Counting (9A)

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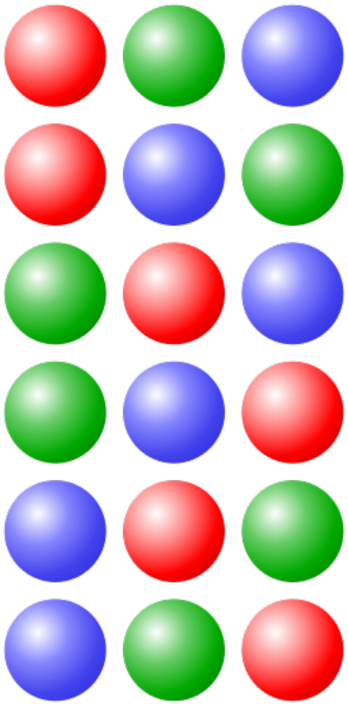
Permutation



Each of the six rows is a different
permutation of three distinct balls

there are six permutations of the set {1,2,3}, namely:
(1,2,3),
(1,3,2),
(2,1,3),
(2,3,1),
(3,1,2),
(3,2,1)

https://en.wikipedia.org/wiki/Permutation

Cauchy's two-line notation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix};$$

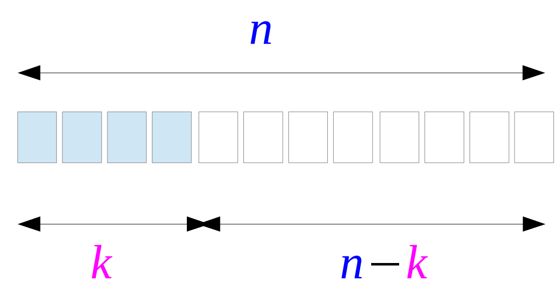
$$\sigma = \begin{pmatrix} 3 & 2 & 5 & 1 & 4 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}.$$

$$\sigma = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ \sigma(x_1) & \sigma(x_2) & \sigma(x_3) & \cdots & \sigma(x_n) \end{pmatrix}.$$

https://en.wikipedia.org/wiki/Permutation

k-permutations of n

$$P(n,k) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}_{k \text{ factors}}$$



$$\frac{n!}{(n-k)!}.$$

$$P(n,k) = \frac{n!}{(n-k)!}$$

$$= (n-0)\cdot(n-1)\cdot(n-2)\cdots\cdot(n-(k-1))$$

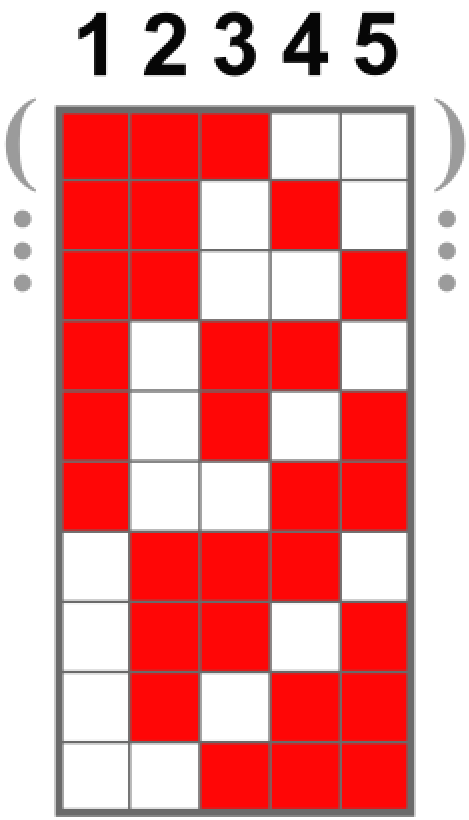
$$C(n,k) = \frac{P(n,k)}{P(k,k)} = \frac{\frac{n!}{(n-k)!}}{\frac{k!}{0!}} = \frac{n!}{(n-k)!\,k!}.$$

