Vector Calculus (H.1) Curl & Div Operators

20160404

∇ | Scalar function | $\text{grad}(f)$
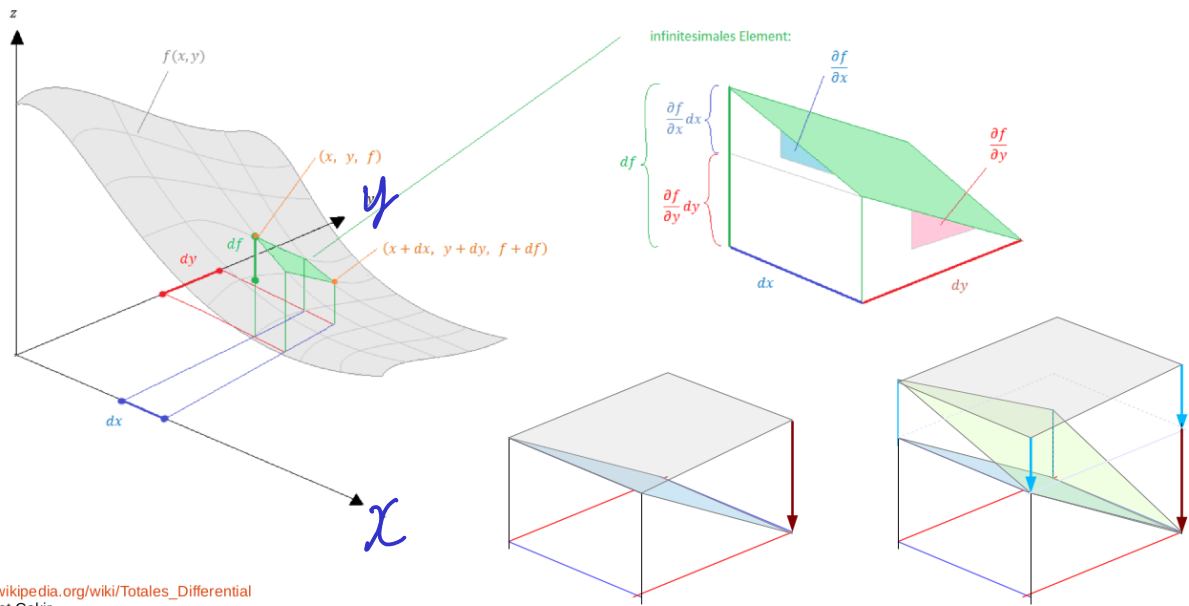
$\nabla \cdot$ | Vector field | $\text{div}(\vec{F})$

$\nabla \times$ | Vector field | $\text{curl}(\vec{F})$

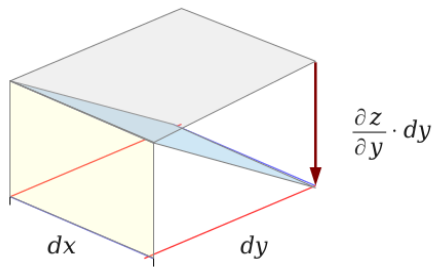Vector field $\vec{F} = P\,\vec{\imath} + Q\,\vec{\jmath} + R\,\vec{k}$

Gradient Vector field $\vec{F} = \dfrac{\partial f}{\partial x}\,\vec{\imath} + \dfrac{\partial f}{\partial y}\,\vec{\jmath} + \dfrac{\partial f}{\partial z}\,\vec{k}$
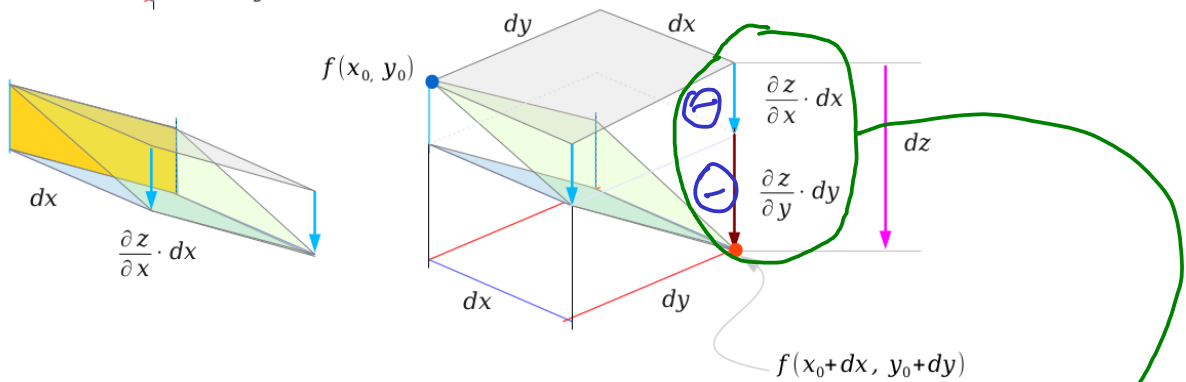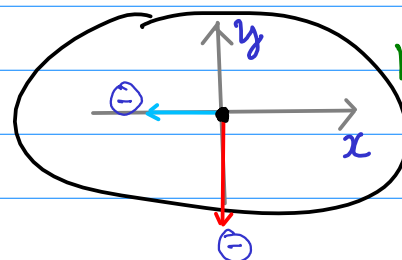
$$= \nabla f$$

Total Differential

$$z = f(x, y)$$

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

differential, or
total differential

a vector representation

Gradient Vector Field

$$z = f(x, y)$$

$$\frac{\partial z}{\partial y} \cdot dy$$

dx dy

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

differential, or
total differential

dy dx

$f(x_0, y_0)$

$$\frac{\partial z}{\partial x} \cdot dx$$

dz

dx

$$\frac{\partial z}{\partial x} \cdot dx$$

$$\frac{\partial z}{\partial y} \cdot dy$$

dx dy

$f(x_0+dx, y_0+dy)$
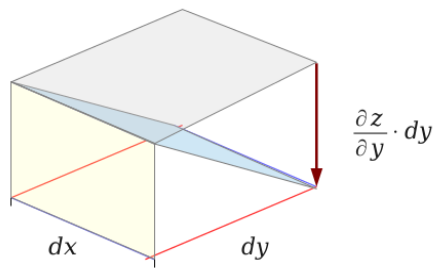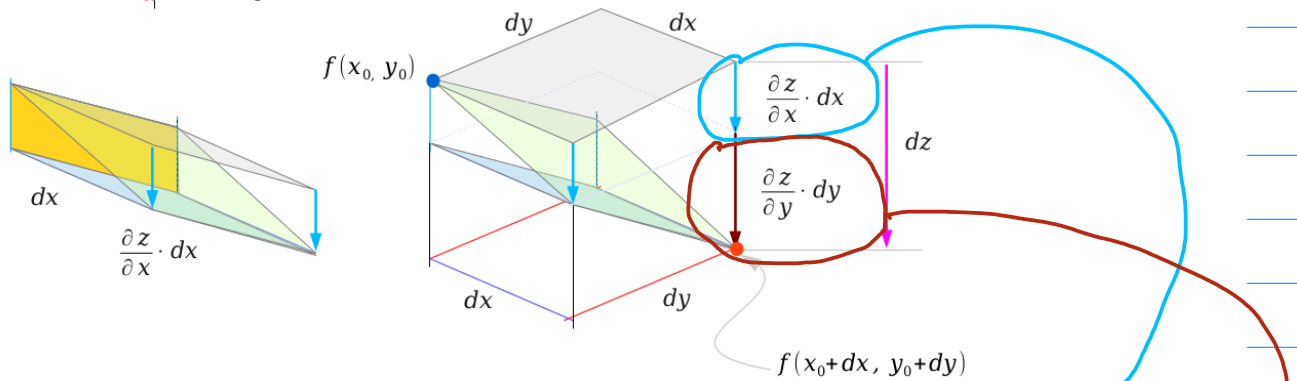
$$\frac{\partial f}{\partial x} < 0 \Rightarrow \text{a negative x direction vector}$$

$$\frac{\partial f}{\partial y} < 0 \Rightarrow \text{a negative y direction vector}$$

A Vector Field Graph

y

(x_0, y_0)

$\vec{F}(x_0, y_0)$

x

increasing f direction
at (x_0, y_0)
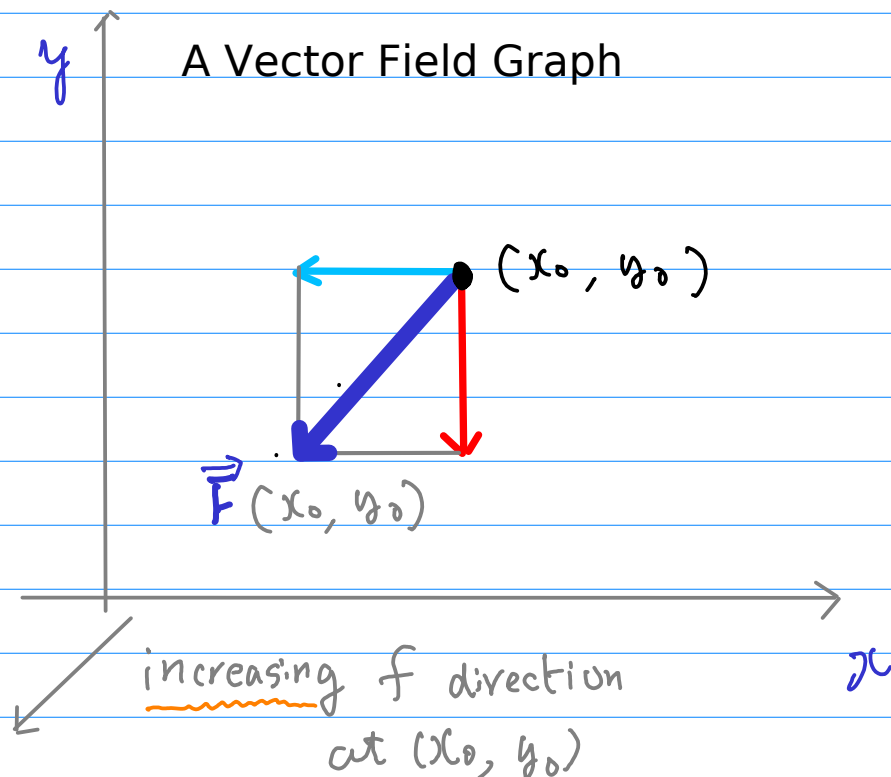
Gradient Vector Field

