$

$\cos\theta \rightarrow \cos\pi\phi$

phase accumulator



M-bit adder

repeatedly increments the phase angle

by F_{cw} at each clock cycle

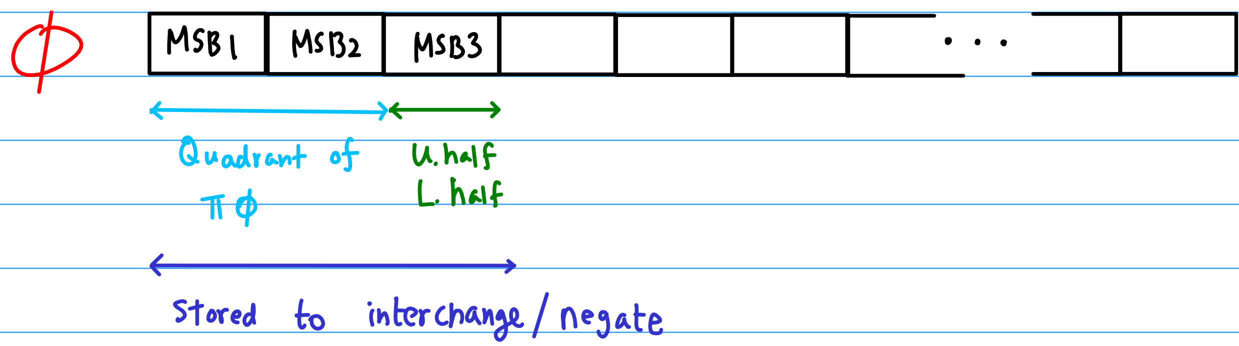
frequency control word

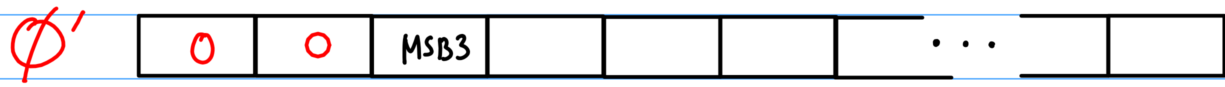
at time n, $\phi = n F_{cw} / 2^M$

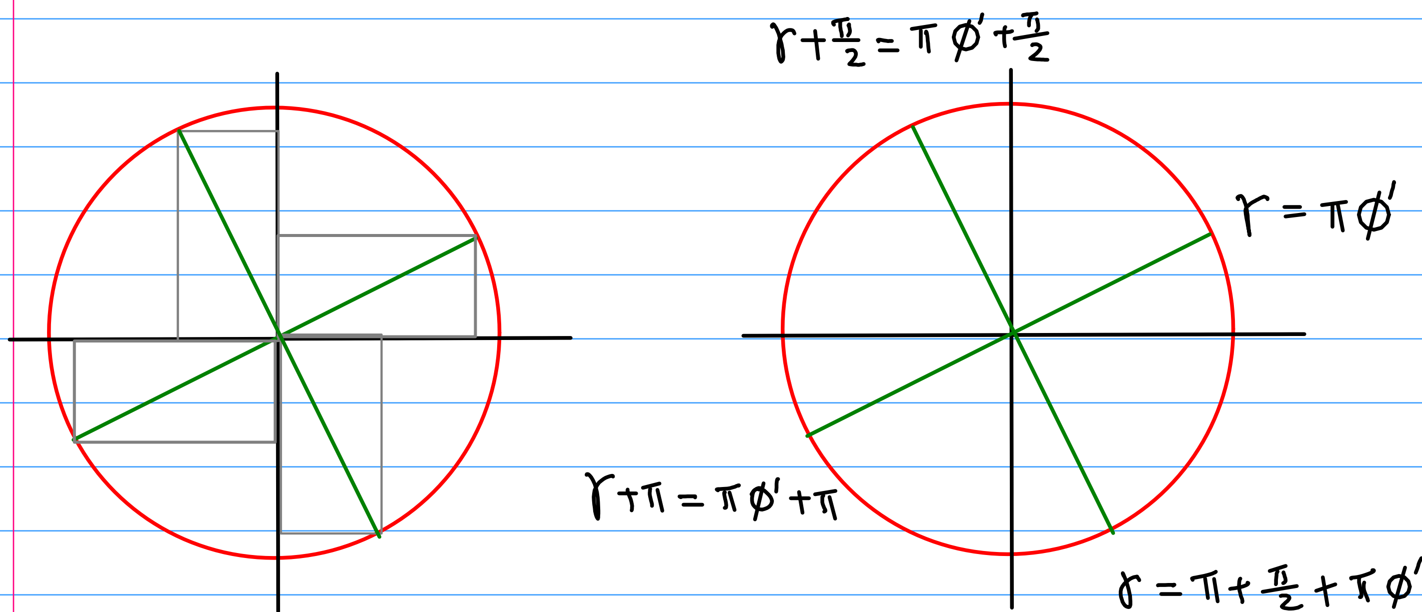
$$\cos\phi = \cos(n F_{cw} / 2^M)$$
$$\sin\phi = \sin(n F_{cw} / 2^M)$$