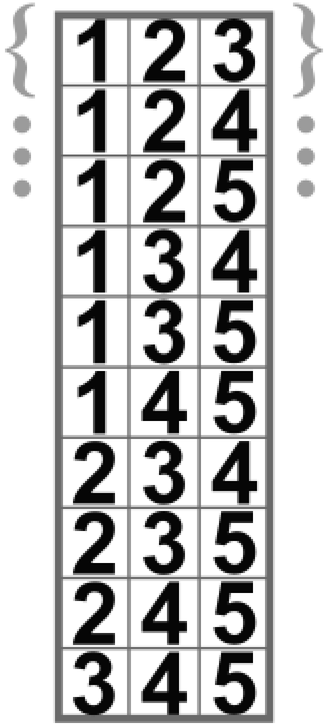
https://en.wikipedia.org/wiki/Combination

k-permutations of 4

$P(4,1)$	$P(4,2)$	$P(4,3)$	$P(4,4)$
$4 = 4!/3!$	$4 \cdot 3 = 4!/2!$	$4 \cdot 3 \cdot 2 = 4!/1!$	$4 \cdot 3 \cdot 2 \cdot 1 = 4!$
1	1, 2	1, 2, 3	1, 2, 3, 4
2	1, 3	1, 2, 4	1, 2, 4, 3
3	1, 4	1, 3, 2	1, 3, 2, 4
4	2, 1	1, 3, 4	1, 3, 4, 2
	2, 3	1, 4, 2	1, 4, 2, 3
	2, 4	1, 4, 3	1, 2, 4, 2
	3, 1	2, 1, 3	2, 1, 3, 4
	3, 2	2, 1, 4	2, 1, 4, 3
	3, 4	2, 3, 1	2, 3, 1, 4
	4, 1	2, 3, 4	2, 3, 4, 1
	4, 2	2, 4, 1	2, 4, 1, 3
	4, 3	2, 4, 3	2, 4, 3, 1
		3, 1, 2	3, 1, 2, 4
		3, 1, 4	3, 1, 4, 2
		3, 2, 1	3, 2, 1, 4
		3, 2, 4	3, 2, 4, 1
		3, 4, 1	3, 4, 1, 2
		3, 4, 2	3, 4, 2, 1
		4, 1, 2	4, 1, 2, 3
		4, 1, 3	4, 1, 3, 2
		4, 2, 1	4, 2, 1, 3
		4, 2, 3	4, 2, 3, 1
		4, 3, 1	4, 3, 1, 2
		4, 3, 2	4, 3, 2, 1

Combination





3-element subsets of a 5-element set

https://en.wikipedia.org/wiki/Combination

k-combination

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}, \qquad \frac{n!}{k!(n-k)!}$$

$$(1+X)^n = \sum_{k\geq 0} \binom{n}{k} X^k,$$

$$\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{k} = 0$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \qquad \binom{n}{k} = \binom{n}{n-k}, \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!},$$

https://en.wikipedia.org/wiki/Combination

k-combination

$$\binom{52}{5} = \frac{52!}{5!47!}$$

$$= \frac{52 \times 51 \times 50 \times 49 \times 48 \times \cancel{47!}}{5 \times 4 \times 3 \times 2 \times \cancel{1} \times \cancel{47!}}$$

$$= \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2}$$

$$= \frac{(26 \times \cancel{2}) \times (17 \times \cancel{3}) \times (10 \times \cancel{5}) \times 49 \times (12 \times \cancel{4})}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2}}$$

$$= 26 \times 17 \times 10 \times 49 \times 12$$

$$= 2{,}598{,}960.$$

https://en.wikipedia.org/wiki/Combination

k-combination

$$\binom{n}{k} = \frac{(n-0)}{1} \times \frac{(n-1)}{2} \times \frac{(n-2)}{3} \times \cdots \times \frac{(n-(k-1))}{k},$$

$$\binom{52}{5} = \frac{52}{1} \times \frac{51}{2} \times \frac{50}{3} \times \frac{49}{4} \times \frac{48}{5} = 2{,}598{,}960$$