$f(x,y) \rightarrow$ 3D graph

$\vec{F}(x,y) \rightarrow$ 2D graph

$\vec{F}(x_0, y_0)$

(x_0, y_0)

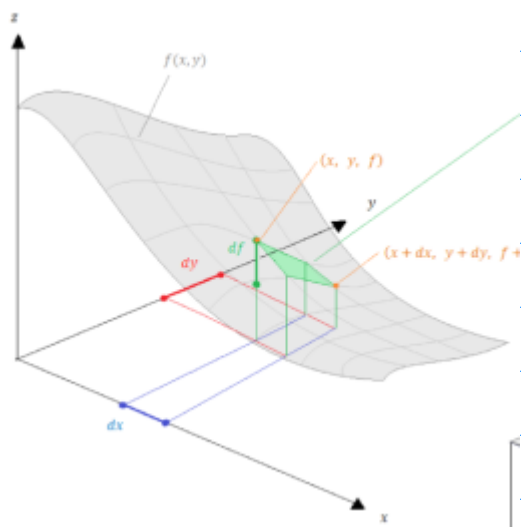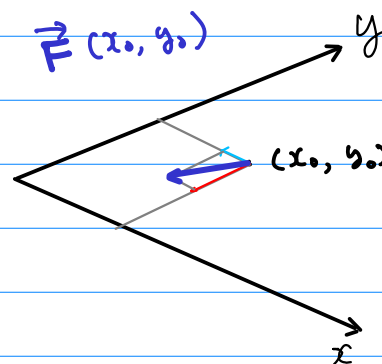mathematics, the **gradient** is a generalization of the usual concept of derivative of a function in one dimension to a function in several dimensions. If $f(x_1, ..., x_n)$ is a differentiable, scalar-valued function of standard Cartesian coordinates in Euclidean space, its gradient is the vector whose components are the n partial derivatives of f. It is thus a vector-valued function.

Similarly to the usual derivative, the gradient represents the slope of the tangent of the graph of the function. More precisely, the gradient points in the direction of the greatest rate of increase of the function and its magnitude is the slope of the graph in that direction. The components of the gradient in coordinates are the coefficients of the variables in the equation of the tangent space to the graph. This characterizing property of the gradient allows it to be defined independently of a choice of coordinate system, as a vector field whose components in a coordinate system will transform when going from one coordinate system to another.

wikipedia.org

Del Operator and a scalar function

$$f(x, y) \qquad\qquad f(x, y, z)$$

$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}$	$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$
$\nabla f = \left(\frac{\partial f}{\partial x}\right)\vec{i} + \left(\frac{\partial f}{\partial y}\right)\vec{j}$	$\nabla f = \left(\frac{\partial f}{\partial x}\right)\vec{i} + \left(\frac{\partial f}{\partial y}\right)\vec{j} + \left(\frac{\partial f}{\partial z}\right)\vec{k}$

Gradient vector

$$\boxed{f} \xrightarrow{\;\nabla f\;} \boxed{\vec{F}}$$

a scalar function   →   a gradient vector field

$$f(x, y) \xrightarrow{\;\nabla f\;} \vec{F}(x\ y) \qquad \mathbb{R}^2$$

$$f(x, y, z) \xrightarrow{\;\nabla f\;} \vec{F}(x, y, z) \qquad \mathbb{R}^3$$

Scalar   Vector

a Gradient Vector Field

$$f(x, y)$$

$$\xrightarrow{\nabla f}$$

$$\vec{F}(x, y)$$

(2) Variable function

(2) component vector in $\mathbb{R}^2$
: 2 component functions
each 2 variable function

This a __gradient__ vector field
not just a vector field

in $\mathbb{R}^2$

$$P(x, y) = \frac{\partial f}{\partial x}$$
$$Q(x, y) = \frac{\partial f}{\partial y}$$

$$f(x, y, z)$$

$$\xrightarrow{\nabla f}$$

$$\vec{F}(x, y, z)$$

(3) Variable function

(3) component vector in $\mathbb{R}^3$
: 3 component functions
each 3 variable functio

This a __gradient__ vector field
not just a vector field

in $\mathbb{R}^3$

$$P(x, y, z) = \frac{\partial f}{\partial x}$$
$$Q(x, y, z) = \frac{\partial f}{\partial y}$$
$$R(x, y, z) = \frac{\partial f}{\partial z}$$

a general Vector Field

~~f(x, y)~~ $\xrightarrow{\nabla f}$ $\boxed{\vec{F}(x, y)}$

even when we

can't find $f(x,y)$

such that

$\begin{cases} P(x,y) = \dfrac{\partial f}{\partial x} \\ Q(x,y) = \dfrac{\partial f}{\partial y} \end{cases}$

2 component functions

$P(x, y)$
$Q(x, y)$

Still a vector field $\vec{F}(x, y)$

~~f(x, y, z)~~ $\xrightarrow{\nabla f}$ $\boxed{\vec{F}(x, y, z)}$

even when we

can't find $f(x,y)$

such that

$P(x, y, z) = \dfrac{\partial f}{\partial x}$
$Q(x, y, z) = \dfrac{\partial f}{\partial y}$
$R(x, y, z) = \dfrac{\partial f}{\partial z}$

3 component functions

$P(x, y, z)$
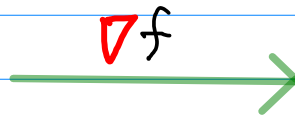$Q(x, y, z)$
$R(x, y, z)$

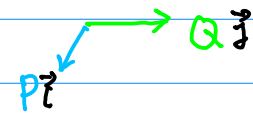Still a vector field $\vec{F}(x, y, z)$

2-d Gradient Vector Field

$$\boxed{f} \xrightarrow{\nabla f} \boxed{\vec{F}}$$

a scalar
function

a gradient
vector field

$f(x,y)$

$P\vec{\imath} \to Q\vec{\jmath}$

2-variable scalar function $\xrightarrow{\nabla f}$ Vectors in R^2

$f(x,y)$

$\vec{F} = \left\langle \text{x-directional derivatives} , \text{y-directional derivatives} \right\rangle$

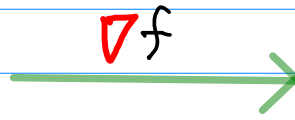$$= \left\langle \frac{\partial f}{\partial x} , \frac{\partial f}{\partial y} \right\rangle$$

$\textcircled{P}$ $\textcircled{Q}$

3-d Gradient Vector Field

$$\boxed{f} \xrightarrow{\nabla f} \boxed{\vec{F}}$$

a scalar
function

a gradient
vector field

$f(x, y, z)$

$R\vec{k}$, $Q\vec{j}$, $P\vec{i}$
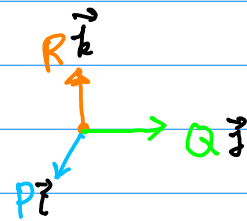
$\boxed{\text{3-variable scalar function}}$ $\xrightarrow{\nabla f}$ Vectors in R^3
$f(x, y, z)$

$\vec{F} = \langle$ x-directional derivatives , y-directional derivatives , z-directional derivatives $\rangle$

$$= \left\langle \underset{P}{\frac{\partial f}{\partial x}}, \underset{Q}{\frac{\partial f}{\partial y}}, \underset{R}{\frac{\partial f}{\partial z}} \right\rangle$$

Vector Field

$\vec{F} = (P)\,\vec{\imath} + (Q)\,\vec{\jmath}$	$\vec{F} = (P)\,\vec{\imath} + (Q)\,\vec{\jmath} + (R)\,\vec{k}$

2D $\qquad \vec{F}(x,y) \qquad\qquad = \quad P(x,y)\,\vec{\imath} + Q(x,y)\,\vec{\jmath}$

3D $\qquad \vec{F}(x,y,z) \qquad\quad = \quad P(x,y,z)\,\vec{\imath} + Q(x,y,z)\,\vec{\jmath} + R(x,y,z)\,\vec{k}$

Vector in $\begin{pmatrix} 2D \\ 3D \end{pmatrix}$ $\qquad\qquad \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ component functions with $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ variables

Gradient Vector Field

$\vec{F} = \nabla f$	$\vec{F} = \nabla f$
$\quad = \dfrac{\partial f}{\partial x}\,\vec{\imath} + \dfrac{\partial f}{\partial y}\,\vec{\jmath}$	$\quad = \dfrac{\partial f}{\partial x}\,\vec{\imath} + \dfrac{\partial f}{\partial y}\,\vec{\jmath} + \dfrac{\partial f}{\partial z}\,\vec{k}$
2 - D	3 - D

Del Operator (vector-like, vector-making)

$$\nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} \qquad \nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$$

Gradient Vector

$$\nabla f = \left(\frac{\partial f}{\partial x}\right)\vec{\imath} + \left(\frac{\partial f}{\partial y}\right)\vec{\jmath} \qquad \nabla f = \left(\frac{\partial f}{\partial x}\right)\vec{\imath} + \left(\frac{\partial f}{\partial y}\right)\vec{\jmath} + \left(\frac{\partial f}{\partial z}\right)\vec{k}$$

$$\nabla \square = \frac{\partial}{\partial x}\square\,\vec{\imath} + \frac{\partial}{\partial y}\square\,\vec{\jmath}$$

$$= \frac{\partial \square}{\partial x}\,\vec{\imath} + \frac{\partial \square}{\partial y}\,\vec{\jmath}$$

$$\nabla \bigcirc = \frac{\partial}{\partial x}\bigcirc\,\vec{\imath} + \frac{\partial}{\partial y}\bigcirc\,\vec{\jmath} + \frac{\partial}{\partial z}\bigcirc\,\vec{k}$$

$$= \frac{\partial \bigcirc}{\partial x}\,\vec{\imath} + \frac{\partial \bigcirc}{\partial y}\,\vec{\jmath} + \frac{\partial \bigcirc}{\partial z}\,\vec{k}$$

