Random Process Background (1C)

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Jun 01, 2024

Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Open Sets and Neighborhoods
Filters
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Open set examples

- The *circle* represents the set of points (x, y)
  satisfying $x^2 + y^2 = r^2$.

  the *circle* set is its **boundary set**

- The *disk* represents the set of points (x, y)
  satisfying $x^2 + y^2 < r^2$.

  The *disk* set is an **open set**

- the union of the *circle* and *disk* sets

  is a **closed set**.

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Open set (1)

- an **open set** is a generalization of
  an **open interval** in the real line.

- a **metric space** is a set along with a distance
  defined between any two points

- in a **metric space**,
  an **open set** is a set that, along with every point P,
  contains all points that are sufficiently near to P
  - all points whose distance to P is
    less than some value depending on P

https://en.wikipedia.org/wiki/Open_set

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Open set (2-1)

- more generally, an **open set** is
  a member of a given collection of subsets of a given set

  - a given set

  - subsets of a given set

  - a given collection of subsets of a given set

  https://en.wikipedia.org/wiki/Open_set

Open set (2-2)

- a **collection** has the following property of containing

  - a **collection** contains
    - every **union** of its members
    - every finite **intersection** of its members
    - the **empty set**
    - the **whole set** itself

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Open set (3)

- These conditions are very <u>loose</u>,
  and allow enormous <u>flexibility</u> in the <u>choice</u> of open sets.

- For example,
    - <u>every</u> subset can be open (the **discrete topology**)
    - <u>no</u> subset can be open (the **indiscrete topology**)
      <u>except</u>
        - the space itself and
        - the empty set

    https://en.wikipedia.org/wiki/Open_set

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Open set (4)

- A set in which such a collection is given
  is called a **topological space**, and
  the collection is called a **topology**.

  - A **set** is a collection of distinct objects.

  - Given a **set** A, we say that a is an **element** of A

    if a is one of the distinct objects in A,
    and we write $a \in A$ to denote this

  - Given two **sets** A and B, we say that A is a **subset** of B

    if every element of A is also an element of B
    write $A \subseteq B$ to denote this.

Open set (5) Open Balls

- An **open ball** $B_r(\boldsymbol{a})$ in $\mathbb{R}^n$
  centered at $\boldsymbol{a} = (a_1, \ldots a_n) \in \mathbb{R}^n$ with radius r

  is the set of all points $\boldsymbol{x} = (x_1, \ldots x_n) \in \mathbb{R}^n$
  such that the distance between x and $\boldsymbol{a}$ is less than r

- In $\mathbb{R}^2$ an **open ball** is often called an **open disk**

  We give these definitions in general, for when one is working in $\mathbb{R}^n$
  since they are really not all that different to define in $\mathbb{R}^n$ than in $\mathbb{R}^2$

  https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAndC

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Open set (6) Interior points

- Suppose that $S \subseteq \mathbb{R}^n$

- A **point** $\boldsymbol{p} \in S$ is an **interior point** of S if there <u>exists</u> an open ball $B_r(\boldsymbol{p}) \subseteq S$

- Intuitively, $\boldsymbol{p}$ is an **interior point** of S if we can *squeeze* an <u>entire</u> open ball centered at $\boldsymbol{p}$ <u>within</u> S

an interior point   a boundary point

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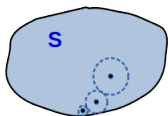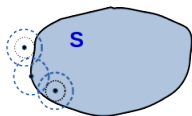
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Open set (7) Boundary points

- A point $\boldsymbol{p} \in \mathbb{R}^n$ is a **boundary point** of S
  if all open balls centered at $\boldsymbol{p}$ contain
  both points in S and points not in S

- The **boundary** of S is the set ∂S
  that consists of all of the **boundary points** of S.

an interior point   a boundary point

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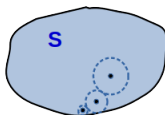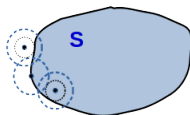
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Open set (8) Open and Closed Sets

- A set $O \subseteq \mathbb{R}^n$ is **open**
  if every point in O is an interior point.