https://en.wikipedia.org/wiki/Combination

k-combination of 4

$P(4,1)$

$4 = 4!/3!$

$C(4,1)$

$4!/3!/1! = 4$

1
2
3
4

$P(4,2)$

$4 \cdot 3 = 4!/2!$

$C(4,2)$

$4!/2!/2! = 6$

1, 2
1, 3
1, 4
2, 3
2, 4
3, 4
4, 3

$P(4,3)$

$4 \cdot 3 \cdot 2 = 4!/1!$

$C(4,3)$

$4!/1!/3! = 4$

1, 2, 3
1, 2, 4
1, 3, 4
2, 3, 4

$P(4,4)$

$4 \cdot 3 \cdot 2 \cdot 1 = 4!$

$C(4,4)$

$4!/4! = 1$

1, 2, 3, 4

k-permutations of 4

$P(4,3)$

$4 \cdot 3 \cdot 2 = 4!/1!$

1, 2, 3
1, 2, 4
~~1, 3, 2~~
1, 3, 4
~~1, 4, 2~~
~~1, 4, 3~~
~~2, 1, 3~~
~~2, 1, 4~~
~~2, 3, 1~~
2, 3, 4
~~2, 4, 1~~
~~2, 4, 3~~
~~3, 1, 2~~
~~3, 1, 4~~
~~3, 2, 1~~
~~3, 2, 4~~
~~3, 4, 1~~
~~3, 4, 2~~
~~4, 1, 2~~
~~4, 1, 3~~
~~4, 2, 1~~
~~4, 2, 3~~
~~4, 3, 1~~
~~4, 3, 2~~

1, 2, 3	1, 2, 4
1, 3, 2	1, 4, 2
2, 1, 3	2, 1, 4
2, 3, 1	2, 4, 1
3, 1, 2	4, 1, 2
3, 2, 1	4, 2, 1

1, 2, 3	2, 3, 4
1, 3, 2	2, 3, 4
2, 1, 3	3, 2, 4
2, 3, 1	3, 4, 1
3, 1, 2	4, 2, 3
3, 2, 1	4, 3, 2

$C(4,3)$

$4!/1!/3! = 4$

1, 2, 3
1, 2, 4
1, 3, 4
2, 3, 4

remove all non-increasing ordering
in order to count only once

k-combination of 4

$C(4,1)$

$4!/3!/1! = 4$

1
2
3
4

$C(4,3)$

$4!/1!/3! = 4$

1, 2, 3
1, 2, 4
1, 3, 4
2, 3, 4

$C(4,1)$

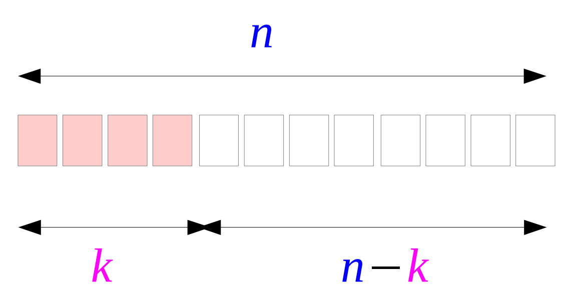
$4!/3!/1! = 4$

1
2
3
4

$C(4,3)$

$4!/1!/3! = 4$

2, 3, 4
1, 3, 4
1, 2, 4
1, 2, 3



$$C(n,k) = C(n,n-k)$$

Binomial Coefficient

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

The binomial coefficients can be arranged to form Pascal's triangle.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad \text{for all integers } n, k : 1 \leq k \leq n-1,$$