Curl & Div operators

Curl $\vec{F}$
div $\vec{F}$

Curl & div operates
over a Vector field $\vec{F}$

$\vec{F}$ must be a Vector field

$$\begin{cases} P\vec{i} + Q\vec{j} & \text{or} \\ P\vec{i} + Q\vec{j} + R\vec{k} \end{cases}$$

(2D)

(3D)

Curl & Div operators

Curl $\vec{F}$: ∇ cross $\times$ product with $\vec{F}$

div $\vec{F}$: ∇ inner $\cdot$ product with $\vec{F}$
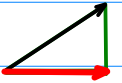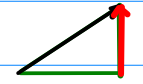
$$\nabla = \begin{cases} \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} & \text{(2D)} \\ \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k} & \text{(3D)} \end{cases}$$

$$\vec{F} = \begin{cases} P\vec{\imath} + Q\vec{\jmath} & \text{(2D)} \\ P\vec{\imath} + Q\vec{\jmath} + R\vec{k} & \text{(3D)} \end{cases}$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

Curl & Div operators

For any vector field $\vec{F}$

$$\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} \qquad \text{cross product}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} \qquad \text{inner product}$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times \vec{F} \qquad \text{a vector}$$

$$\text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \vec{F} \qquad \text{a scalar}$$

Curl & Div over a gradient field

⑥ $\vec{F}$ can be a **gradient** vector field

$$\vec{F} = \boxed{P}\,\vec{i} + \boxed{Q}\,\vec{j} + \boxed{R}\,\vec{k}$$

$$= \boxed{\frac{\partial f}{\partial x}}\,\vec{i} + \boxed{\frac{\partial f}{\partial y}}\,\vec{j} + \boxed{\frac{\partial f}{\partial z}}\,\vec{k}$$

$\exists f$ a scalar function ⓕ s.t.

$$\vec{F} = \nabla f$$

For a **gradient** vector field $\vec{F}$

$$\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\cdot\vec{k} = \nabla f$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \boxed{\nabla \times \nabla f}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \boxed{\nabla \cdot \nabla f}$$

3D Curl & Div

a vector field $\vec{F}$

$$\vec{F} = \left(\boxed{P}\vec{\imath} + \boxed{Q}\vec{\jmath} + \boxed{R}\vec{k} \right)$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k} \right) \times \left(\boxed{P}\vec{\imath} + \boxed{Q}\vec{\jmath} + \boxed{R}\vec{k} \right)$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k} \right) \cdot \left(\boxed{P}\vec{\imath} + \boxed{Q}\vec{\jmath} + \boxed{R}\vec{k} \right)$$

$$\vec{F} = \left(\boxed{P}\vec{\imath} + \boxed{Q}\vec{\jmath} + \boxed{R}\vec{k} \right)$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\vec{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\vec{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\vec{k}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)$$

a gradient vector field $\vec{F}$ $\left(\vec{F} = \nabla f \right)$

$$\vec{F} = \left(\frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k} \right)$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right)\vec{\imath} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right)\vec{\jmath} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)\vec{k}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right)$$

$$\nabla \cdot (\nabla f) = \boxed{\nabla^2 f} \qquad \nabla \times (\nabla f) = \vec{0}$$

2D Curl & Div

a Vector field $\vec{F}$

$$\vec{F} = \qquad \left(\boxed{P}\vec{\imath} + \boxed{Q}\vec{\jmath} + O\,\vec{k} \right)$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + O\,\vec{k} \right) \times \left(\boxed{P}\vec{\imath} + \boxed{Q}\vec{\jmath} + O\,\vec{k} \right)$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + O\,\vec{k} \right) \cdot \left(\boxed{P}\vec{\imath} + \boxed{Q}\vec{\jmath} + O\,\vec{k} \right)$$

$$\vec{F} = \left(\boxed{P}\vec{\imath} + \boxed{Q}\vec{\jmath} \right)$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\vec{k}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)$$

a gradient Vector field $\vec{F}$ $\quad \left(\vec{F} = \nabla f \right)$

$$\vec{F} = \left(\frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} \right)$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)\vec{k} = \nabla \times (\nabla f) = \vec{0}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) = \nabla \cdot (\nabla f) = \boxed{\nabla^2 f}$$
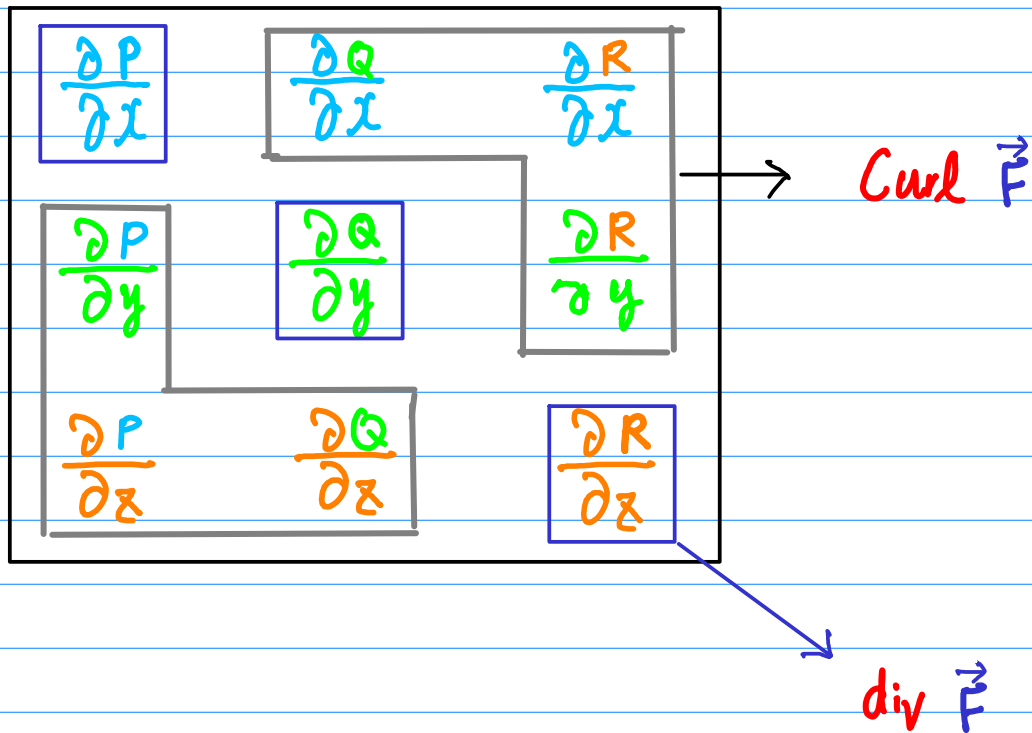
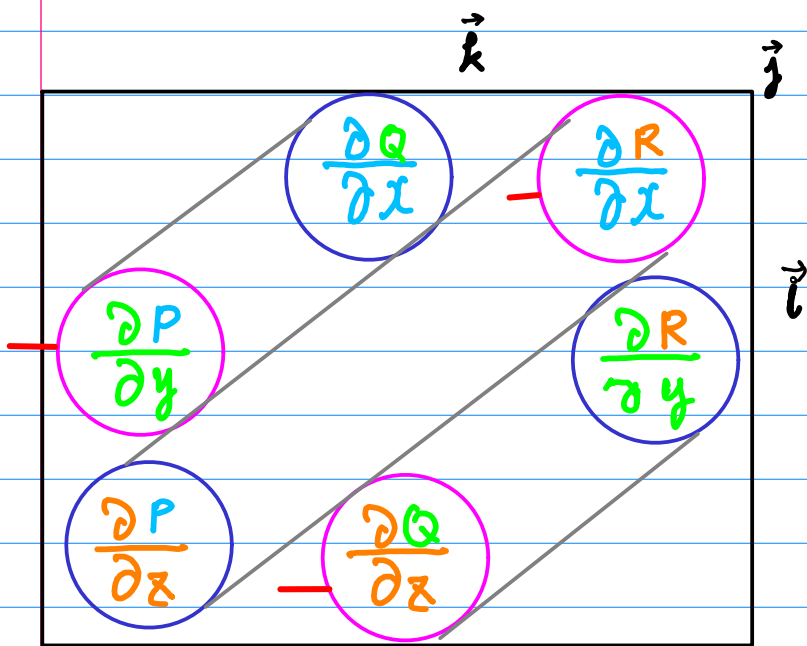# Combination of Partial Derivatives