Radian Converter

normalized angle ϕ







$\phi \rightarrow \phi' \rightarrow$

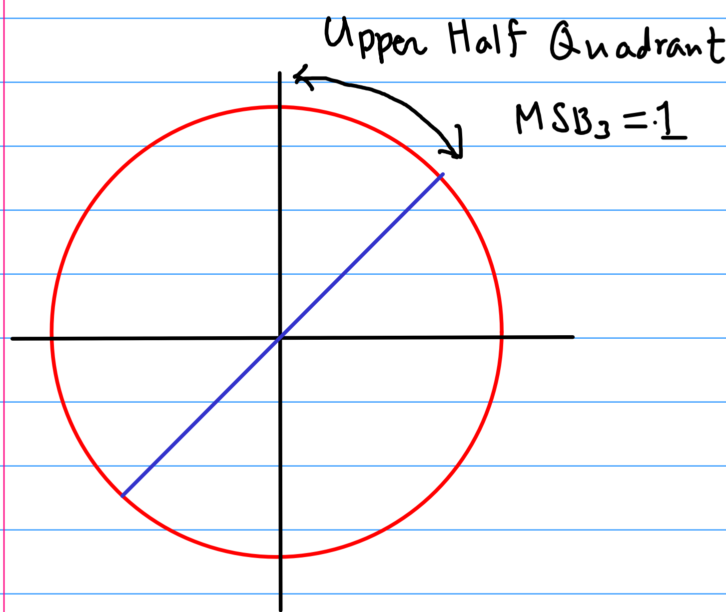
↑ 1st Quad

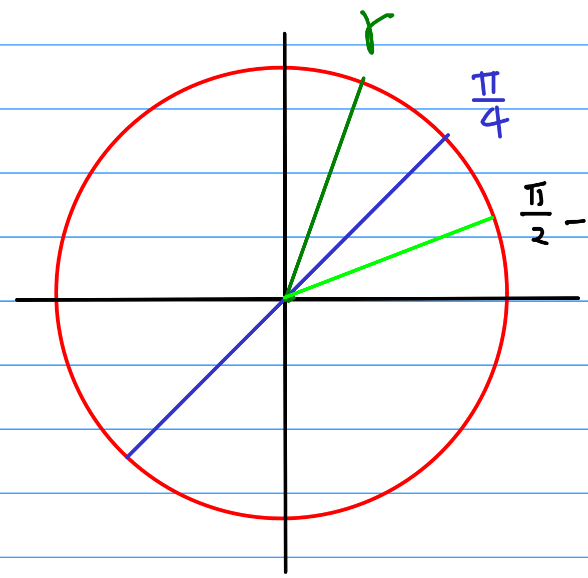
$$\pi\phi' + 0 \cdot \frac{\pi}{2} \quad 00$$
$$\pi\phi' + 1 \cdot \frac{\pi}{2} \quad 01$$
$$\pi\phi' + 2 \cdot \frac{\pi}{2} \quad 10$$
$$\pi\phi' + 3 \cdot \frac{\pi}{2} \quad 11$$

1st Quadrant

ϕ'

0	0	MSB3					...		



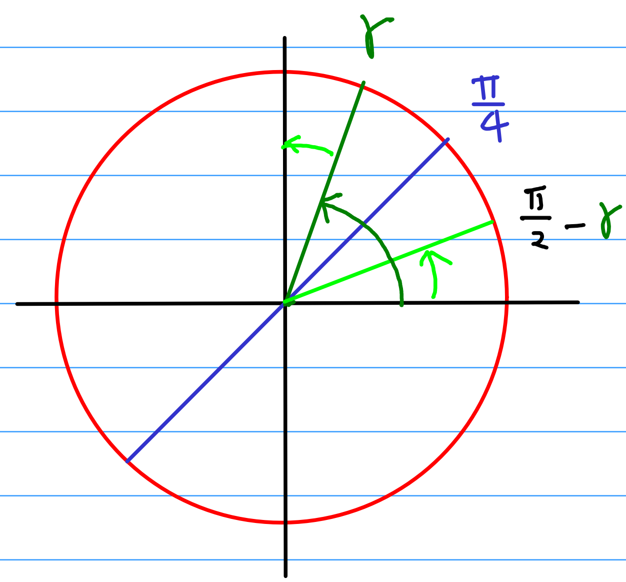


$\gamma > \frac{\pi}{4}$: Upper Half ($MSB_3 = 1$)

$\gamma < \frac{\pi}{4}$: Lower Half ($MSB_3 = 0$)

$$\cos\gamma = \sin\left(\frac{\pi}{2} - \gamma\right)$$

$$\sin\gamma = \cos\left(\frac{\pi}{2} - \gamma\right)$$



$\gamma > \frac{\pi}{4}$ $\quad \frac{\pi}{2} - \gamma < \frac{\pi}{4}$.

