

Angle Recoding CORDIC

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Extended EAS (EEAS) - Wu

more flexible way of decomposing the rotation angle

better

the number of iterations
the error performance

$$S_{EAS} = \{ (\sigma \cdot \tan^{-1}(2^{-r})) : \sigma \in \{+1, 0, -1\}, r \in \{1, 2, \dots, n-1\} \}$$

$$S_{EEAS} = \{ (\sigma_1 \cdot \tan^{-1}(2^{-r_1}) + \sigma_2 \cdot \tan^{-1}(2^{-r_2})) : \\ \sigma_1, \sigma_2 \in \{+1, 0, -1\}, r_1, r_2 \in \{1, 2, \dots, n-1\} \}$$

The pseudo-rotation
for i -th micro rotations

$$\begin{aligned}x_{i+1} &= x_i - [\sigma_1(i) \cdot 2^{-r_1(i)} + \sigma_2(i) \cdot 2^{-r_2(i)}] y_i \\y_{i+1} &= y_i + [\sigma_1(i) \cdot 2^{-r_1(i)} + \sigma_2(i) \cdot 2^{-r_2(i)}] x_i\end{aligned}$$

The pseudo-rotated vector $[x_{R_m}, y_{R_m}]$
after R_m (the required number of micro-rotations)

needs to be scaled by a factor $K = \prod K_i$

$$K_i = \left[1 + (\sigma_1(i) \cdot 2^{-r_1(i)} + \sigma_2(i) \cdot 2^{-r_2(i)})^2 \right]^{-\frac{1}{2}}$$

$$\begin{aligned}\tilde{x}_{i+1} &= \tilde{x}_i - [k_1(i) \cdot 2^{-s_1(i)} + k_2(i) \cdot 2^{-s_2(i)}] \tilde{y}_i \\ \tilde{y}_{i+1} &= \tilde{y}_i + [k_1(i) \cdot 2^{-s_1(i)} + k_2(i) \cdot 2^{-s_2(i)}] \tilde{x}_i\end{aligned}$$

$$\tilde{x}_0 = x_{R_m}$$

$$\tilde{y}_0 = y_{R_m}$$

$$k_1, k_2 \in \{-1, 0, 1\}$$

$$s_1, s_2 \in \{1, 2, \dots, n-1\}$$

- [21] C.-S. Wu, A.-Y. Wu, and C.-H. Lin, "A high-performance/low-latency vector rotational CORDIC architecture based on extended elementary angle set and trellis-based searching schemes," *IEEE Trans. Circuits Syst. II: Anal. Digital Signal Process.*, vol. 50, no. 9, pp. 589–601, Sep. 2003.

A Unified View for Vector Rotational CORDIC Algorithms and Architectures Based on Angle Quantization Approach

An-Yeu Wu and Cheng-Shing Wu

AR : to approximate θ
with the combination
of selected angle elements
from a pre-defined EAS
(Elementary Angle Set)

EAS : all possible values of $\theta(j)$

$$\text{EAS } \hat{S}_1 = \{ \tan^{-1}(\alpha^* \cdot 2^{-s^*}) : \alpha^* \in \{-1, 0, +1\}, \\ s^* \in \{0, 1, \dots, N-1\} \}$$

EAS \hat{S}_1 consists of $\tan^{-1}(\text{Single signed power of two})$
 $\tan^{-1}(\text{Single SPT})$
 $\tan^{-1}(\alpha^* \cdot 2^{-s^*})$

SPT-based digital filter design

to increase the coefficient resolution

→ employ more SPT terms to represent filter coefficients

[12] H. Samuelli, "An improved search algorithm for the design of multiplierless FIR filters with power-of-two coefficients," *IEEE Trans. Circuits Syst.*, vol. 36, pp. 1044–1047, July 1989.

[13] Y. C. Lim, R. Yang, D. Li, and J. Song, "Signed power-of-two term allocation scheme for the design of digital filters," *IEEE Trans. Circuits Syst. II*, vol. 46, pp. 577–584, May 1999.