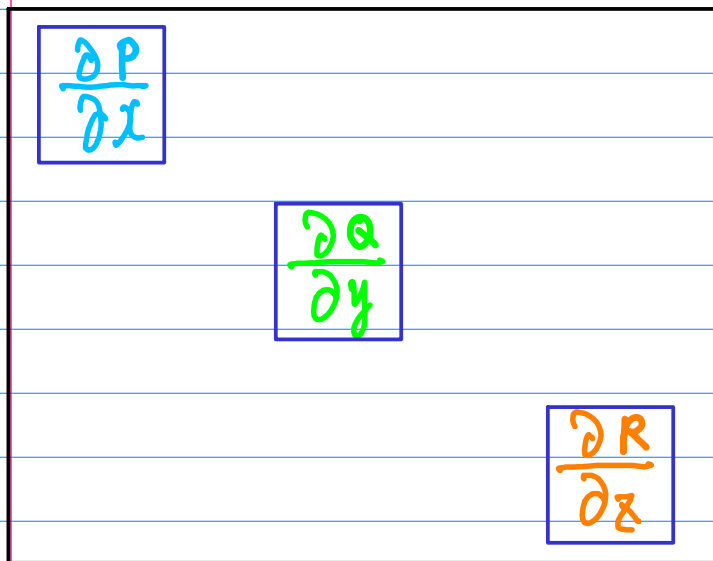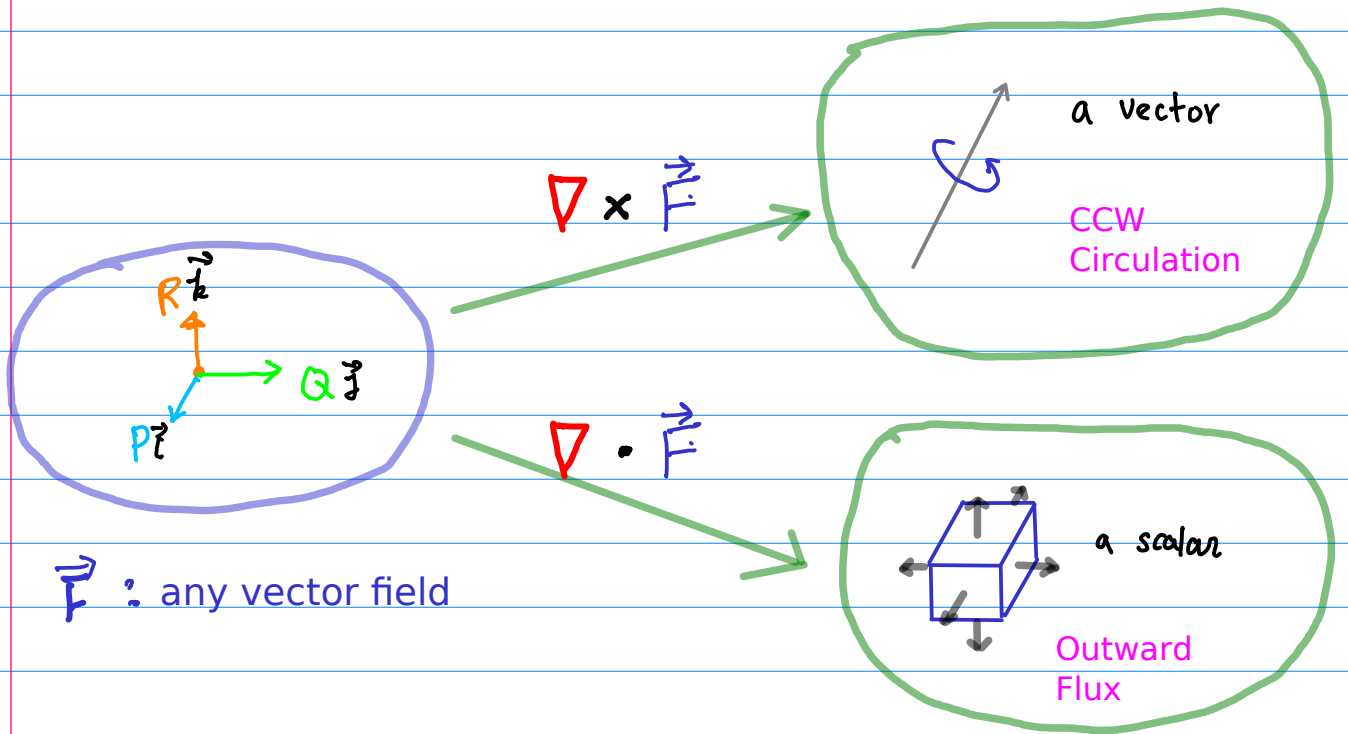$P(x,y,z)$ $Q(x,y,z)$ $R(x,y,z)$

$$\frac{\partial}{\partial x} \qquad \frac{\partial}{\partial y} \qquad \frac{\partial}{\partial z}$$

$\dfrac{\partial P}{\partial x}$	$\dfrac{\partial Q}{\partial x}$	$\dfrac{\partial R}{\partial x}$
$\dfrac{\partial P}{\partial y}$	$\dfrac{\partial Q}{\partial y}$	$\dfrac{\partial R}{\partial y}$
$\dfrac{\partial P}{\partial z}$	$\dfrac{\partial Q}{\partial z}$	$\dfrac{\partial R}{\partial z}$

$\longrightarrow$ Curl $\vec{F}$

$\searrow$ div $\vec{F}$

$\vec{k}$ $\vec{j}$

$$\frac{\partial Q}{\partial x} \qquad \frac{\partial R}{\partial x}$$

$\vec{i}$

$$\frac{\partial P}{\partial y} \qquad\qquad \frac{\partial R}{\partial y}$$
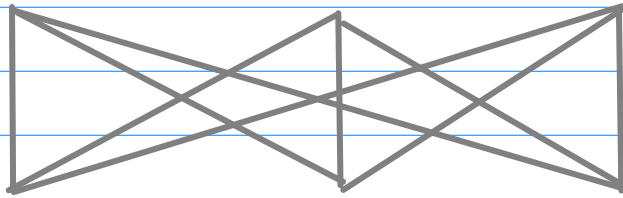
$$\frac{\partial P}{\partial z} \qquad \frac{\partial Q}{\partial z}$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$
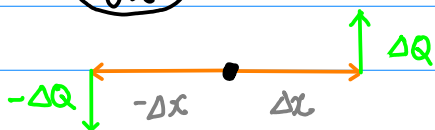
$$\frac{\partial P}{\partial x}$$

$$\frac{\partial Q}{\partial y}$$

$$\frac{\partial R}{\partial z}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)$$

Physical Interpretations

$$\nabla \times \vec{F}$$

a vector

CCW Circulation

$\vec{R}\vec{k}$

$\vec{Q}\vec{j}$

$\vec{P}\vec{i}$

$$\nabla \cdot \vec{F}$$

a scalar

Outward Flux

$\vec{F}$: any vector field

Component Functions of a Vector Field

$$\vec{F} = \boxed{P}\,\vec{i} + \boxed{Q}\,\vec{j}$$

component functions

$$P(x,y) \qquad Q(x,y)$$

$\vec{j}$

$\vec{i}$
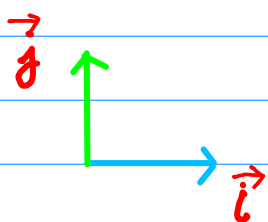
partial derivatives

$$\frac{\partial P}{\partial x} \qquad \frac{\partial Q}{\partial y}$$

$$\frac{\partial P}{\partial y} \qquad \frac{\partial Q}{\partial x}$$

$\vec{i}$ component $\vec{j}$ component $\vec{i}$ component $\vec{j}$ component

$$P(x,y) \qquad Q(x,y) \qquad\qquad P(x,y) \qquad Q(x,y)$$

$$\frac{\partial}{\partial x} \qquad \frac{\partial}{\partial y} \qquad\qquad \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial y}$$

Vector Field & Partial Derivatives

$$\vec{F} = \boxed{P}\,\vec{i} + \boxed{Q}\,\vec{j}$$

Δx

$(x, y+\Delta y)$ $(x+\Delta x, y+\Delta y)$

Δy

Δy

$\boxed{Q(x,y)\,\vec{j}}$

Δx

(x, y) $(x+\Delta x, y)$

$\boxed{P(x,y)\,\vec{i}}$

$\left(\dfrac{\partial P}{\partial x}\right)$

$-\Delta P \quad -\Delta x \quad \Delta x \quad \Delta P$

$\left(\dfrac{\partial Q}{\partial y}\right)$

ΔQ

Δy

$-\Delta y$

$-\Delta Q$

$\left(\dfrac{\partial Q}{\partial x}\right)$

ΔQ

$-\Delta Q \quad -\Delta x \quad \Delta x$

$\left(\dfrac{\partial P}{\partial y}\right)$

ΔP

Δy

$-\Delta y$

$-\Delta P$
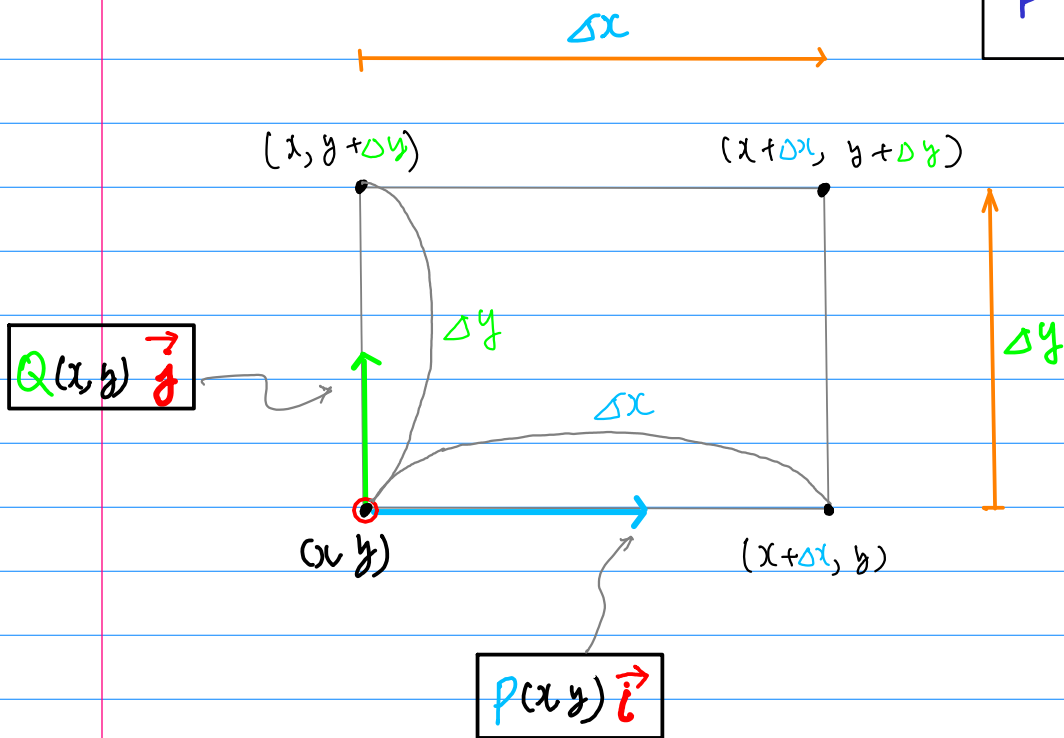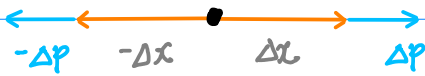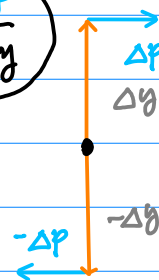
Differentials $\quad$ dP & dQ