- A set $C \subseteq \mathbb{R}^n$ is **closed**
  if it contains <u>all</u> of its boundary points.

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAndC

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Open set (9) Bounded and Unbounded

- A set S is **bounded** if there is an open ball $B_M(0)$ such that

$$S \subseteq B.$$

  intuitively, this means that we can enclose
  all of the set S within a large enough ball
  centered at the origin, $B_M(0)$

- A set that is not **bounded** is called **unbounded**

Family of sets (1)

- a **collection** F of subsets of a given set S is called
  a **family** of **subsets** of S, or
  a **family** of **sets** over S.

- More generally,
  a **collection** of any sets whatsoever is called
  a **family** of **sets**,
  set family, or
  a **set system**

  https://en.wikipedia.org/wiki/Family_of_sets

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Family of sets (2)

- The term "**collection**" is used here because,
    - in some contexts,
      a **family** of **sets** may be allowed
      to contain repeated copies of any given member, and
    - in other contexts
      it may form a proper class rather than a set.

  https://en.wikipedia.org/wiki/Family_of_sets

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Examples of family of sets (1)

- The set of all subsets of a given set S
  is called the **power set** of S
  and is denoted by $\wp(S)$.

  The **power set** $\wp(S)$ of a given set S
  is a **family** of **sets** over S.

- A subset of S having k elements
  is called a k-subset of S.

  The k-subset $S^{(k)}$ of a set S
  form a **family** of **sets**.

Examples of family of sets (2)

- Let $S = \{a, b, c, 1, 2\}$.

    An example of a **family** of **sets** over S

    (in the multiset sense) is given by $F = \{A_1, A_2, A_3, A_4\}$, where

    $A_1 = \{a, b, c\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2\}$, and $A_4 = \{a, b, 1\}$.

    https://en.wikipedia.org/wiki/Family_of_sets

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Neighbourhood basis (1)

- A **neighbourhood basis** or **local basis**
  (or **neighbourhood base** or **local base**) for a point x
  is a filter base of the neighbourhood filter;

- this means that it is a subset $\mathscr{B} \subseteq \mathscr{N}(x)$ such that for all
  $V \in \mathscr{N}(x)$, there exists some $B \in \mathscr{B}$ such that $B \subseteq V$.
  That is, for any **neighbourhood** V we can find a **neighbourhood**
  B in the **neighbourhood basis** that is contained in V.

  https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis

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Neighbourhood basis (2)

- Equivalently, $\mathscr{B}$ is a local basis at x if and only if the neighbourhood filter $\mathscr{N}$ can be recovered from $\mathscr{B}$ in the sense that the following equality holds:

$$\mathscr{N}(x) = \{V \subseteq X \ : \ B \subseteq V \text{ for some } B \in \mathscr{B}\}$$

- A family $\mathscr{B} \subseteq \mathscr{N}(x)$ is a neighbourhood basis for x if and only if $\mathscr{B}$ is a cofinal subset of $(\mathscr{N}(x), \supseteq)$ with respect to the partial order $\supseteq$ (importantly, this partial order is the superset relation and not the subset relation).

https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis

A collection of sets around x

- In general, one refers to the family of sets containing 0, used to **approximate** 0, as a neighborhood basis;

- a member of this neighborhood basis is referred to as an **open set**.

- In fact, one may generalize these notions to an arbitrary set (X); rather than just the real numbers.

- In this case, given a point (x) of that set (X), one may define a **collection** of sets "around" (that is, containing) x, used to **approximate** x.

https://en.wikipedia.org/wiki/Open_set

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Smaller sets containing x

- Of course, this collection would have to
  satisfy certain properties (known as axioms)
  for otherwise we may <u>not</u> have a *well-defined method*
  to <u>measure</u> distance.

- For example, every point in X should
  approximate x to some degree of accuracy.

- Thus X should be in this family.

- Once we begin to define "<u>smaller</u>" sets containing x,
  we tend to **approximate** x to a greater degree of accuracy.

- Bearing this in mind, one may <u>define</u> the remaining axioms
  that the family of sets about x is required to satisfy.

  https://en.wikipedia.org/wiki/Open_set

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Open ball (1)

- a **ball** is the solid figure bounded by a **sphere**;
  it is also called a **solid sphere**.