ϕ'

0	0	MSB3					...		

MSB3 = 1 $\quad \phi' > \frac{\pi}{4}$

$$\phi'' = \frac{\pi}{2} - \phi'$$

0.5

1	0	0					...		

$\ominus$ ϕ'

0	0	MSB3					...		

$$\begin{cases} MSB3 = 0 & \phi'' = \phi' \\ MSB_3 = 1 & \phi'' = 0.5 - \phi' \end{cases}$$

$\theta = \pi \phi''$ (Hardwired Multiplier)

$$0 < \theta < \frac{\pi}{4}$$

$$\phi \longrightarrow \phi' \longrightarrow \phi''$$

1st Quad $\quad$ Lower Half

Sine / Cosine Generator

Subrotation

$$X_{k+1} = X_k - (r_k \tan\theta_k) Y_k$$

$$Y_{k+1} = Y_k + (r_k \tan\theta_k) X_k$$

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

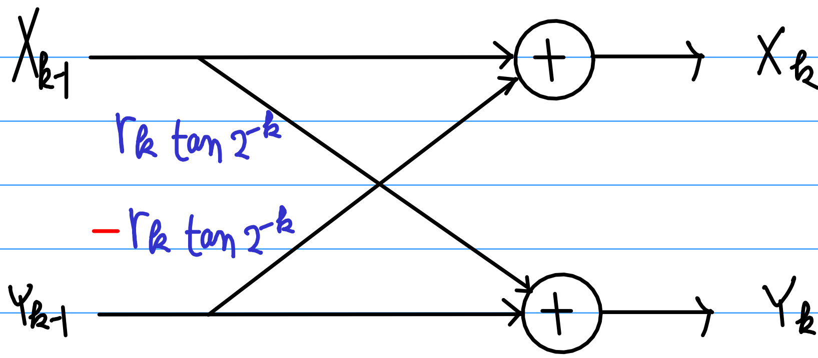
$$= \cos\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

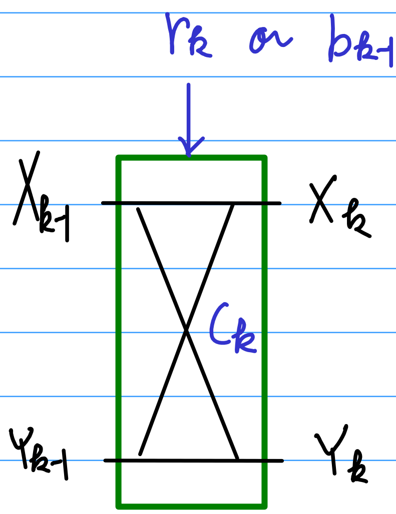
$$\theta = \sigma_0\theta_0 + \sigma_1\theta_1 + \cdots + \sigma_N\theta_N$$

$$\sigma_k = \{-1, 0, +1\}$$

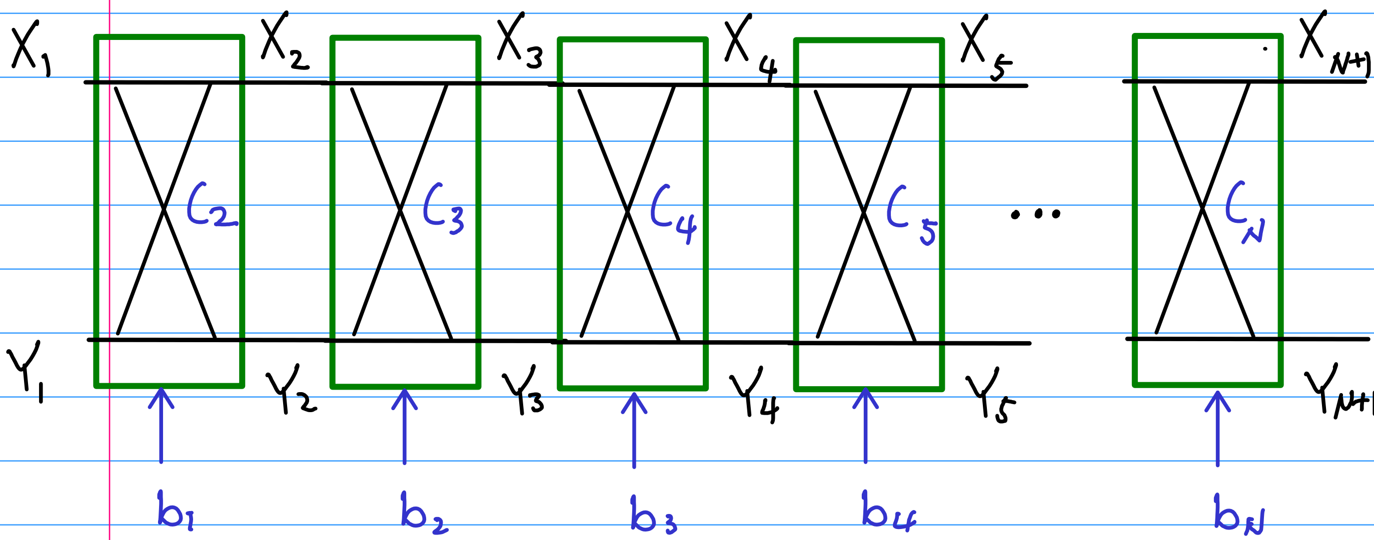
$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = K \begin{bmatrix} 1 & -\tan\sigma_N\theta_N \\ \tan\sigma_N\theta_N & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan\sigma_0\theta_0 \\ \tan\sigma_0\theta_0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$K = \cos\sigma_0\theta_0 \cdot \cos\sigma_1\theta_1 \cdots \cos\sigma_N\theta_N$$