EAS \hat{S}_1 consists of $\tan^{-1}(\text{Single signed power of two})$
 $\tan^{-1}(\text{Single SPT})$
 $\tan^{-1}(\alpha^* \cdot 2^{-s^*})$

EAS \hat{S}_2 consists of $\tan^{-1}(\text{two signed power of two})$
 $\tan^{-1}(\text{two SPT})$
 $\tan^{-1}(\alpha_0^* \cdot 2^{-s_0^*} + \alpha_1^* \cdot 2^{-s_1^*})$

Two Signed - Power - of - Two terms

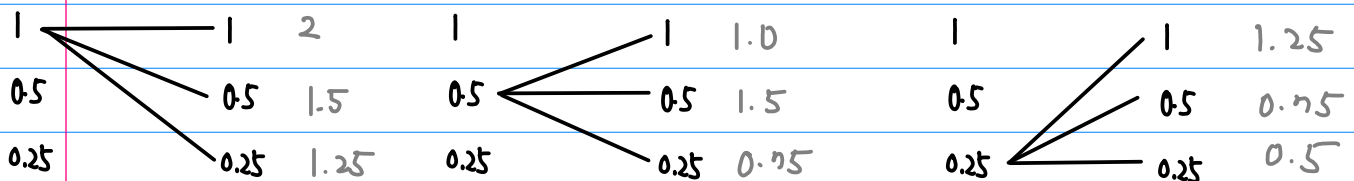
$$S_2 = \left\{ \tan^{-1} \left(\alpha_0^* \cdot 2^{-s_0^*} + \alpha_1^* \cdot 2^{-s_1^*} \right) : \right. \\ \left. \begin{array}{l} \alpha_0^*, \alpha_1^* \in \{-1, 0, +1\} \\ s_0^*, s_1^* \in \{0, 1, \dots, W-1\} \end{array} \right\}$$

S_1

1	$1 = 2^{-0}$	$\tan^{-1}(2^{-0})$
0.5	$\frac{1}{2} = 2^{-1}$	$\tan^{-1}(2^{-1})$
0.25	$\frac{1}{4} = 2^{-2}$	$\tan^{-1}(2^{-2})$

S_2

2	$1+1 = 2^{-0} + 2^{-0}$	$\pm \tan^{-1}(2^{-0} + 2^{-0})$
1.5	$1+\frac{1}{2} = 2^{-0} + 2^{-1}$	$\pm \tan^{-1}(2^{-0} + 2^{-1})$
1.25	$1+\frac{1}{4} = 2^{-0} + 2^{-2}$	$\pm \tan^{-1}(2^{-0} + 2^{-2})$
1.0	$1 = 2^{-0}$	$\pm \tan^{-1}(2^{-0})$
0.75	$\frac{1}{2}+\frac{1}{4} = 2^{-1} + 2^{-2}$	$\pm \tan^{-1}(2^{-1} + 2^{-2})$
0.5	$\frac{1}{2} = 2^{-1}$	$\pm \tan^{-1}(2^{-1})$
0.25	$\frac{1}{4} = 2^{-2}$	$\pm \tan^{-1}(2^{-2})$



$$2^{-0}, 2^{-1}, 2^{-2}$$

$$\{0, 1, 2\} = \{0, 1, w-1\}$$

$$w=3$$

$$s_0^*, s_i^* \in \{0, 1, 2\}$$

$$2^{s_0^*}, 2^{s_i^*} \in \{2^{-0}, 2^{-1}, 2^{-2}\}$$

as the wordsize w increases,
the size of the set S_2 increases exponentially

$$\theta_i = \tan^{-1} (\alpha_0(j) \cdot 2^{-s_0(j)} + \alpha_1(j) \cdot 2^{-s_1(j)})$$

R_m : the number of the subangle N_A

$$S_2 = \{ \theta_i \mid i = 0, 1, \dots, R_m \}$$

$$S_2 = \left\{ \tan^{-1} (\alpha_0^* \cdot 2^{-s_0^*} + \alpha_1^* \cdot 2^{-s_1^*}) : \right. \\ \left. \begin{array}{l} \alpha_0^*, \alpha_1^* \in \{-1, 0, +1\} \\ s_0^*, s_1^* \in \{0, 1, \dots, w-1\} \end{array} \right\}$$

the optimization problem of the EEAS-based CORDIC algorithm

given θ and R_m

find $\alpha_0(j)$, $\alpha_1(j)$, $s_0(j)$, and $s_1(j)$

the combination of elementary angles
from EEAS S_2

Minimize the angle quantization error

$$\left| \sum_{m, EEAS} \right| \triangleq \theta - \sum_{j=0}^{R_m-1} \tan^{-1} (\alpha_0(j) 2^{-s_0(j)} + \alpha_1(j) 2^{-s_1(j)})$$