$\Delta P \;\vec{i}$
$\Delta x \;\vec{i}$

$\Delta Q \;\vec{j}$
$\Delta y \;\vec{j}$

$$dP = \frac{\partial P}{\partial x}\, dx$$

$$dQ = \frac{\partial Q}{\partial y}\, dy$$

ΔQ
$+\Delta y$
$-\Delta y$
$-\Delta Q$

$-\Delta P \quad\longleftarrow\quad -\Delta x \quad +\Delta x \quad\longrightarrow\quad \Delta P$

$$dQ = \frac{\partial Q}{\partial x}\, dx$$

$$dP = \frac{\partial P}{\partial y}\, dy$$

$-\Delta x \quad \Delta x \quad \Delta Q$
$-\Delta Q$

ΔP
$+\Delta y$
$-\Delta y$
$-\Delta P$
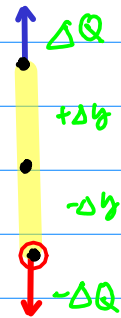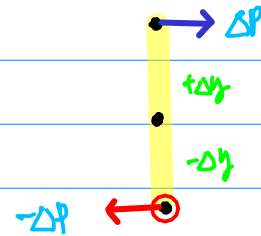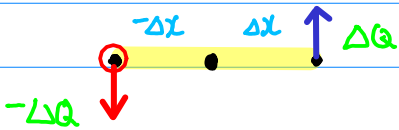
Interpretations of Partial Derivatives

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

outward flux
magnitude only

$$\Delta P \; \vec{i}$$

$$\Delta x \; \vec{i}$$

$$\Delta Q \; \vec{j}$$

$$\Delta y \; \vec{j}$$
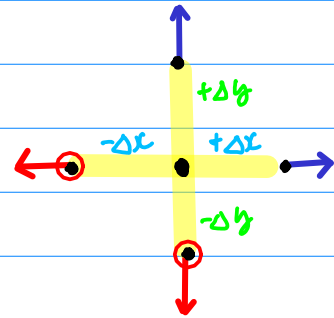
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

C.C.W circulation
magnitude & direction

$$- \quad \Delta P \; \vec{i}$$

$$\Delta x \; \vec{i}$$

$$\Delta Q \; \vec{j}$$

$$\Delta y \; \vec{j}$$

$$\vec{i} \quad \leftarrow \quad \frac{\partial P}{\partial x} \quad + \quad \frac{\partial Q}{\partial y} \quad \rightarrow \quad \vec{j}$$
$$\vec{i} \quad \leftarrow \qquad\qquad\qquad\qquad \rightarrow \quad \vec{j}$$

$$\boxed{\vec{i} \cdot \vec{i} = 1}$$

$$\boxed{\vec{j} \cdot \vec{j} = 1}$$

$$+\Delta y$$
$$-\Delta x \qquad +\Delta x$$
$$-\Delta y$$

$$\vec{j} \quad \leftarrow \quad \frac{\partial Q}{\partial x} \quad - \quad \frac{\partial P}{\partial y} \quad \rightarrow \quad \vec{i}$$
$$\vec{i} \quad \leftarrow \qquad\qquad\qquad\qquad \rightarrow \quad \vec{j}$$

$$\boxed{\vec{i} \times \vec{j} = \vec{k}}$$

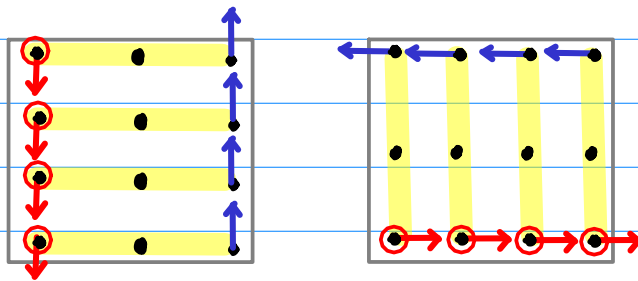$$- \boxed{\vec{j} \times \vec{i} = -\vec{k}}$$

$$+\Delta y$$
$$-\Delta x \qquad \Delta x$$
$$-\Delta y$$

$$\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \Delta x \, \Delta y$$
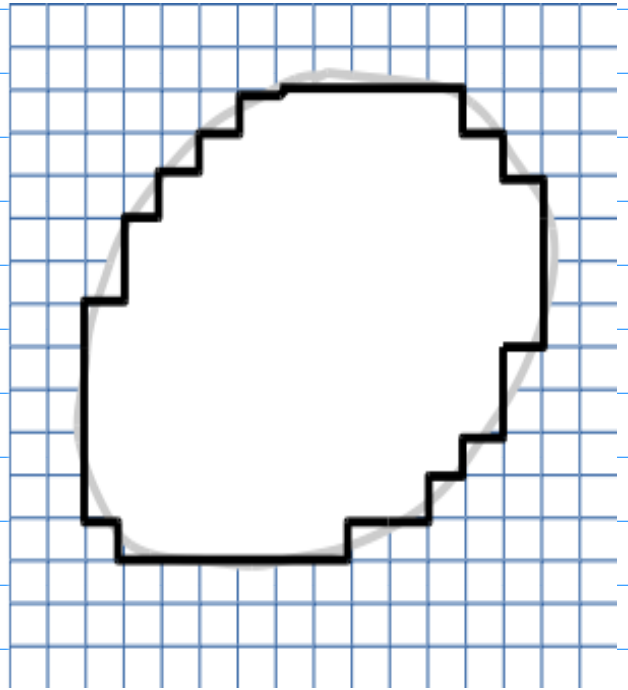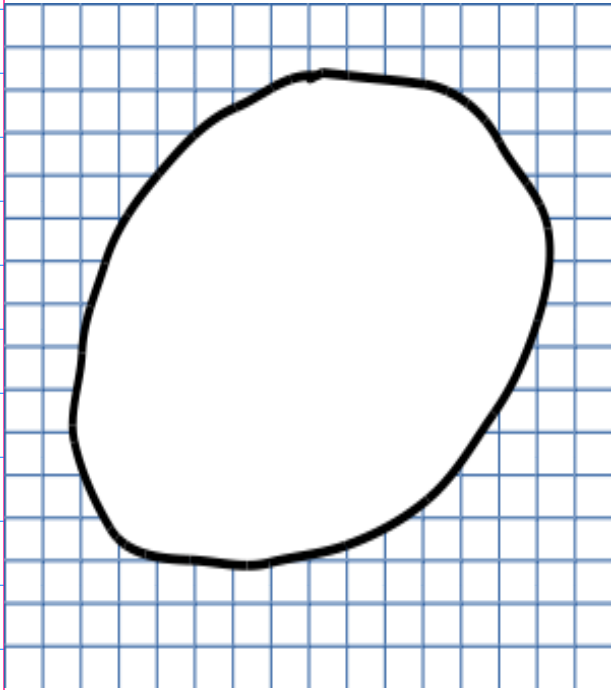

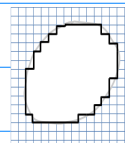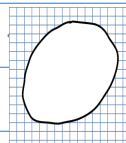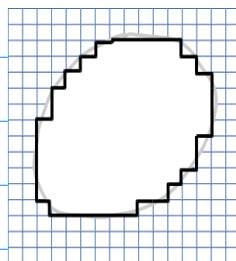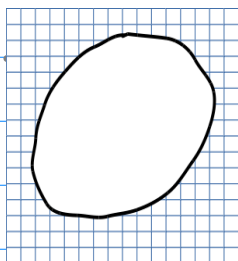$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \Delta x \, \Delta y$$

Approximation   (Digitization)

Approximated by a collection of rectangles

Some Identity Equations

Gradient

Curl of the gradient

$$\nabla \times (\nabla f) = \vec{0} \qquad \text{zero vector}$$

Divergence of the gradient

$$\nabla \cdot (\nabla f) = \boxed{\nabla^2 f} \qquad \text{Laplacian}$$

Curl

Curl of the curl

$$\nabla \times (\nabla \times \vec{F}) = \nabla (\nabla \cdot \vec{F}) - \boxed{\nabla^2 \vec{F}}$$

Vector
Laplacian

Divergence of the curl

$$\nabla \cdot (\nabla \times \vec{F}) = 0 \qquad \text{zero scalar}$$

Div

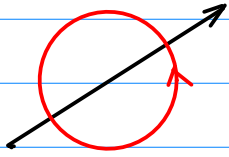### Curl of the divergence $\neq$ Curl (a vector)

$$\nabla \times (\nabla \cdot \vec{F}) \quad \text{✗} \qquad \text{must be } \nabla \times (\text{Vector})$$

Divergence of the divergence $\neq$ Div (a vector)

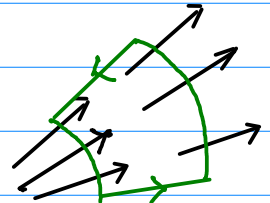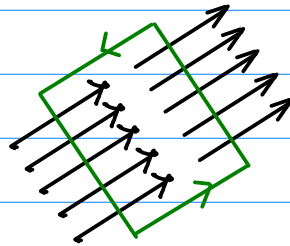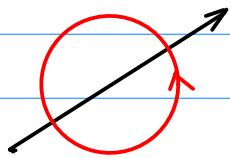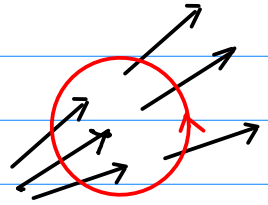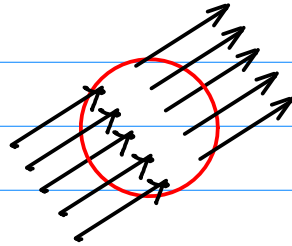$$\nabla \cdot (\nabla \cdot \vec{F}) \quad \text{✗} \qquad \text{must be } \nabla \cdot (\text{Vector})$$

$$\nabla \cdot \vec{F} : \underline{\text{scalar}}$$

curl (grad)

orthogonal

— can always find orthogonal contour
— path independence

Laplacian ∇^2

$$\nabla f = \left\langle \frac{\partial}{\partial x} f, \ \frac{\partial}{\partial y} f, \ \frac{\partial}{\partial z} f \right\rangle$$
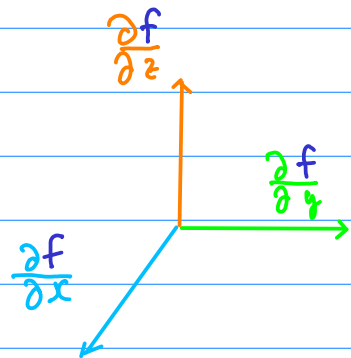
$$\nabla^2 f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f$$