Butterfly



$c_2 = \tan\left(\frac{1}{2^2}\right)$ $\quad c_3 = \tan\left(\frac{1}{2^3}\right)$ $\quad c_4 = \tan\left(\frac{1}{2^4}\right)$ $\quad c_5 = \tan\left(\frac{1}{2^5}\right)$

$K\cos\phi_0 \rightarrow X_1$ $\qquad X_{N+1} \rightarrow \cos\theta$

$K\sin\phi_0 \rightarrow Y_1$ $\qquad Y_{N+1} \rightarrow \sin\theta$

$\theta \rightarrow \{b_1, b_2, \dots, b_N\}$

the initial (X_0, Y_0) always the same

merge the first B/3 butterflies

→ $2^{B/3}$ words ROM implementation

→ no need $\tan \theta_k$ multipliers

→ $\{b_1, b_2, \cdots, b_{B/3}\}$ ⇒ address

accesses

$$\cos\left(\phi_0 + \sum_{k=1}^{B/3} b_k 2^{-k+1}\right)$$

$$\sin\left(\phi_0 + \sum_{k=1}^{B/3} b_k 2^{-k+1}\right)$$

Lower Half of the 1st Quadrant

- all positive X_k & Y_k
- no need sign extension
- reduce the loads
- high speed

Merging Butterflies

merge m final butterflies

$$\begin{pmatrix} X_k \\ Y_k \end{pmatrix} \longrightarrow \begin{pmatrix} X_{k+m} \\ Y_{k+m} \end{pmatrix} \quad \text{directly}$$

$$X_{k+m} = X_k - Y_k \sum_{i=k}^{k+m-1} r_i \tan 2^{-i}$$

$$Y_{k+m} = Y_k + X_k \sum_{i=k}^{k+m-1} r_i \tan 2^{-i}$$

valid merging $\quad k \geq (B-1)/2$

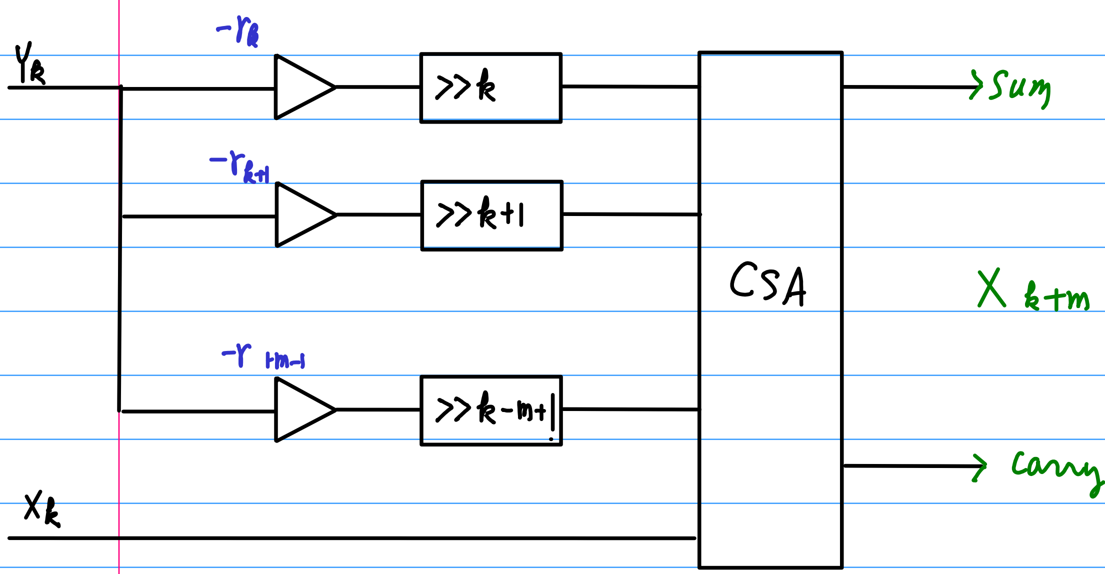
$\tan(2^{-i}) = 2^{-i} \quad k \geq B/3$