    - a **closed ball**
      includes the *boundary points* that constitute the sphere
    - an **open ball**
      excludes them

    https://en.wikipedia.org/wiki/Ball_(mathematics)

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Open ball (2)

- A **ball** in *n dimensions* is called a **hyperball** or **n-ball** and is bounded by a **hypersphere** or $(\boldsymbol{n} - 1)$-**sphere**

- One may talk about **balls** in any topological space X, not necessarily induced by a metric.

- An *n-dimensional* **topological ball** of X is any subset of X which is homeomorphic to an **Euclidean n-ball**.

https://en.wikipedia.org/wiki/Ball_(mathematics)

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Neighborhood (1)

- a neighbourhood is one of the basic *concepts* in a topological space.

- It is closely related to the *concepts* of open set and interior.

- Intuitively speaking, a neighbourhood of a point is a set of points containing that point where one can move some amount in any direction away from that point without leaving the set.

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

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Interior

- the interior of a subset S of a topological space X
  is the union of *all* subsets of S that are open in X.

- A point that is in the interior of S is an interior point of S.

- The interior of S is the complement
  of the closure of the complement of S.
  the closure of (boundary + exterior)

- In this sense, interior and closure are dual notions.

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Exterior

- The exterior of a set S is the complement of the closure of S; the closure of S = boundary + interior

- it consists of the points that are in neither the set nor its boundary.

- The interior, boundary, and exterior of a subset together partition the whole space into three blocks

- fewer when one or more of these is empty

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Interior Point (1)

- If S is a subset of a Euclidean space,
  then x is an interior point of S

  if there exists an open ball centered at x
  which is <u>completely contained</u> in S.

- This definition <u>generalizes</u> to
  any subset S of a metric space X with metric d:

  x is an interior point of S if there <u>exists</u> a real number $r > 0$,
  such that y is in S whenever the distance $d(x, y) < r$.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

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Interior Point (2)

- This definition <u>generalizes</u> to topological spaces by replacing "open ball" with "open set".

  - if there exists an *open ball* centered at x which is <u>completely contained</u> in S.

  - if x is <u>contained</u> in an *open subset* of X that is <u>completely contained</u> in S.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

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Interior Point (3)

- If S is a subset of a topological space X
  then x is an interior point of S in X

  if x is <u>contained</u> in an open subset of X
  that is <u>completely contained</u> in S.

- Equivalently, x is an interior point of S
  if S is a neighbourhood of x.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

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Interior of a Set (1)

- The interior of a subset S of a topological space X, can be defined in any of the following equivalent ways:

    - the largest open subset of X contained in S.

    - the union of all open sets of X contained in S.

    - the set of all interior points of S.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

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Interior of a Set (2)

- The interior of a subset S of a topological space X, denoted by $int_X S$ or $int S$ or S°

- If the space X is understood from context then the shorter notation $int S$ is usually preferred to $int_X S$.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

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Neighborhood of a point (1-1)

- If X is a topological space and p is a point in X, then a neighbourhood of p is a subset V of X that includes an open set U containing p,

$$p \in U \subseteq V \subseteq X.$$

  - X : a topological space
  - V : a subset of X
  - U : an open set containing p
  - p : a point in X
  - V : a neighbourhood of p

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Neighborhood of a point (1-2)

- This is also equivalent to the point $p \in X$
  belonging to the topological interior of V in X.

- The neighbourhood V need not be an open subset of X,
  but when V is open in X
  then it is called an open neighbourhood.

- Some authors have been known
  to require neighbourhoods to be open,
  so it is important to note conventions.

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

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Neighborhood of a point (2)

- A set that is a neighbourhood of each of its points is open since it can be expressed as
  the union of open sets containing each of its points.

- A closed rectangle, as illustrated in the figure,
  is not a neighbourhood of all its points;

  - points on the edges or corners of the rectangle
    are not contained in any open set
    that is contained within the rectangle.

- The collection of all neighbourhoods of a point
  is called the neighbourhood system at the point.