$$\vec{F} = \langle \ P, \ \ Q, \ \ R \ \rangle$$

$$\nabla^2 \vec{F} = \langle \nabla^2 P, \ \nabla^2 Q, \ \nabla^2 R \rangle$$

$$= \left\langle \frac{\partial^2}{\partial x^2} P + \frac{\partial^2}{\partial y^2} P + \frac{\partial^2}{\partial z^2} P, \ \frac{\partial^2}{\partial x^2} Q + \frac{\partial^2}{\partial y^2} Q + \frac{\partial^2}{\partial z^2} Q, \ \frac{\partial^2}{\partial x^2} R + \frac{\partial^2}{\partial y^2} R + \frac{\partial^2}{\partial z^2} R \right\rangle$$
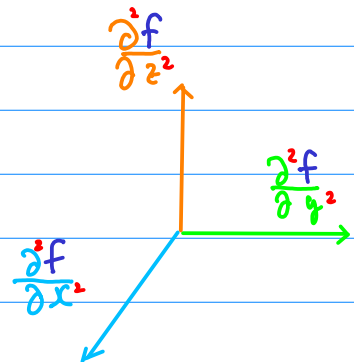
curl (curl)

conservative field   (irrotational)

orthogonal   always $\vec{0}$

curl → all $\vec{0}$ curl → $\vec{0}$
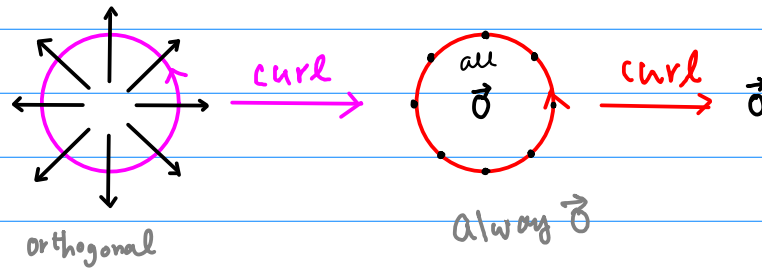
non-conservative field   (rotational)

curl →   curl → 0

in this case
curl becomes
conservative
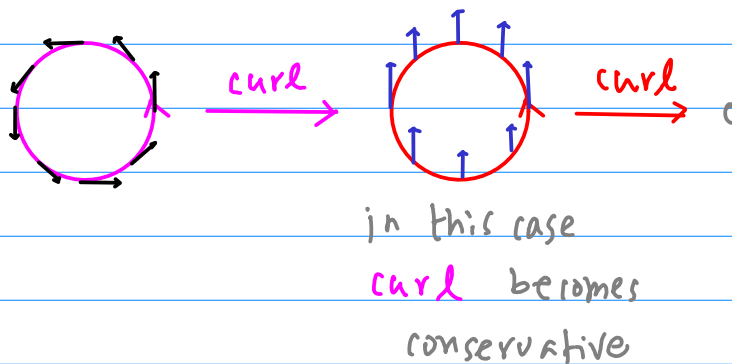
Generally

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

$$\nabla \times (\nabla \times \vec{F}) = (\nabla \cdot \vec{F})\nabla - (\nabla \cdot \nabla)\vec{F}$$

$$= \nabla(\nabla \cdot \vec{F}) - (\nabla \cdot \nabla)\vec{F}$$

$$\vec{F} = y \, \vec{i} - x \, \vec{j}$$

$\text{Curl}(\vec{F}) \neq 0 \qquad \vec{F} \neq \nabla f$
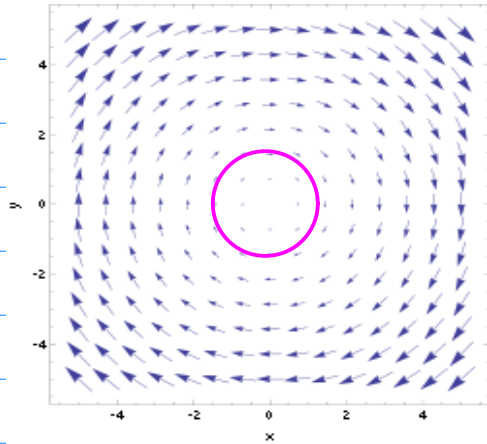
non-gradient
vector field

Can be rotating

✳ Gradient Vector Field

always
$\text{Curl}(\nabla f) = 0$

always
$\text{Curl}(\nabla f) = 0$

conservative
→ irrotational
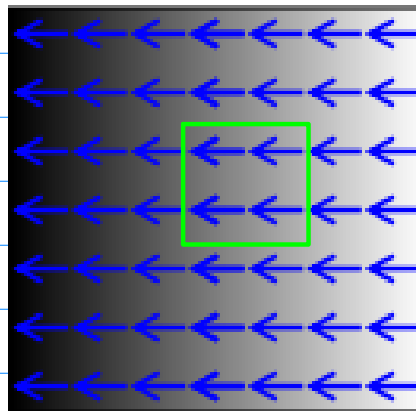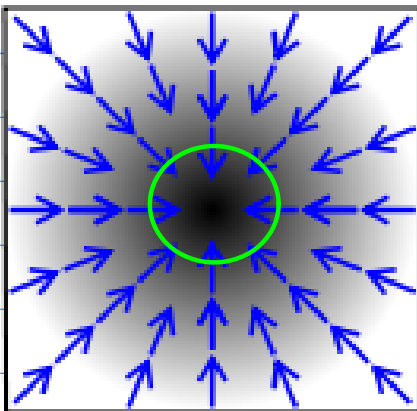
$\nabla^2 f = \text{div}(\nabla f) < 0$

$\nabla^2 f = \text{div}(\nabla f) = 0$

Sink

source ✗
sink ✗

$$\vec{F} = y\,\vec{i} - x\,\vec{j}$$

$$\nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = (-1\ -1)\vec{k}$$
$$= -2\,\vec{k}$$

Vector field $\vec{F}$
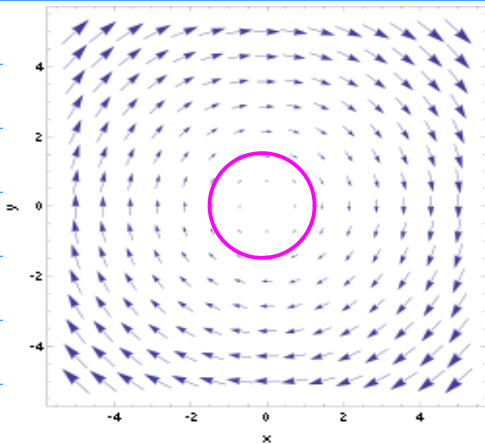
curl $(\vec{F})$
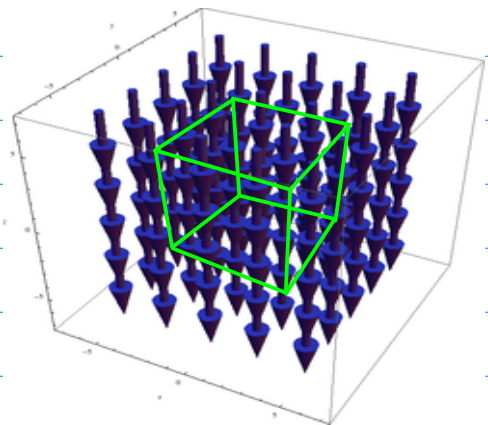
$$\frac{\text{non-gradient}}{\text{vector field}}$$

curl (grad) = 0

div (curl) = 0
always

can't be applied
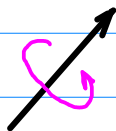
curl $\neq 0$

div (curl) = 0
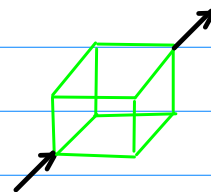
① Curl (grad) = $\vec{0}$

$$Curl\left(\nabla f\right) = \nabla \times \nabla f = \vec{0}$$

$$Curl\left(\vec{F}\right) = \nabla \times \nabla f = \vec{0}$$

$$\vec{F} = \nabla f$$

$$Curl \; \vec{F} = \nabla \times \vec{F}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\vec{F} = \nabla f \implies \quad P = \frac{\partial f}{\partial x} \qquad Q = \frac{\partial f}{\partial y} \qquad R = \frac{\partial f}{\partial z}$$

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \implies \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = 0$$

$$\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \implies \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = 0$$

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial r}{\partial y}\right) \implies \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0$$

Schwarz Theorem   continuous 2nd order   : symmetric
partial derivatives

② $\text{div} (\text{grad}) = \nabla^2 f$

$$\text{div} (\nabla f) = \nabla \cdot \nabla f = \nabla^2 f$$
$$\text{div} (\vec{F}) = \nabla \cdot \nabla f = \nabla^2 f$$

$$\vec{F} = \nabla f$$

$$\vec{F} = \nabla f = \left(\frac{\partial}{\partial x} \vec{\imath} + \frac{\partial}{\partial y} \vec{\jmath} + \frac{\partial}{\partial z} \vec{k} \right) f(x, y, z)$$
$$= \left(\frac{\partial f}{\partial x} \vec{\imath} + \frac{\partial f}{\partial y} \vec{\jmath} + \frac{\partial f}{\partial z} \vec{k} \right)$$