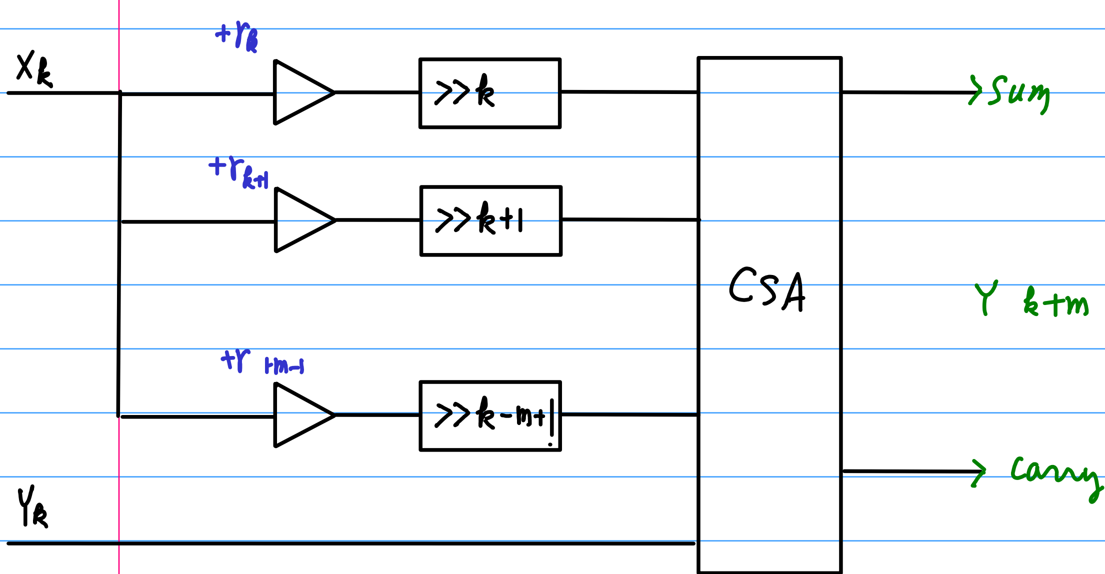
look ahead by m

the individual terms in the summation can be computed independently and summed in parallel

$$X_{k+m} = X_k - Y_k \sum_{i=k}^{k+m-1} r_i \tan 2^{-i}$$

$$Y_{k+m} = Y_k + X_k \sum_{i=k}^{k+m-1} r_i \tan 2^{-i}$$





+ reduced latency
+ reduced routing
+ only the half for a single-ended system.

Output Stage

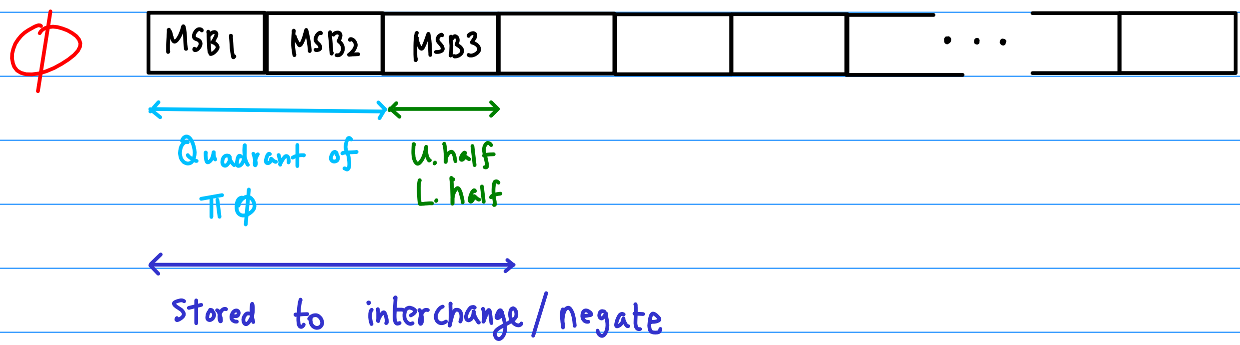
$\sin \theta$, $\cos \theta$ $\rightarrow$ $\sin \pi\phi$, $\cos \pi\phi$