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Neighborhood of a set (1-1)

- If S is a subset of a topological space X,
  then a neighbourhood of S is a set V
  that includes an open set U containing S,

$$S \subseteq U \subseteq V \subseteq X.$$

- It follows that a set V is a neighbourhood of S
  if and only if it is a neighbourhood of all the points in S.

- Furthermore, V is a neighbourhood of S
  if and only if S is a subset of the interior of V.

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

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Neighborhood of a set (1-2)

- A neighbourhood of S that is also an open subset of X is called an open neighbourhood of S.

- The neighbourhood of a point is just a special case of this definition.

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Neighborhood definition (1)

- the open set axioms are often taken as
  the <u>definition</u> of a topology,
  when they are quite *unintuitive*,
  though extremely useful in the long run.

- the neighbourhood <u>definition</u>,
  while somewhat *cumbersome*,
  has the advantage of being closely related to ideas from
  analysis, and has a *historical basis*

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-

and-open-set-in-topology

Neighborhood definition (2-1)

- A neighbourhood topology on a set X
  assigns to each element $x \in X$
  a non empty set $N(x)$ of subsets of X,
  called neighbourhoods of x

- with the following properties:

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-

and-open-set-in-topology

Neighborhood definition (2-2)

- the <u>properties</u> of a neighbourhood topology:

  - If N is a neighbourhood of x then $x \in X$
  - If M is a neighbourhood of x and $M \subseteq N \subseteq X$,
    then N is a neighbourhood of x
  - The intersection of two neighbourhoods of x
    is a neigbourhood of x
  - If N is a neighbourhood of x,
    then N contains a neighbourhood M of x
    such that N is a neighbourhood of each point of M.

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-

and-open-set-in-topology

Neighborhood definition (3-1)

- Then one says a function $f : X \to Y$ is continuous
  wrt neighbourhoods on X and Y
  if for each $x \in X$ and neighbourhood N of $f(x)$
  there is a neighbourhood M of x such that $f(M) \subseteq N$.

- The open set definition of continuity
  is then justified as being equivalent
  to this definition in terms of neighbourhoods.

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-

and-open-set-in-topology

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Neighborhood definition (3-2)

- One also says a set U in X is open
  if U is a neighbourhood of all of its points.
  THEN one can develop the open set axioms
  and show that one can <u>recover</u> the neighbourhoods.

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-

and-open-set-in-topology

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Class (1)

- a **class** is a collection of sets

  (or sometimes other mathematical objects)

  that can be unambiguously <u>defined</u>

  <u>by</u> a property that all its members share.

- **Classes** act as a way to have set-like collections

  while differing from sets so as to avoid **Russell's paradox**

  https://en.wikipedia.org/wiki/Class_(set_theory)

Class (2)

- A **class** *that is not a set* is called a **proper class**, and
- a **class** *that is a set* is sometimes called a **small class**.
- the followings are **proper classes** in many formal systems
    - the **class** of all sets
    - the **class** of all ordinal numbers
    - the **class** of all cardinal numbers

https://en.wikipedia.org/wiki/Class_(set_theory)

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Class (3)

- consider "the set of all sets with property X."

- especially when dealing with categories,
  since the objects of a concrete category
  are all sets with certain additional structure.

- However, **if** we're <u>not</u> *careful* about this
  we can get into serious *trouble* –

  https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-

  objects-and-a-class-of-objects

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Class (4)

- let X be the set of all sets
  which do not contain *themselves*

- Since X is a set, we can ask
  whether X is an element of *itself*.

- But then we run into a paradox –
  if X contains *itself* as an element,
  then it does not, and vice versa.

  https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-

  objects-and-a-class-of-objects

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Class (5)

- In order to avoid this paradox,
  we <u>cannot</u> consider the collection of <u>all</u> sets
  to be itself a set.

- This means we have to *throw out*
  the whole "the set of all sets with property X" construction.
  But we wanted that.

- So the way we get around it is to say that
  there's something called a **class**,
  which is <u>like</u> a set but <u>not</u> a set.

  https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-

  objects-and-a-class-of-objects

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Class (6)

- Then we can talk about
  "the **class** X of all sets with property Y."

- Since X is <u>not</u> a set,
  it can't be an element of itself, and we're fine.