$\boxed{\text{Laplace Operator}}$ $\nabla \cdot \nabla = \nabla^2$

$$\text{div} (\nabla f) = \nabla \cdot \nabla f = \nabla^2 f$$
$$= \left(\frac{\partial}{\partial x} \vec{\imath} + \frac{\partial}{\partial y} \vec{\jmath} + \frac{\partial}{\partial z} \vec{k} \right) \cdot \left(\frac{\partial f}{\partial x} \vec{\imath} + \frac{\partial f}{\partial y} \vec{\jmath} + \frac{\partial f}{\partial z} \vec{k} \right)$$
$$= \boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}}$$
$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f$$
$$= \left(\frac{\partial^2}{\partial x^2} f(x,y,z) + \frac{\partial^2}{\partial y^2} f(x,y,z) + \frac{\partial^2}{\partial z^2} f(x,y,z) \right)$$

$$\text{③} \quad \text{curl (curl)} = \nabla(\nabla \cdot \vec{F}) - \boxed{\nabla^2 \vec{F}}$$

$$\nabla \times (\nabla \times \vec{F})$$

$$= \nabla \times \left(\vec{B} \right)$$

$$\vec{B} = \nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\nabla \times (\nabla \times \vec{F}) = \nabla \times \vec{B}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) & \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) & \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \right]\vec{i}$$

$$- \left[\frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \right]\vec{j}$$

$$\left[\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \right]\vec{k}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial x}\right) - \boxed{\frac{\partial^2 P}{\partial y^2}} - \boxed{\frac{\partial^2 P}{\partial z^2}} + \frac{\partial}{\partial z}\left(\frac{\partial R}{\partial x}\right) \right]\vec{i}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right) - \boxed{\frac{\partial^2 Q}{\partial x^2}} - \boxed{\frac{\partial^2 Q}{\partial z^2}} + \frac{\partial}{\partial z}\left(\frac{\partial R}{\partial y}\right) \right]\vec{j}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial z}\right) - \boxed{\frac{\partial^2 R}{\partial x^2}} - \frac{\partial^2 R}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial z}\right) \right]\vec{k}$$

$$\nabla \times (\nabla \times \vec{F}) - \nabla(\nabla \cdot \vec{F}) = -\nabla^2 \vec{F}$$

$$\nabla \cdot \vec{F} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)$$

$$\nabla(\nabla \cdot \vec{F}) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \right) \cdot \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)\vec{i} +$$

$$\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)\vec{j} +$$

$$\frac{\partial}{\partial z}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right)\vec{k}$$

$$= \left(\boxed{\frac{\partial^2 P}{\partial x^2}} + \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial z}\right) \right)\vec{i} +$$

$$\left(\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right) + \boxed{\frac{\partial^2 Q}{\partial y^2}} + \frac{\partial}{\partial y}\left(\frac{\partial R}{\partial z}\right) \right)\vec{j} +$$

$$\left(\frac{\partial}{\partial z}\left(\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial y}\right) + \boxed{\frac{\partial^2 R}{\partial z^2}} \right)\vec{k}$$

$$\nabla \times (\nabla \times \vec{F}) - \nabla(\nabla \cdot \vec{F})$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial x}\right) - \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\partial R}{\partial x}\right) \right] \vec{i}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right) - \frac{\partial^2 Q}{\partial x^2} - \frac{\partial^2 Q}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{\partial R}{\partial y}\right) \right] \vec{j}$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial z}\right) - \frac{\partial^2 R}{\partial x^2} - \frac{\partial^2 R}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial Q}{\partial z}\right) \right] \vec{k}$$

$$- \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial z}\right) \right) \vec{i}$$

$$- \left(\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right) + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial R}{\partial z}\right) \right) \vec{j}$$

$$- \left(\frac{\partial}{\partial z}\left(\frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial y}\right) + \frac{\partial^2 R}{\partial z^2} \right) \vec{k}$$

$$= -\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} \right) \vec{i}$$

$$-\left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2} \right) \vec{j}$$

$$-\left(\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 R}{\partial z^2} \right) \vec{k}$$

$$= -\nabla^2 \vec{F}$$

$$\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

④ div (curl) = 0

$$\text{Curl } \vec{F} = \nabla \times \vec{F}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$\text{div}\left(\text{Curl } \vec{F}\right) = \nabla \cdot \left(\nabla \times \vec{F}\right)$$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot$$

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= 0$$

Laplacian Operator

$$\nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k} = \qquad \text{vector}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla \cdot \nabla \qquad \text{scalar}$$

$$\nabla \,\square = \frac{\partial \square}{\partial x}\vec{\imath} + \frac{\partial \square}{\partial y}\vec{\jmath} + \frac{\partial \square}{\partial z}\vec{k} \qquad \text{vetor}$$

$$\nabla^2 \,\square = \frac{\partial^2 \square}{\partial x^2} + \frac{\partial^2 \square}{\partial y^2} + \frac{\partial^2 \square}{\partial z^2} \qquad \text{scalar}$$

$$\boxed{\nabla^2 f = \nabla \cdot \nabla f}$$

Laplacian Operator

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla \cdot \nabla = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$(\nabla \cdot \nabla)f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$\nabla \cdot (\nabla f) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 f = (\nabla \cdot \nabla)f = \nabla \cdot (\nabla f)$$

$$\nabla^2 \vec{F} = \nabla^2 \left(\boxed{P}\vec{i} + \boxed{Q}\vec{j} + \boxed{R}\vec{k} \right)$$

$$= \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} \right) \vec{i}$$

$$+ \left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2} \right) \vec{j}$$

$$+ \left(\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 R}{\partial z^2} \right) \vec{k}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left(\boxed{P}\vec{i} + \boxed{Q}\vec{j} + \boxed{R}\vec{k} \right)$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \boxed{P}\vec{i} +$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \boxed{Q}\vec{j} +$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \boxed{R}\vec{k}$$

$$= \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} \right) \vec{i}$$

$$+ \left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2} \right) \vec{j}$$

$$+ \left(\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 R}{\partial z^2} \right) \vec{k}$$

Vector Laplacian

$$\nabla^2 f = (\nabla \cdot \nabla) f = \nabla \cdot (\nabla f) \qquad \text{div}(\text{grad}(f))$$

$$\nabla^2 \,\blacksquare = \left(\frac{\partial^2 \blacksquare}{\partial x^2} + \frac{\partial^2 \blacksquare}{\partial y^2} + \frac{\partial^2 \blacksquare}{\partial z^2} \right)$$

$$\nabla^2 \vec{F} = \nabla^2 \left(\boxed{P}\vec{i} + \boxed{Q}\vec{j} + \boxed{R}\vec{k} \right)$$

$$= \left(\frac{\partial^2 \blacksquare}{\partial x^2} + \frac{\partial^2 \blacksquare}{\partial y^2} + \frac{\partial^2 \blacksquare}{\partial z^2} \right) \left(\boxed{P}\vec{i} + \boxed{Q}\vec{j} + \boxed{R}\vec{k} \right)$$

$$\nabla^2 \vec{F} = \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} \right)\vec{i} \qquad \text{div}(\text{grad}(P))\,\vec{i}$$
$$+ \left(\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2} \right)\vec{j} \qquad + \text{div}(\text{grad}(Q))\,\vec{j}$$
$$+ \left(\frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 R}{\partial z^2} \right)\vec{k} \qquad + \text{div}(\text{grad}(R))\,\vec{k}$$

$$\nabla^2 f = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right) \qquad \text{div}(\text{grad}(f))$$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla \bigcirc = \frac{\partial}{\partial x}\bigcirc\vec{i} + \frac{\partial}{\partial y}\bigcirc\vec{j} + \frac{\partial}{\partial z}\bigcirc\vec{k} \qquad \bigcirc \quad f(x,y,z)$$

$$\nabla^2 \bigcirc = \frac{\partial^2}{\partial x^2}\bigcirc + \frac{\partial^2}{\partial y^2}\bigcirc + \frac{\partial^2}{\partial z^2}\bigcirc \qquad \bigcirc \quad f(x,y,z)$$

$$\nabla^2 \square = \frac{\partial^2}{\partial x^2}\square + \frac{\partial^2}{\partial y^2}\square + \frac{\partial^2}{\partial z^2}\square \qquad \square \quad P\vec{i} + Q\vec{j} + R\vec{k}$$

$$\begin{array}{ccc}
\dfrac{\partial^2 P}{\partial x^2} & \dfrac{\partial^2 Q}{\partial x^2} & \dfrac{\partial^2 R}{\partial x^2} \\[2ex]
\dfrac{\partial^2 P}{\partial y^2} & \dfrac{\partial^2 Q}{\partial y^2} & \dfrac{\partial^2 R}{\partial y^2} \\[2ex]
\dfrac{\partial^2 P}{\partial z^2} & \dfrac{\partial^2 Q}{\partial z^2} & \dfrac{\partial^2 R}{\partial z^2} \\[2ex]
\vec{i} & \vec{j} & \vec{k}
\end{array}$$

Vector Laplacian

$$= -\,\mathrm{Curl}\left(\mathrm{Curl}\ \vec{F}\right) + \mathrm{grad}\left(\mathrm{Div}\ \vec{F}\right)$$

$$\begin{array}{ccc}
\dfrac{\partial P}{\partial x} & \dfrac{\partial Q}{\partial x} & \dfrac{\partial R}{\partial x} \\[2ex]
\dfrac{\partial P}{\partial y} & \dfrac{\partial Q}{\partial y} & \dfrac{\partial R}{\partial y} \\[2ex]
\dfrac{\partial P}{\partial z} & \dfrac{\partial Q}{\partial z} & \dfrac{\partial R}{\partial z}
\end{array}$$

$\longrightarrow$ Curl $\vec{F}$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\begin{aligned}
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \\
&= (\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla)\mathbf{A} - (\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla)\mathbf{B} \\
&= \nabla \cdot (\mathbf{B}\mathbf{A}^{\mathrm{T}}) - \nabla \cdot (\mathbf{A}\mathbf{B}^{\mathrm{T}}) \\
&= \nabla \cdot (\mathbf{B}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{B}^{\mathrm{T}})
\end{aligned}$$

- The scalar triple product is invariant under a circular shift of its three operands (**a**, **b**, **c**):

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

- Swapping the positions of the operators without re-ordering the operands leaves the triple product unchanged. This follows from the preceding property and the commutative property of the dot product.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

- Swapping any two of the three operands negates the triple product. This follows from the circular-shift property and the anticommutativity of the cross product.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$$
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$$
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$$

- The scalar triple product can also be understood as the determinant of the 3 × 3 matrix (thus also its inverse) having the three vectors either as its rows or its columns (a matrix has the same determinant as its transpose):

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

- If the scalar triple product is equal to zero, then the three vectors **a**, **b**, and **c** are coplanar, since the "parallelepiped" defined by them would be flat and have no volume.