given $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$

$$\begin{bmatrix} x(j+1) \\ y(j+1) \end{bmatrix} = \begin{bmatrix} 1 & \alpha_0(j) 2^{-s_0(j)} + \alpha_1(j) 2^{-s_1(j)} \\ \alpha_0(j) 2^{-s_0(j)} + \alpha_1(j) 2^{-s_1(j)} & 1 \end{bmatrix} \begin{bmatrix} x(j) \\ y(j) \end{bmatrix}$$

$$\begin{bmatrix} x_f \\ y_f \end{bmatrix} = P \begin{bmatrix} x(R_m) \\ y(R_m) \end{bmatrix} = \frac{1}{\prod_{j=0}^{R_m-1} \sqrt{1 + [\alpha_0(j) \cdot 2^{-s_0(j)} + \alpha_1(j) \cdot 2^{-s_1(j)}]^2}} \begin{bmatrix} x(R_m) \\ y(R_m) \end{bmatrix}$$

Micro rotation procedure
the scaling operation

↳ additions

increased hardware
reduced iteration steps

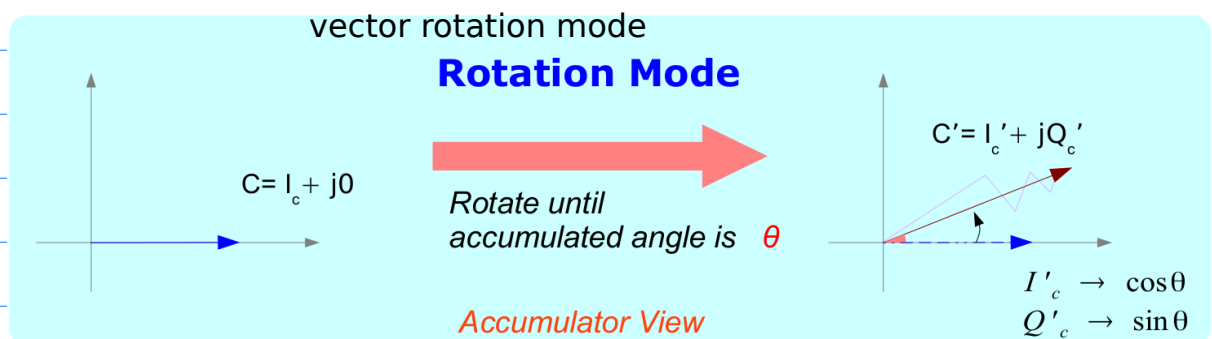
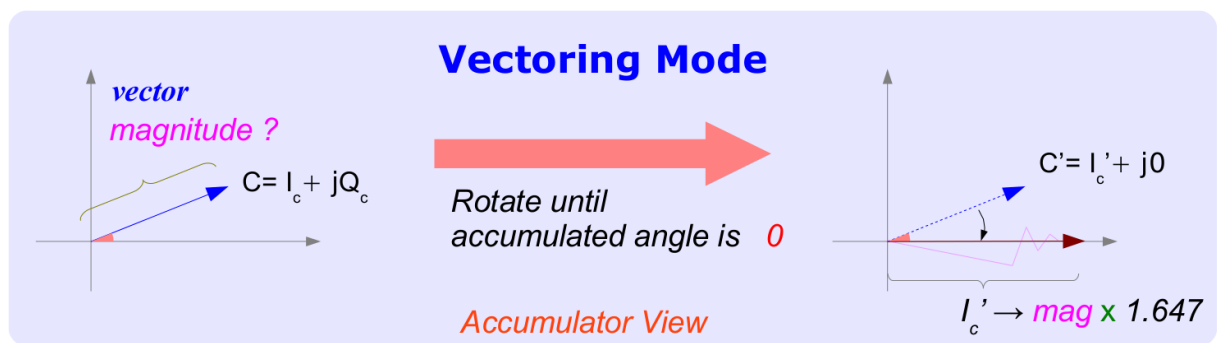
MVR (Modified Vector Rotation)

1) Repeat of Elementary Angles θ_i, θ_i

2) fixed total micro-rotation Number R_m

* Vector Rotation Mode

* and the rotation angles are known in advance



Modified Vector Rotational MVR CORDIC

- reduce the iteration number
- maintaining the SQNR performance
- modifying the basic microrotation procedure

Three Searching Algorithm

- ① the selective prerotation
- ② the selective scaling
- ③ iteration-tradeoff scheme

