Example Random Processes

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Gaussian Random Process

N Gaussian random variables

Definition

$$f_X(x_1, \cdots, x_N; t_1, \cdots, t_N) =$$

$$\frac{exp\left\{-(1/2)\left[x - \overline{X}\right]^t [C_X]^{-1} \left[x - \overline{X}\right]\right\}}{\sqrt{(2\pi)^N \left|[C_X]\right|}}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \qquad \overline{X} = \begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \\ \vdots \\ \overline{X}_N \end{bmatrix} \qquad [x - \overline{X}] = \begin{bmatrix} x_1 - \overline{X}_1 \\ x_2 - \overline{X}_2 \\ \vdots \\ x_N - \overline{X}_N \end{bmatrix}$$

The Covariance Matrix (1)

N Gaussian random variables

Definition

$$\overline{X}_i = E[X_i] = E[X(t_i)]$$

$$\overline{X} = \begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \\ \vdots \\ \overline{X}_N \end{bmatrix} = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_N] \end{bmatrix} = \begin{bmatrix} E[X(t_1)] \\ E[X(t_2)] \\ \vdots \\ E[X(t_N)] \end{bmatrix}$$

The Covariance Matrix (2)

N Gaussian random variables

Definition

$$C_{ik} = C_{X_i X_k} = E\left[\left(X_i - \overline{X}_i\right)\left(X_k - \overline{X}_k\right)\right]$$
$$= E\left[\left(X(t_i) - E\left[X(t_i)\right]\right)\left(X(t_k) - E\left[X(t_k)\right]\right)\right]$$

$$C_{ik} = C_{X_i X_k} = C_{XX}(t_i, t_k)$$
$$= R_{XX}(t_i, t_k) - E\left[X(t_i)\right] E\left[X(t_k)\right]$$

Stationary Gaussian Process

N Gaussian random variables

Definition

$$\overline{X}_i = E[X_i] = E[X(t_i)] = \overline{X} = const$$

$$C_{XX}(t_i, t_k) = C_{XX}(t_k - t_i)$$

$$R_{XX}(t_i, t_k) = R_{XX}(t_k - t_i)$$

Jointly Gaussian Process

N Gaussian random variables

Definition

the two random processes $X(t)$ and $Y(t)$ are jointly Gaussian
if the random variables $X(t_1), \ldots, X(t_N), Y(t'_1), \ldots, Y(t'_M)$
at times $t_1, \ldots, t_N$ for $X(t)$ and $t'_1, \ldots, t'_M$ for $Y(t)$
are jointly gaussian for any N, $t_1, \ldots, t_N$, and M, $t'_1, \ldots, t'_M$

Stationary Gaussian Markov Process

N Gaussian random variables

Definition

$$C_{XX}(\tau) = \sigma^2 e^{-\beta|\tau|}$$

$$C_{XX}[k] = \sigma^2 a^{-|k|}$$

$$a = e^{\beta T_S}$$

Poisson Random Process

N Gaussian random variables

Definition

$$p\left[X(t=k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}\right], \qquad k = 0, 1, 2, \cdots$$

$$f_X(x) = \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k)$$

Poisson Random Process - mean and 2nd moment

N Gaussian random variables

Definition

$$E[X(t)] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k) dx$$

$$= \sum_{k=0}^{\infty} \frac{k(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t$$

$$E\left[X^2 t)\right] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_{-\infty}^{\infty} x^2 \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k) dx$$

$$= \sum_{k=0}^{\infty} \frac{k^2(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t(1+\lambda t)$$

Poisson Random Process - joint probability density

N Gaussian random variables

Definition

$$P[X(t_1) = k_1] = \frac{(\lambda t_1)^{k_1} e^{-\lambda t_1}}{k_1!} \qquad k = 0, 1, 2, \cdots$$

$$P[X(t_2) = k_2 | X(t_1) = k_1] = \frac{(\lambda (t_2 - t_1))^{k_2 - k_1} e^{-\lambda (t_2 - t_1)}}{(k_2 - k_1)!}$$

$$P(k_1, k_2) = P[X(t_2) = k_2 | X(t_1) = k_1] \cdot P[X(t_1) = k_1]$$

$$= \frac{(\lambda t_1)^{k_1} (\lambda (t_2 - t_1))^{k_2 - k_1} e^{-\lambda t_2}}{k_1! (k_2 - k_1)!}$$

$$f_X(x_1, x_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=k_1}^{\infty} P(k_1, k_2) \delta(x_1 - k_1) \delta(x_2 - k_2)$$

Bernoulli Random Process

N Gaussian random variables

Definition

$$X[n] = \sum_{m=1}^{n} I[m]$$

$$f_X(x) = \sum_{k=0}^{n} P(k)\delta(x-k)$$

$$P(k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k}$$

$$E\left[X\left[n\right]\right] = np$$

$$Var\left[X\left[n\right]\right] = np(1-p)$$

Binomial Counting Process
N Gaussian random variables

Definition

$$f_X(x_1, x_2) = \sum_{k_1=0}^{n_1} \sum_{k_2=k_1}^{n_2} P(k_1, k_2)\delta(x_1 - k_1)\delta(x_2 - k_2)$$

$$P(k_1, k_2) = P\left[x[n_1] = k_1, x[n_2] = k_2\right]$$

$$= \left(\begin{array}{c} n_2 - n_1 \\ k_2 - k_1 \end{array} \right) \left(\begin{array}{c} n_1 \\ k_1 \end{array} \right) p^{k_2}(1-p)^{n_2-k_2}$$

$$P(k) = \frac{(np)^k e^{-np}}{k!} = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$