Z Transform (H.2) Inverse

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Based on
Complex Analysis for Mathematics and Engineering
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z - Transform

$$X(z) = \sum_{k=-\infty}^{+\infty} x[k]\, z^{-k}$$

$$z = |r|\, e^{j2\pi F} = |r|\, e^{j\Omega}$$

$$x[n] \longleftrightarrow X(z)$$

One Sided z-transform

$$X(z) = \sum_{k=0}^{+\infty} x[k]\, z^{-k}$$

Inverse z-Transform

$$X(z) = \mathcal{Z}\left[\{x_n\}_{n=0}^{\infty}\right]$$

$$x_n = x[n] = \mathcal{Z}^{-1}\left[X(z)\right]$$

$$\boxed{\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x[n]\, z^{-n} \\ x[n] &= \frac{1}{2\pi i}\int_C X(z)\, z^{n-1}\, dz \end{aligned}}$$

$$x[n] = \frac{1}{2\pi i}\int_C X(z)\, z^{n-1}\, dz$$

$$X(z) = \sum_{k=0}^{\infty} x_k\, z^{-k}$$

$$z^{n-1} X(z) = \left(\sum_{k=0}^{\infty} x_k\, z^{-k}\right) z^{n-1} \qquad \boxed{\int z^{n-1}\, \mathrm{LHS}\, dz = \int \mathrm{RHS}\, z^{n-1}\, dz}$$

$$= \sum_{k=0}^{\infty} x_k\, z^{-k+n-1} \qquad \boxed{[0,\infty) = [0, n-1] \cup [n] \cup [n+1, \infty)}$$

$$= \sum_{k=0}^{n-1} x_k\, z^{-k+n-1} + \sum_{k=n}^{n} x_k\, z^{-k+n-1} + \sum_{k=n+1}^{\infty} x_k\, z^{-k+n-1}$$

$$= \sum_{k=0}^{n-1} x_k\, z^{-k+n-1} + \frac{x_k}{z^1} + \sum_{k=n+1}^{\infty} \frac{x_k}{z^{k-n+1}}$$

$$\int_c X(z)\, z^{n-1}\, dz = \int_c \sum_{k=0}^{n-1} x_k\, z^{-k+n-1}\, dz + \int_c \frac{x_k}{z^1}\, dz + \int_c \sum_{k=n+1}^{\infty} \frac{x_k}{z^{k-n+1}}\, dz$$

$$= \sum_{k=0}^{n-1} x_k \int_c z^{-k+n-1}\, dz + x_k \int_c \frac{1}{z^1}\, dz + \sum_{k=n+1}^{\infty} x_k \int_c \frac{1}{z^{k-n+1}}\, dz$$

$$= \sum_{k=0}^{n-1} x_k \cdot 0 + x_k \cdot 2\pi i + \sum_{k=n+1}^{\infty} x_k \cdot 0$$

$$x_k = \frac{1}{2\pi i}\int_c X(z)\, z^{n-1}\, dz$$

D: simply connected domain

C: simple closed contour (CCW) in D

if $F(z)$ is analytic inside C and on C

except at the points $z_1, z_2, \cdots, z_k$ in C

then

$$\frac{1}{2\pi i}\int_C F(z)\,dz = \sum_{j=1}^{k} \mathrm{Res}\left(F(z), z_j\right)$$

$$\frac{1}{2\pi i}\int_C \boxed{X(z)\,z^{n-1}}\,dz = \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z)\,z^{n-1}}, z_j\right)$$

$$x[n] = \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z)\,z^{n-1}}, z_j\right)$$

Residues at Poles

If $F(z)$ has a simple pole at z_0

$$\mathrm{Res}(F(z), z_0) = \lim_{z \to z_0} (z - z_0) F(z)$$

If $F(z)$ has a pole of order 2 at z_0

$$\mathrm{Res}(F(z), z_0) = \lim_{z \to z_0} \frac{d}{dz}\left((z - z_0)^2 F(z)\right)$$

If $F(z)$ has a pole of order 3 at z_0

$$\mathrm{Res}(F(z), z_0) = \frac{1}{2!} \lim_{z \to z_0} \frac{d^2}{dz^2}\left((z - z_0)^3 F(z)\right)$$

$$x[n] = \frac{1}{2\pi i} \int_C \boxed{X(z) z^{n-1}} \, dz = \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z) z^{n-1}} \; z_j\right)$$

simple pole z_j $\qquad \lim_{z \to z_j} \frac{d}{dz}\left[(z - z_j) X(z) z^{n-1}\right]$

m-th order pole z_j $\qquad \frac{1}{(m-1)!} \lim_{z \to z_j} \frac{d^{m-1}}{dz^{m-1}}\left[(z - z_j)^m X(z) z^{n-1}\right]$

"Transforms and Applications Handbook"

① power series method

② Partial Fraction Expansion

③ Inverse Transform by Integration

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} \, dz = \sum_{i=1}^{k} \mathrm{Res}\left(X(z) z^{n-1}, z_i \right)$$

④ Simple Poles

$$\lim_{z \to a} (z-a) X(z) z^{n-1}$$

⑤ Multiple Poles   m-th order

$$\lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m X(z) z^{n-1} \right]$$

⑥ Simple Poles Not Factorable

$$\left. \frac{F(z)}{\frac{dG(z)}{dz}} z^{n-1} \right|_{z=a} \qquad X(z) = \frac{F(z)}{G(z)}$$