Angle Quantization

Quantization process on the rotational angle θ

decompose θ into several subangles θ_i 's

the angle quantization error

$$\xi_m \triangleq \theta - \sum_{i=0}^{N_A-1} \theta_i$$

N_A the number of subangles
 $\theta_0, \theta_1, \dots, \theta_{N_A-1}$

$$\theta = \theta_0 + \theta_1 + \dots + \theta_{N_A-1} + \xi_m$$

data : W -bit word length

the iteration number : N $N \leq W$

the restricted iteration number : R_m $R_m \ll W$

AQ Process : 2 Design Issues

① need to determine the sub-angles θ_i

② select / combine sub angles
to minimize the angle quantization angle ξ_m

CSD (Canonical Signed Digit) Quantization

digital filter designs

coefficients are recoded

in terms of SPT (Signed Power of Two) terms

multiplication can be easily realized
with shift-and-add operations

$$h_2 = (-0.156249)_{10} \Rightarrow (0.0\bar{1}011)_2$$

$w=8$, 3 non-zero digits

- ① CSD quantization decomposes coefficients into several SPT terms (sub-coefficients)
- ② the multiplication of a coefficient can be reformed through the combination of the non-zero SPT sub-coefficients

Quantize the rotation angle θ

decompose the rotation angle θ
into several sub-angles θ_i 's

the rotational operation of each θ_i
should be easily realized

If each θ_i can be realized
using only shift-and-add operations

the rotation of θ can be performed
through successive applications of
sub-angle rotations
in a cost-effective way

Approximation target	Coefficient r_i	Rotation angle θ
Basic Element	Non-zero digit 2^{-i}	Sub-angle $\alpha(i) = \tan^{-1}(2^{-i})$
Basic Operation	shift-and-add operation	2 shift-and-add operations
Approximation Equation	$r_i \approx \sum_{j=0}^{N_D-1} g_j \cdot 2^{-d_j}$	$\theta \approx \sum_{j=0}^{N_A-1} \alpha(j) \cdot a(s_j)$
	$g_j \in \{-1, 0, +1\}$ $d_j \in \{0, 1, \dots, w-1\}$ <p>$N_D =$ the number of non-zero digits</p>	<p>$N_A =$ the number of sub-angles</p>

Vector Rotation CORDIC Family

① Conventional CORDIC

② AR

③ MVR

④ EEAS

① Conventional CORDIC

elementary angle $\alpha(i) = \tan^{-1}(2^{-i})$

the number of elementary angles N

the rotation sequence $\mu(i) = \{-1, +1\}$
 $+1, -1, -1, +1, +1, \dots$

the i -th rotation angle $\alpha(i)$

the w -bit word length

the iteration number $N \leq w$

the angle quantization error

$$\xi_{m, \text{CORDIC}} \equiv \theta - \sum_{i=0}^{M-1} \mu(i) \alpha(i)$$

① AR [Hu]

skip certain micro rotations

the rotation sequence $\mu(i) = \{-1, 0, +1\}$

$$\mu(i) = 0 \rightarrow \text{skip}$$

desire to minimize

$$\sum_{i=0}^N |\mu(i)|$$

so that the total number of CORDIC iterations can be minimized

Angle Recoding \leftarrow Multiplier Recoding

angle recoding method for efficient implementation of the CORDIC algorithm
Hu & Naganathan, ISCAS 89

Greedy algorithm

$$\theta(0) = \theta, \{ \mu(i) = 0, 0 \leq i \leq N-1 \}, k=0$$

repeat until $|\theta(k)| < a(N-1)$ Do

choose $i_k, 0 \leq i_k \leq N-1$

$$| |\theta(k)| - a(i_k) | = \text{Min}_{0 \leq i \leq N-1} | |\theta(k)| - a(i) |$$

$$\theta(k+1) = \theta(k) - \mu(i_k) a(i_k)$$

$$\mu(i_k) = \text{Sign}(\theta(k))$$

try to approach the target rotation angle θ
step by step

decisions are made in each step
by choosing the best combination of $\alpha(i)$ $a(s(i))$

So as to minimize $|\xi_m|$

$\alpha(i)$, $a(i)$ are determined such that
the error function is minimized

$$J(i) = |\theta(i) - \alpha(i)a(s(i))|$$

$$\theta(i) = \theta - \sum_{m=0}^{i-1} \alpha(m) a(s(m))$$

terminated if no further improvement can be found

$$J(i) \geq J(i-1)$$

or $\alpha(R_m-1)$ and $s(R_m-1)$
are determined at the end

$$\xi_{m, \text{CORDIC}} \equiv \theta - \sum_{i=0}^M \mu(i) \alpha(i) \quad \mu(i) = \{-1, 0, +1\}$$