- Of course, if we need to talk about the collection of all classes,
  we need to create another name that goes another step back,
  and so forth.

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Class Examples (1)

- The **collection** of all algebraic structures of a given type
  will usually be a **proper class**.
  (a **class** *that is not a set* is called a **proper class**)
  - the **class** of all groups
  - the **class** of all vector spaces
  - and many others.

- Within set theory, many **collections** of sets
  turn out to be proper classes.

  https://en.wikipedia.org/wiki/Class_(set_theory)

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Class Examples (2)

- One way to *prove* that a **class** is **proper** is
  to place it in bijection
  with the class of all ordinal numbers.

  - **Cardinal numbers** indicate an amount
    how many of something we have: one, two, three, four, five.

  - **Ordinal numbers** indicate position in a series:
    first, second, third, fourth, fifth.

  https://en.wikipedia.org/wiki/Class_(set_theory)
  https://editarians.com/cardinals-ordinals/

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Class Paradoxes (1)

- The **paradoxes** of naive set theory can be explained
  in terms of the *inconsistent tacit assumption*
  that "all classes are sets".

- These paradoxes do not arise with classes
  because there is no notion of classes containing classes.

- Otherwise, one could, for example,
  define a class of all classes
  that do not contain themselves,
  which would lead to a Russell paradox for classes.

  https://en.wikipedia.org/wiki/Class_(set_theory)

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Class Paradoxes (2)

- With a rigorous foundation,
  these **paradoxes** instead *suggest proofs*
  that certain classes are proper (i.e., that they are not sets).
  - **Russell's paradox** *suggests* a *proof*
    that the class of all sets
    which do not contain themselves is proper
  - the **Burali-Forti paradox** *suggests*
    that the class of all ordinal numbers is proper.

  https://en.wikipedia.org/wiki/Class_(set_theory)

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Russell's Paradox (1)

- According to the unrestricted comprehension principle,
  for any sufficiently well-defined property,
      there is the set of all
  and only the objects that have that property.

  https://en.wikipedia.org/wiki/Russell%27s_paradox

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Russell's Paradox (2)

- Let R be the set of all sets ($R = \{x \mid x \notin x\}$)
  that are <u>not</u> members of themselves ($R \notin R$).

  - ***if*** R is <u>not</u> a member of itself ($R \notin R$),
    then its definition (the set of all sets) entails
    <u>that</u> it is a member of itself ($R \in R$);

  - yet, ***if*** it is a member of itself ($R \in R$),
    then it is <u>not</u> a member of itself ($R \notin R$),
    since it is the set of all sets
    <u>that</u> are <u>not</u> members of themselves ($R \notin R$)

- the resulting contradiction is **Russell's paradox**.
- Let $R = \{x \mid x \notin x\}$, then $R \in R \iff R \notin R$

https://en.wikipedia.org/wiki/Russell%27s_paradox

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Russell's Paradox (3)

- *Most sets* commonly encountered
  are <u>not</u> members of themselves.

- For example, consider the set of all <u>squares</u> in a plane.

- This set is <u>not</u> itself a <u>square</u> in the plane,
  thus it is <u>not</u> a member of itself.

- Let us call a set "**normal**" if it is <u>not</u> a member of itself,
  and "**abnormal**" if it is a member of itself.

  https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (4)

- Clearly *every* set must be either normal or abnormal.

- The set of squares in the plane is normal.

- In contrast, the complementary set
  that contains everything which is not a square in the plane
  is itself not a square in the plane,
  and so it is one of its own members
  and is therefore abnormal.

  https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (5)

- Now we consider the set of all normal sets, R,
  and try to determine whether R is normal or abnormal.

  - **If** R were normal,
    it would be contained
    in the set of all normal sets (itself),
    and therefore be abnormal;

  - on the other hand **if** R were abnormal,
    it would not be contained
    in the set of all normal sets (itself),
    and therefore be normal.

- This leads to the conclusion that
  R is neither normal nor abnormal: **Russell's paradox**.

  https://en.wikipedia.org/wiki/Russell%27s_paradox

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Binary Relation (1)

- a **binary relation** associates elements of one set, called the domain, with elements of another set, called the codomain.