- If any two vectors in the triple scalar product are equal, then its value is zero:

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{a} \times \mathbf{a}) = 0$$

- Moreover,

$$[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]\mathbf{a} = (\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c})$$

- The simple product of two triple products (or the square of a triple product), may be expanded in terms of dot products:[1]

$$((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c})\, ((\mathbf{d} \times \mathbf{e}) \cdot \mathbf{f}) = \det \left[\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{d} & \mathbf{e} & \mathbf{f} \end{pmatrix} \right] = \det \left[\right.$$

The **vector triple product** is defined as the cross product of one vector with the cross product of the other two. The following relationship holds:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

This is known as **triple product expansion**, or **Lagrange's formula**,[2][3] although the latter name is also used for several other formulae. Its right hand side can be remembered by using the mnemonic "BAC − CAB", provided one keeps in mind which vectors are dotted together. A proof is provided below.

Since the cross product is anticommutative, this formula may also be written (up to permutation of the letters) as:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -(\mathbf{c} \cdot \mathbf{b})\mathbf{a} + (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

From Lagrange's formula it follows that the vector triple product satisfies:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$$

which is the Jacobi identity for the cross product. Another useful formula follows:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \times (\mathbf{a} \times \mathbf{c})$$
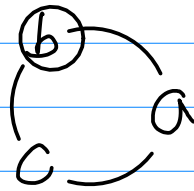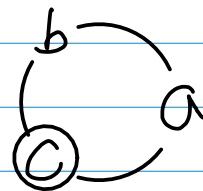
These formulas are very useful in simplifying vector calculations in physics. A related identity regarding gradients and useful in vector calculus is Lagrange's formula of vector cross-product identity:[4]

$$\nabla \times (\nabla \times \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - (\nabla \cdot \nabla)\mathbf{f}$$

$(a \times b) \times c = a \times (b \times c) - b \times (a \times c)$

$$\vec{a} \times (\vec{b} \times \vec{c})$$
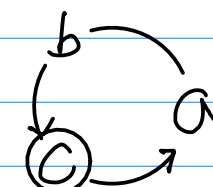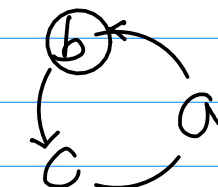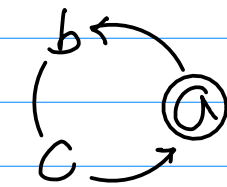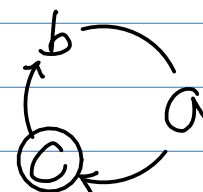
$$\vec{b} \times (\vec{c} \times \vec{a})$$

$$\vec{c} \times (\vec{a} \times \vec{b})$$

$$b(a \cdot c) \qquad c(b \cdot a) \qquad a(c \cdot b)$$
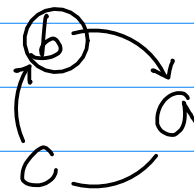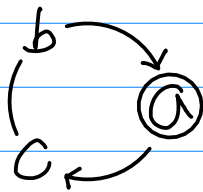
$$-c(a \cdot b) \qquad -a(b \cdot c) \qquad -b(c \cdot a)$$

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$$

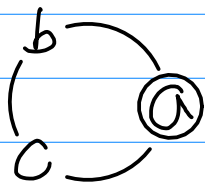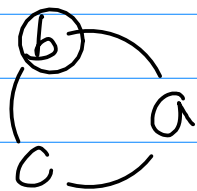$$\parallel \qquad\qquad \parallel \qquad\qquad \parallel$$

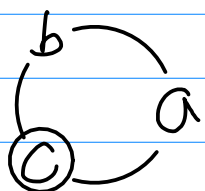$$b(a \cdot c) \qquad c(b \cdot a) \qquad a(c \cdot b)$$

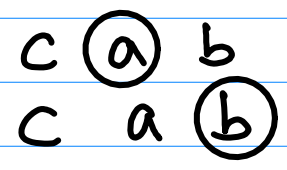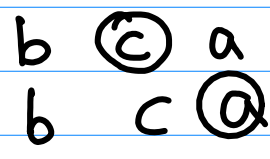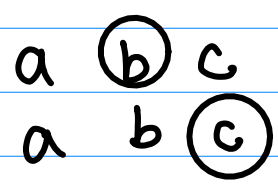$$-c(a \cdot b) \qquad -a(b \cdot c) \qquad -b(c \cdot a)$$

$$\vec{a} \times (\vec{b} \times \vec{c})$$

$$\vec{b} \times (\vec{c} \times \vec{a})$$

$$\vec{c} \times (\vec{a} \times \vec{b})$$

a ⓑ c   b ⓒ a   c ⓐ b

a b ⓒ   b c ⓐ   c a ⓑ

$$b(a \cdot c)$$
$$-c(a \cdot b)$$

$$c(b \cdot a)$$
$$-a(b \cdot c)$$

$$a(c \cdot b)$$
$$-b(c \cdot a)$$

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c})} + \boxed{\vec{b} \times (\vec{c} \times \vec{a})} + \boxed{\vec{c} \times (\vec{a} \times \vec{b})} = 0$$

$$\parallel \qquad\qquad \parallel \qquad\qquad \parallel$$

$b(a \cdot c)$	$c(b \cdot a)$	$a(c \cdot b)$
$-c(a \cdot b)$	$-a(b \cdot c)$	$-b(c \cdot a)$

$$-\boxed{\vec{a} \times (\vec{b} \times \vec{c})} - \boxed{\vec{b} \times (\vec{c} \times \vec{a})} = \boxed{\vec{c} \times (\vec{a} \times \vec{b})}$$

$$\parallel \qquad\qquad \parallel \qquad\qquad \parallel$$

$- \quad b(a \cdot c)$	$c(b \cdot a)$	$a(c \cdot b)$
$-c(a \cdot b)$	$- \quad -a(b \cdot c)$	$-b(c \cdot a)$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$$
$$= (\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla)\mathbf{A} - (\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla)\mathbf{B}$$
$$= \nabla \cdot (\mathbf{B}\mathbf{A}^{\mathrm{T}}) - \nabla \cdot (\mathbf{A}\mathbf{B}^{\mathrm{T}})$$
$$= \nabla \cdot (\mathbf{B}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{B}^{\mathrm{T}})$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \times (\mathbf{a} \times \mathbf{c})$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -(\mathbf{c} \cdot \mathbf{b})\mathbf{a} + (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

$$\begin{aligned}
\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \\
&= (\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla)\mathbf{A} - (\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla)\mathbf{B} \\
&= \nabla \cdot (\mathbf{B}\mathbf{A}^{\mathrm{T}}) - \nabla \cdot (\mathbf{A}\mathbf{B}^{\mathrm{T}}) \\
&= \nabla \cdot (\mathbf{B}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{B}^{\mathrm{T}})
\end{aligned}$$

$$\nabla \times (A \times B) = \nabla \times (A \times B) + \nabla \times (A \times B)$$

$$\begin{aligned}
\nabla \times (A \times B) &= A(\nabla \cdot B) - B(\nabla \cdot A) \\
&= (B \cdot \nabla)A - B(\nabla \cdot A)
\end{aligned}$$

$$\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A)$$
$$(B \cdot \nabla)A - (A \cdot \nabla)B$$

$$\begin{aligned}
\nabla \times (A \times B) &= A(\nabla \cdot B) - B(\nabla \cdot A) \\
&\quad + (B \cdot \nabla)A - (A \cdot \nabla)B
\end{aligned}$$

Directional Derivatives

$$B \cdot \nabla = \langle B_x, B_y, B_z \rangle \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle$$

$$= B_x \tfrac{\partial}{\partial x} + B_y \tfrac{\partial}{\partial y} + B_z \tfrac{\partial}{\partial z}$$

$$\neq \tfrac{\partial B_x}{\partial x} + \tfrac{\partial B_y}{\partial y} + \tfrac{\partial B_z}{\partial z}$$

$$(B \cdot \nabla) f = \langle B_x, B_y, B_z \rangle \cdot \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle f$$

$$= B_x \tfrac{\partial f}{\partial x} + B_y \tfrac{\partial f}{\partial y} + B_z \tfrac{\partial f}{\partial z}$$

$$= B \cdot (\nabla f) \qquad f_B$$
$$\text{grad}$$

$$\vec{u} = \langle a, b, c \rangle$$

$$D_{\vec{u}} f(x,y,z) = a \cdot f_x(x,y,z) + b \cdot f_y(x,y,z) + c \cdot f_z(x,y,z)$$

$$= \langle f_x, f_y, f_z \rangle \cdot \langle a, b, c \rangle$$

$$B \cdot \nabla = \quad B_x, \; B_y, \; B_z$$

$$= \quad B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}$$

$$\neq \quad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

$$\nabla \, \square = \frac{\partial \square}{\partial x} \vec{\imath} + \frac{\partial \square}{\partial y} \vec{\jmath} + \frac{\partial \square}{\partial z} \vec{k}$$

(Op)
$$B \cdot \nabla = \left(B_x \vec{\imath} + B_y \vec{\jmath} + B_z \vec{k} \right) \cdot \left(\frac{\partial \square}{\partial x} \vec{\imath} + \frac{\partial \square}{\partial y} \vec{\jmath} + \frac{\partial \square}{\partial z} \vec{k} \right)$$

$$= B_x \frac{\partial \square}{\partial x} + B_y \frac{\partial \square}{\partial y} + B_z \frac{\partial \square}{\partial z}$$

(Scalar)
$$\nabla \cdot B = \left(\frac{\partial \square}{\partial x} \vec{\imath} + \frac{\partial \square}{\partial y} \vec{\jmath} + \frac{\partial \square}{\partial z} \vec{k} \right) \cdot \left(B_x \vec{\imath} + B_y \vec{\jmath} + B_z \vec{k} \right)$$

$$= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

Dot Del Operator $\dot{\nabla}$

Op

$$B \cdot \nabla = B_x \frac{\partial \square}{\partial x} + B_y \frac{\partial \square}{\partial y} + B_z \frac{\partial \square}{\partial z}$$

Scalar

$$\nabla \cdot B = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

$$B \cdot \nabla \neq \nabla \cdot B$$

$$B \cdot \nabla = \dot{\nabla} \cdot B$$

dot del