⑦ Irational Function

eg) $X(z) = \left(\frac{z+1}{z} \right)^{\alpha}$

Power Series Method

$$X(z) = \frac{1+z^{-1}}{1+2z^{-1}+3z^{-2}} = \frac{z^2+z}{z^2+2z+3}$$

$$
\begin{array}{r l}
 & 1 - z^{-1} - z^{-2} + 5z^{-3} \cdots \\
\hline
z^2+2z+3 \,\big) & z_2 + z \\
 & \underline{z^2 + 2z + 3} \\
 & -z - 3 \\
 & \underline{-z - 2 - 3z^{-1}} \\
 & -1 + 3z^{-1} \\
 & \underline{-1 - 2z^{-1} - 3z^{-2}} \\
 & 5z^{-1} + 3z^{-2} \\
 & \underline{5z^{-1} + 10z^{-2} + 15} \\
 & -7z^{-2} - 15 \cdots
\end{array}
$$

$$X(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + \cdots + p_n z^{-n}}{q_0 + q_1 z^{-1} + q_2 z^{-2} + \cdots + q_n z^{-n}}$$

$$= x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

Partial Fraction

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{z^2 + 2z + 1}{z^2 - \frac{3}{2}z + \frac{1}{2}} = \frac{z^2 - \frac{3}{2}z + \frac{1}{2} + \frac{7}{2}z + \frac{1}{2}}{z^2 - \frac{3}{2}z + \frac{1}{2}}$$

$$= 1 + \frac{\frac{7}{2}z + \frac{1}{2}}{z^2 - \frac{3}{2}z + \frac{1}{2}}$$

$$\frac{\frac{7}{2}z + \frac{1}{2}}{z^2 - \frac{3}{2}z + \frac{1}{2}} = \frac{\frac{7}{2}z + \frac{1}{2}}{(z-1)(z-\frac{1}{2})} = \frac{A}{(z-1)} + \frac{B}{(z-\frac{1}{2})}$$

$$A = (z-1)\frac{\frac{7}{2}z + \frac{1}{2}}{(z-1)(z-\frac{1}{2})}\Bigg|_{z=1} = \frac{\frac{7}{2}+\frac{1}{2}}{1-\frac{1}{2}} = 8$$

$$B = (z-\tfrac{1}{2})\frac{\frac{7}{2}z + \frac{1}{2}}{(z-1)(z-\frac{1}{2})}\Bigg|_{z=\frac{1}{2}} = \frac{\frac{7}{4}+\frac{1}{2}}{\frac{1}{2}-1} = -\frac{9}{2}$$

$$X(z) = 1 + \frac{8}{(z-1)} - \frac{9}{2}\frac{1}{(z-\frac{1}{2})}$$

$$= 1 + \frac{8z}{(z-1)}z^{-1} - \frac{9}{2}\frac{z}{(z-\frac{1}{2})}z^{-1}$$

$$x[n] = \delta[n] + 8u[n-1] - \frac{9}{2}\left(\frac{1}{2}\right)^{n-1}u[n-1]$$

$$\delta[n] \longleftrightarrow 1$$

$$u[n] \longleftrightarrow \frac{z}{z-1}$$

$$b^n \longleftrightarrow \frac{z}{z-b}$$

$$b^{n-1}u[n-1] \longleftrightarrow \frac{1}{z-b}$$

$$x_{n-1}\, u[n-1] \longleftrightarrow z^{-1}X(z)$$

$$x_{n-m}\, u[n-m] \longleftrightarrow z^{-m}X(z)$$

Partial Fraction in general

$$X(z) = X_1(z) + X_2(z) + X_3(z) + \cdots$$

$$X_i(z) = \frac{F(z)}{(z-p)^n} = \frac{A_1}{(z-p)} + \frac{A_2}{(z-p)^2} + \cdots + \frac{A_n}{(z-p)^n}$$

$$A_n = (z-p)^n F(z) \Big|_{z=p}$$

$$A_{n-1} = \frac{d}{dz}\left[(z-p)^n F(z)\right]\Big|_{z=p}$$

$$A_{n-k} = \frac{1}{k!}\frac{d^k}{dz^k}\left[(z-p)^n F(z)\right]\Big|_{z=p}$$

$$A_1 = A_{n-(n-1)} = \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-p)^n F(z)\right]\Big|_{z=p}$$

$$X(z) = X_1(z) + X_2(z) + X_3(z) + \cdots$$

$$X_i(z) = \frac{F(z)}{(z-p)^n} = \frac{A_1}{(z-p)} + \frac{A_2}{(z-p)^2} + \cdots + \frac{A_n}{(z-p)^n}$$

$$\frac{A_n}{(z-p)^n} = \left\{ (z-p)^n F(z) \Big|_{z=p} \right\} \cdot \frac{1}{(z-p)^n}$$

$$\frac{A_{n-1}}{(z-p)^{n-1}} = \left\{ \frac{d}{dz}\left[(z-p)^n F(z)\right] \Big|_{z=p} \right\} \frac{1}{(z-p)^{n-1}}$$

$$\frac{A_{n-k}}{(z-p)^{n-k}} = \left\{ \frac{1}{k!} \frac{d^k}{dz^k}\left[(z-p)^n F(z)\right] \Big|_{z=p} \right\} \frac{1}{(z-p)^{n-k}}$$

$$\frac{A_{n-(n-1)}}{(z-p)^{n-(n-1)}} = \left\{ \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}\left[(z-p)^n F(z)\right] \Big|_{z=p} \right\} \frac{1}{(z-p)^{n-(n-1)}}$$