$$\binom{n}{0} = \binom{n}{n} = 1 \quad \text{for all integers } n \geq 0,$$

https://en.wikipedia.org/wiki/Binomial_coefficient

Binomial Coefficient

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

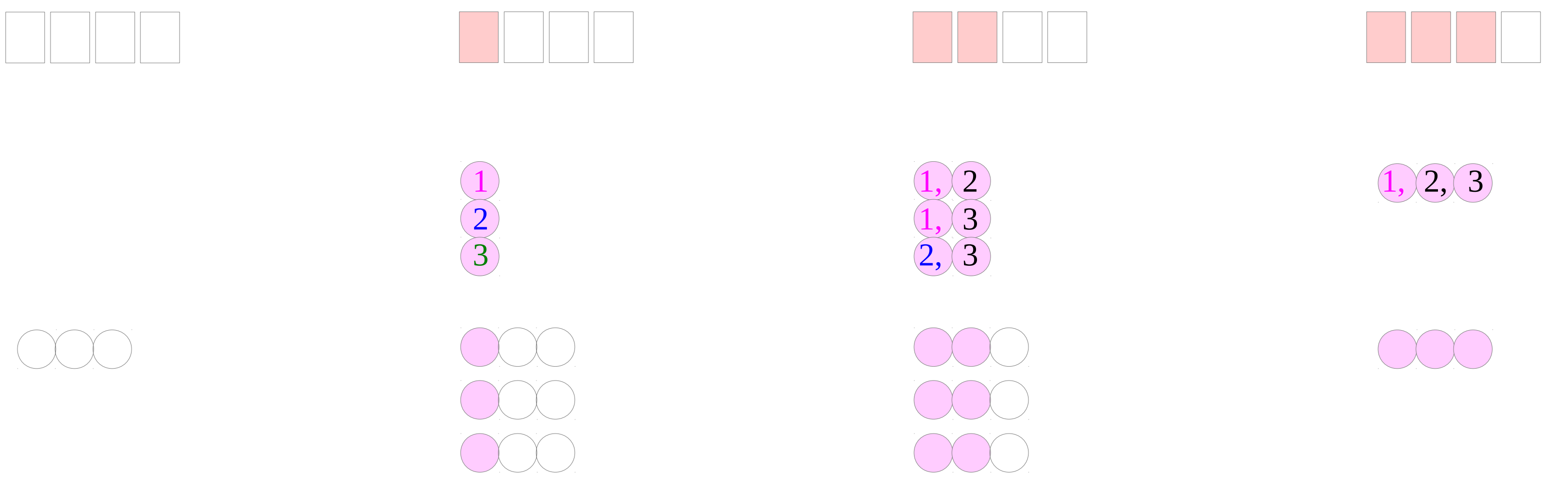
$$= xxx + 3xx\,y + 3x\,yy + yyy$$

$C(3,0)$

$C(3,1)$ $\quad 3!/2!/1! = 3$

$C(3,2)$ $\quad 3!/1!/2! = 6$

$C(3,3)$ $\quad 3!/1!/3! = 1$



1 *way*   3 *ways*   3 *ways*   1 *way*

Binomial Coefficient

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$= xxx + 3\,xx\,y + 3\,x\,yy + yyy$$

$$= \binom{3}{0}xxx + \binom{3}{1}xx\,y + \binom{3}{2}x\,yy + \binom{3}{3}yyy$$

$C(3,0)$

$C(3,1)$

$3!/2!/1! = 3$

1
2
3

$C(3,2)$

$3!/1!/2! = 6$

1, 2
1, 3
2, 3

$C(3,3)$

$3!/1!/3! = 1$

1, 2, 3

Binomial Coefficient

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$= xxx + 3xx\,y + 3x\,yy + yyy$$

$$= \binom{3}{0}xxx + \binom{3}{1}xx\,y + \binom{3}{2}x\,yy + \binom{3}{3}yyy$$

$$2^3 = \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$$

Pascal’s Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Rows zero to five of Pascal's triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Six rows Pascal's triangle as binomial coefficients

https://en.wikipedia.org/wiki/Pascal%27s_triangle

Pascal's Triangle

$$(x+1)^n = \sum_{i=0}^{n} a_i x^i.$$

$$(x+1)^{n+1} = (x+1)(x+1)^n = x(x+1)^n + (x+1)^n = \sum_{i=0}^{n} a_i x^{i+1} + \sum_{i=0}^{n} a_i x^i.$$

$$\begin{aligned}
&\sum_{i=0}^{n} a_i x^{i+1} + \sum_{i=0}^{n} a_i x^i \\
&= \sum_{i=1}^{n+1} a_{i-1} x^i + \sum_{i=0}^{n} a_i x^i \\
&= \sum_{i=1}^{n} a_{i-1} x^i + \sum_{i=1}^{n} a_i x^i + a_0 x^0 + a_n x^{n+1} \\
&= \sum_{i=1}^{n} (a_{i-1} + a_i) x^i + a_0 x^0 + a_n x^{n+1} \\
&= \sum_{i=1}^{n} (a_{i-1} + a_i) x^i + x^0 + x^{n+1}
\end{aligned}$$

https://en.wikipedia.org/wiki/Pascal%27s_triangle

Pascal's Triangle

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$

$$\mathbf{C}(n,k) = \mathbf{C}_k^n = {}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

https://en.wikipedia.org/wiki/Euclidean_algorithm

Upper and Lower Bounds

https://en.wikipedia.org/wiki/Algorithm

References

[1] http://en.wikipedia.org/
[2]