$\theta \in [0, \frac{\pi}{4}] \rightarrow \phi \in [-1, +1]$

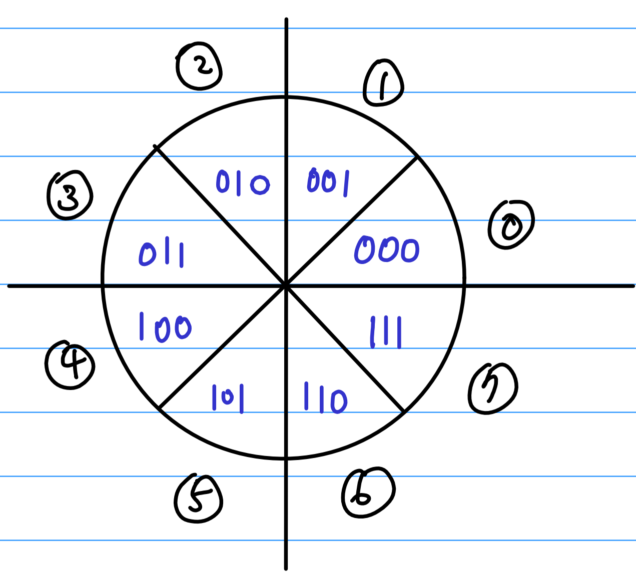
- negation
- interchange

* n-gation before interchange

normalized angle ϕ



MSB of ϕ	ϕ	X_{inv}	Y_{inv}	Swap	$\cos\pi\phi$	$\sin\pi\phi$
0 0 0	⓪	0	0	0	$\cos\theta$	$\sin\theta$
0 0 1	①	0	0	1	$\sin\theta$	$\cos\theta$
0 1 0	②	0	1	1	$-\sin\theta$	$\cos\theta$
0 1 1	③	1	0	0	$-\cos\theta$	$\sin\theta$
1 0 0	④	1	1	0	$-\cos\theta$	$-\sin\theta$
1 0 1	⑤	1	1	1	$-\sin\theta$	$-\cos\theta$
1 1 0	⑥	1	0	1	$\sin\theta$	$-\cos\theta$
1 1 1	⑦	0	1	0	$\cos\theta$	$-\sin\theta$



IC Implementation

clock : 100 MHz
acc : 36-bit (22-bit + 14-bit)
precision : 16-bit
advantage over traditional ROM lookup table approach