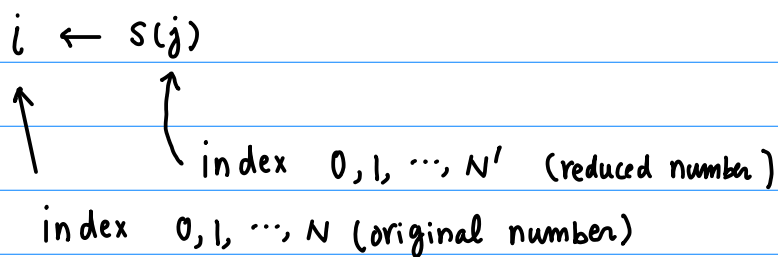
$$= \theta - \sum_{j=0}^{N'} \tilde{\theta}(j)$$

$$N' \equiv \sum_{i=0}^{N-1} |\mu(i)| \quad \{+1, 0, +1\}$$

the effective iteration number N'

$S(j)$ the rotational sequence

determines the micro-rotation angle in the j -th iteration



$$\mu(S(j)) \leftarrow \alpha(j)$$

$$\downarrow \qquad \uparrow$$

$$\{-1, +1\}$$

$$\mu(i) = \begin{cases} \mu(S(j)) & i = S(j) \\ 0 & i \neq S(j) \text{ --- reduced index} \end{cases}$$

er

$$\begin{aligned}
 i &= 0, \overset{\text{see}}{\boxed{1, 2}}, 3, \dots, N-1 \\
 s(j) &= 0, \boxed{1, 2}, 3, \dots, N-1 && \text{rotational sequence} \\
 \alpha(j) &= -1, \boxed{0, 0}, +1, \dots, -1 && \text{directional sequence} \\
 j &= 0, -, -, 1, \dots, N'-1 && \text{effective iteration number} \\
 N' &= N-2
 \end{aligned}$$

the j -th micro-rotation of $a(s(j))$

elementary angle

$$a(i) = \tan^{-1}(2^{-i})$$

$$a(s(j)) = \tan^{-1}(2^{-s(j)})$$

$$\alpha(j) a(s(j)) = \alpha(j) \tan^{-1}(2^{-s(j)}) \quad \alpha(j) \in \{-1, +1\}$$

$$\Leftrightarrow \mu(i) a(i) \quad \mu(i) \in \{-1, 0, +1\}$$

$$\sum_{m, \text{CORDIC}} \equiv \theta - \sum_{i=0}^{N-1} \mu(i) a(i) \quad \mu(i) \in \{-1, 0, +1\}$$

$$= \theta - \left[\sum_{j=0}^{N'} \tilde{\theta}(j) \right]$$

$$= \theta - \left[\sum_{j=0}^{N'} \tan^{-1}(\alpha(j) \cdot 2^{-s(j)}) \right] \quad \alpha(j) \in \{-1, +1\}$$

$$\tilde{\theta}(j) = \alpha(j) \tan^{-1}(2^{-s(j)})$$

$$= \tan^{-1}(\alpha(j) \cdot 2^{-s(j)})$$

$$S_1 = \{ \tan^{-1}(\alpha \cdot 2^{-s}) \mid \alpha \in \{-1, 0, +1\}, s \in \{0, 1, 2, \dots, N-1\} \}$$

② MVR (Modified Vector Rotational)

two modifications

① repeat of elementary angles

each micro-rotation of elementary angle
can be performed repeatedly

- more possible combinations
- smaller ξ_m

② confines of total micro-rotation number

confine the iteration number
in the micro-rotation phase
to R_m ($R_m \ll W$)