accumulator :: 36-bit = 22-bit + 14 bit
carry select adder

Speed & layout consideration

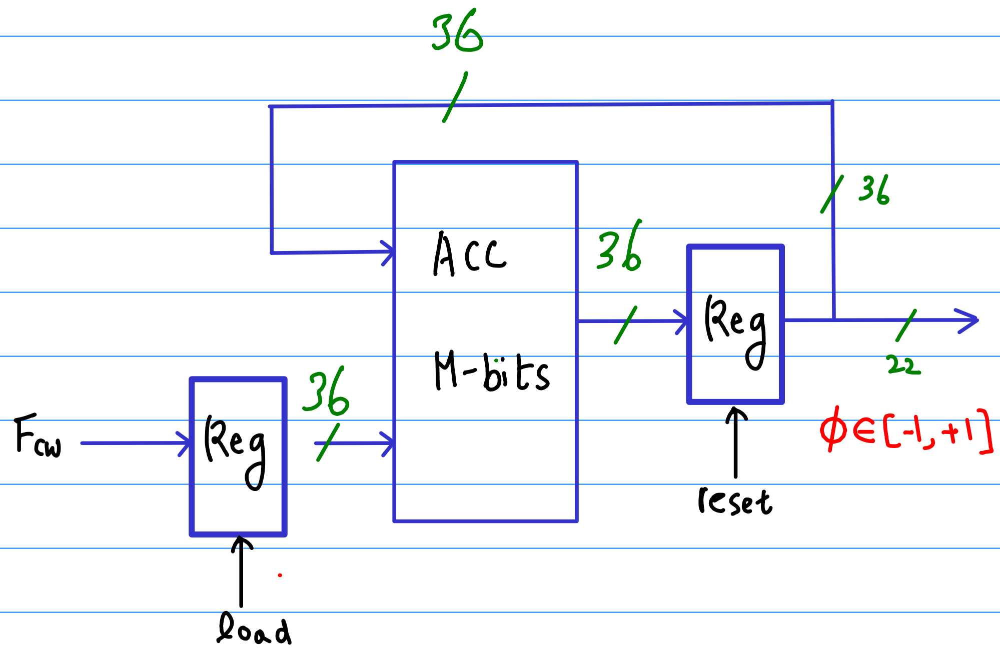
36-bit output → truncated to 22-bit

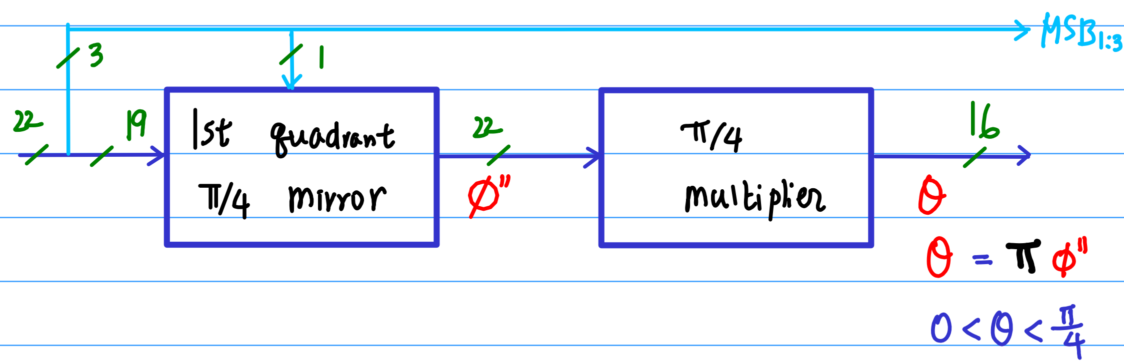
22-bit radian converter
$\pi/4$ multiplier
SDFR ≥ 100 dBc

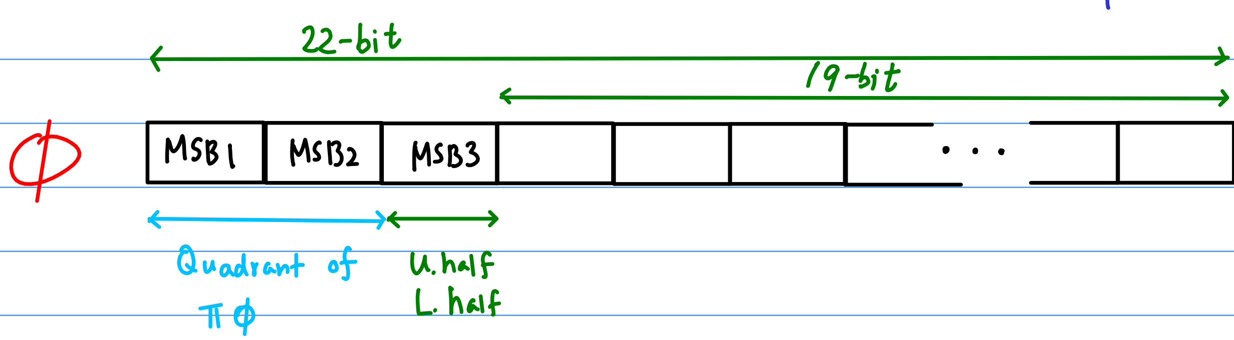
19-bit ROM address
for the similar performance
⇒ 2^{19} words HUGE!

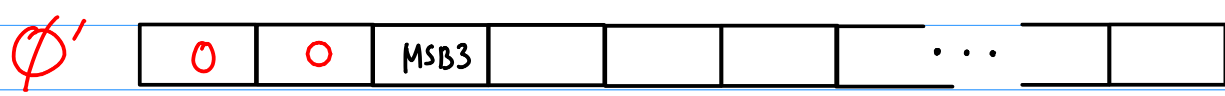
Coarse / fine ROM arch

[2] H. T. Nicholas and H. Samueli, "A 150 MHz direct digital frequency synthesizer in 1.25-μm CMOS with −90 dBc spurious performance," *IEEE J. Solid-State Circuits,* vol. 26, pp. 1959–1969, Dec. 1991.