The role of R_m is quite similar
to the number of non-zero digit
 N_D in CSD recoding scheme

$$\sum_{m, \text{MVR}} \triangleq \theta - \sum_{i=0}^{R_m-1} \alpha(i) a(s(i))$$

the rotational sequence

$$s(i) \in \{0, 1, \dots, W-1\}$$

determines the micro-rotation angle
in the i -th iteration

the directional sequence

$$\alpha(i) \in \{-1, 0, +1\}$$

controls the direction of the i -th
micro-rotation of $a(s(i))$

$$\alpha(i) a(s(i)) = \tilde{\theta}(j)$$

$$\xi_{m, MVR} \cong \theta - \sum_{j=0}^{R_m-1} \alpha(j) a(s(j))$$

the rotational sequence $s(j)$

$$j = 0, 1, 2, \dots, R_m-1$$



$$s(j) \in \{0, 1, \dots, W-1\}$$

determines the micro-rotation angle $a(s(j))$
in the j -th iteration

the directional sequence $\alpha(j)$

$$\alpha(j) \in \{-1, 0, +1\}$$

controls the direction of the j -th
micro-rotation of $a(s(j))$

$$\alpha(j) a(s(j)) = \tilde{\theta}(j)$$

$i = 0, 1, 2, 3, \dots, W-1$	
$s(j) = 0, 1, 2, 3, \dots, W-1$	rotational sequence
$\alpha(j) = -1, 0, 0, +1, \dots, -1$	directional sequence
$j = 0, \dots, R_m-1$	effective iteration number
$R_m \ll W$	

sub-angle $(\alpha(j) a(s(j))) \sim \tilde{\theta}(j)$

$$\xi_{m,AR} = \theta - \left[\sum_{j=0}^{N'-1} \tan^{-1}(\alpha(j) \cdot 2^{-s(j)}) \right]$$
$$= \theta - \left[\sum_{j=0}^{N'-1} \tilde{\theta}(j) \right], \quad \tilde{\theta}(j) = \tan^{-1}(\alpha(j) \cdot 2^{-s(j)})$$

$$N' \triangleq \sum_{j=0}^{N-1} |\mu(j)| \quad \text{the effective iteration number}$$

EAS formed by MVR-CORDIC
is the same as AR
also performs AQ

The major difference

1) the total number of sub-angles N_A

the total iteration number

in the micro-rotation phase

is kept fixed to a pre-defined value of R_m

$$N_A = R_m$$

2) the sub-angle θ_j corresponds to $\alpha(j) a(s(j))$

$$\theta_j = \alpha(j) a(s(j)) = \tilde{\theta}_j$$

Optimization Problem

EAS point of view

Given θ , find the combination of R_m elementary angles from EAS S_i , such that the angle quantization error $|\xi_{m, \text{MUR}}|$ is minimized.

Semi-greedy algorithm
trade offs between computational complexities
and performance

Key issue in the MVR-CORDIC

is to find the best sequences of

$s(i)$ and $\alpha(i)$ to minimize $|\xi_m|$

subject to the constraint that

the total iteration number is confined to R_m

1) Greedy Algorithm

2) Exhaustive Algorithm

3) Semigreedy Algorithm

1) Greedy Algorithm

try to approach the target rotation angle, Θ ,
step by step

in each step, decisions are made on $\alpha(i)$ and $s(i)$
by choosing the best combination of $\alpha(i)$ and $s(i)$
so as to minimize $|\xi_m|$

$\alpha(i)$ and $s(i)$ are determined such that

the error function $J(i) = |\Theta(i) - \alpha(i) a(s(i))|$ is minimized

$\Theta(i)$: the residue angle in the i -th step

$$\Theta(i) = \Theta - \sum_{m=0}^{i-1} \alpha(m) a(s(m))$$

the searching is terminated

if no further improvements can be found

$$J(i) \geq J(i-1)$$

$\alpha(R_m-1)$ and $s(R_m-1)$ are determined

at the end of the searching

the greedy algorithm terminates

only when the residue angle error
cannot be further reduced.

2) Exhaustive Algorithm

search for the entire solution space

all possible combinations of

$$\sum_{i=0}^{R_m-1} \alpha(i) a(s(i))$$

in a single step

decisions for $\alpha(i)$ and $s(i)$, $0 \leq i \leq R_m-1$
by minimizing the error function

$$J = \left| 0 - \sum_{i=0}^{R_m-1} \alpha(i) a(s(i)) \right|$$

global optimal solution

in the i -th block

decision of $\alpha(k)$ and $s(k)$ for $iD \leq k \leq (i+1)D-1$

$$\text{minimizes } J = \left| \theta(i) - \sum_{k=iD}^{(i+1)D-1} \alpha(k) a(s(k)) \right|$$

$$\text{where } \theta(i) = \theta - \sum_{m=0}^{i-1} \left[\sum_{k=mD}^{(m+1)D-1} \alpha(k) a(s(k)) \right]$$

the residue angle in the i -th step