$\gamma > \frac{\pi}{4}$: Upper Half $(MSB_3 = 1)$ $\quad \phi'' = \phi'$

$\gamma < \frac{\pi}{4}$: Lower Half $(MSB_3 = 0)$ $\quad \phi'' = 0.5 - \phi'$

radian converter $\theta = \pi \phi''$

all internal angle
~ represented as
fractional binary 2's complement numbers

$\theta = (\pi/4)(4\phi'')$

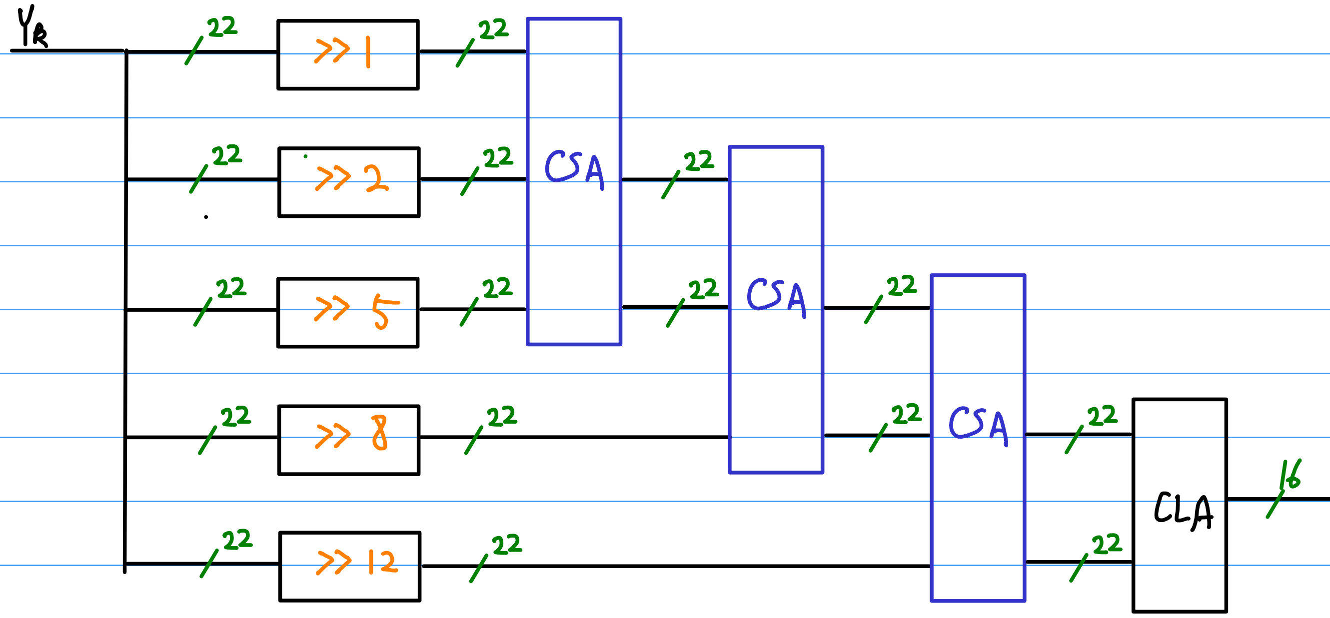
$(\pi/4) = 2^{-1} + 2^{-2} + 2^{-5} + 2^{-8} + 2^{-12}$

1	2	3	4	5	6	7	8	9	10	11	12
1	1	0	0	1	0	0	1	0	0	0	1

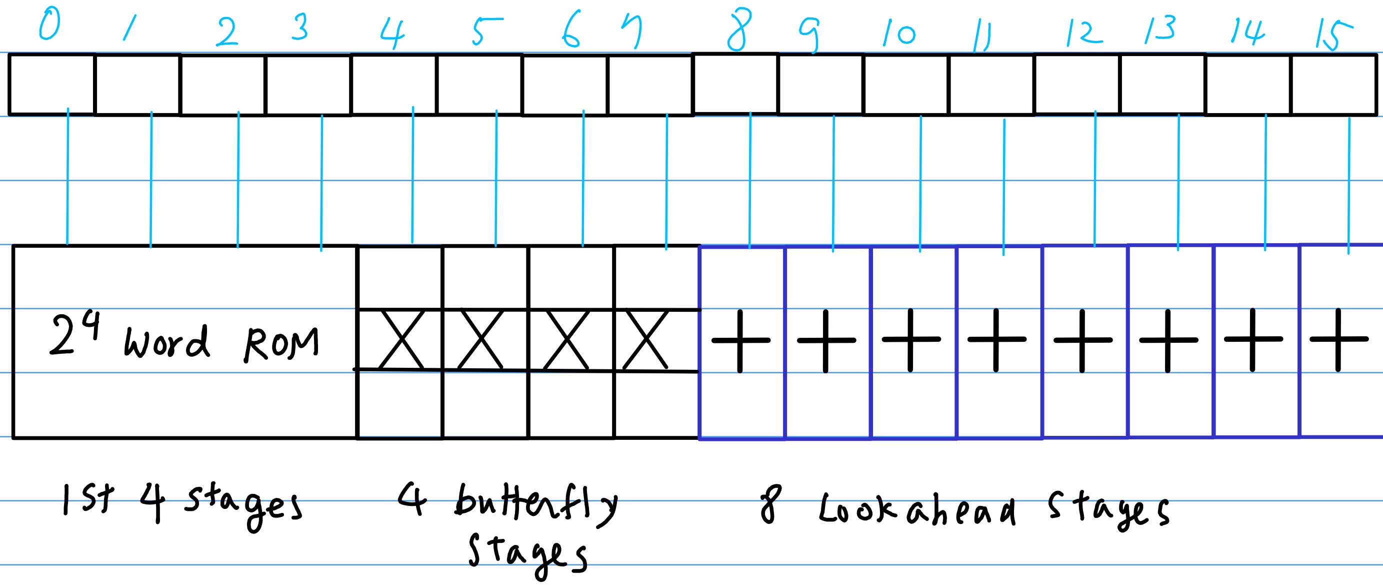
$2^{-1} = 0.5$
$2^{-2} = 0.25$
$2^{-5} = 0.3125$
$2^{-8} = 0.00390625$
$2^{-12} = 0.000244141$

$\pi/4 = 0.785398163 = 0.785400391$

only 1st 5 partial products



Y_R
22
>> 1
>> 2
>> 5
>> 8
>> 12
CSA
CSA
CSA
CLA
16


0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
2^4 Word ROM
X
X
X
X
+
+
+
+
+
+
+
+
1st 4 stages
4 butterfly stages
8 Lookahead stages