- A **binary relation** over sets X and Y is a new set of ordered pairs (x, y) consisting of elements x from X and y from Y.

  https://en.wikipedia.org/wiki/Binary_relationelation

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Binary Relation (2)

- It is a generalization of a unary function.
- It encodes the common concept of relation:
- an element x is related to an element y,
  if and only if the pair (x, y) belongs
  to the set of ordered pairs that defines the binary relation.
- A binary relation is the most studied special case $n = 2$
  of an n-ary relation over sets $X_1, ..., X_n$,
  which is a subset of the Cartesian product $X_1 \times \cdots \times X_n$.

  https://en.wikipedia.org/wiki/Binary_relationelation

Homogeneous Relation

- a **homogeneous relation** (also called endorelation) on a set X is
  a binary relation between X and itself, i.e.
  it is a subset of the Cartesian product $X \times X$.

- This is commonly phrased as "a **relation** on X"
  or "a (**binary**) **relation** over X".

- An example of a **homogeneous relation** is the relation of kinship,
  where the relation is between people.

https://en.wikipedia.org/wiki/Homogeneous_relation

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Partially Ordered Set (1-1)

- a **partial order** on a set is an arrangement such that,
  for certain pairs of elements, one precedes the other.

- The word **partial** is used to indicate
  that not every pair of elements needs to be comparable;
  that is, there may be pairs for which neither element precedes the
  other.

- **Partial orders** thus generalize **total orders**,
  in which every pair is comparable.

  https://en.wikipedia.org/wiki/Partially_ordered_set

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Partially Ordered Set (1-2)

- Formally, a **partial order** is a homogeneous binary relation that is reflexive, transitive and antisymmetric.

- A **partially ordered set** (**poset** for short) is a set on which a **partial order** is defined.

- A reflexive, weak, or non-strict **partial order**, commonly referred to simply as a **partial order**, is a homogeneous relation $\leq$ on a set P that is reflexive, antisymmetric, and transitive.

  https://en.wikipedia.org/wiki/Partially_ordered_set

Partially Ordered Set (2)

- a homogeneous relation $\leq$ on a set P
  that is reflexive, antisymmetric, and transitive.

- That is, for all $a, b, c \in P$, it must satisfy:
  - Reflexivity:
    $a \leq a$, i.e. every element is related to itself.
  - Antisymmetry:
    if $a \leq b$ and $b \leq a$ then $a = b$,
    i.e. no two distinct elements precede each other.
  - Transitivity:
    if $a \leq b$ and $b \leq c$ then $a \leq c$.

- A non-strict **partial order** is also known as an antisymmetric preorder.

  https://en.wikipedia.org/wiki/Partially_ordered_set

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Filter in Set Theory (1-1)

- A **filter** on a set may be thought of
  as representing a "collection of *large* subsets",
  one intuitive example being the **neighborhood filter**.

- keep *large* grains excluding *small* impurities

    https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

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Filter in Set Theory (1-2)

- When you put a filter in your sink,
  the idea is that you filter out the *big* chunks of food,
  and let the water and the *smaller* chunks go through
  (which can, in principle, be washed through the pipes) .

- You filter out the *larger parts*.

- A filter filters out the *larger* sets.

- It is a way to say "these sets are *'large'*"

  https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

Filter in Set Theory (1-3)

- a **filter** on a set X is a family $\mathscr{B}$ of subsets such that:

  1. $X \in \mathscr{B}$ and $\emptyset \notin \mathscr{B}$

  2. if $A \in \mathscr{B}$ and $B \in \mathscr{B}$,
     then $A \cap B \in \mathscr{B}$

  3. If $A, B \subset X, A \in \mathscr{B}$, and $A \subset B$,
     then $B \in \mathscr{B}$

https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

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Filter in Set Theory (1-4)

- The set of "everything" is definitely *large*

  $$X \in \mathscr{B}$$

- and "nothing" is definitely not;

  $$\emptyset \notin \mathscr{B}$$

- if something is *larger* than a *large* set,
  then it is also *large*;

  If $A, B \subset X, A \in \mathscr{B}$, and $A \subset B$, then $B \in \mathscr{B}$

- and two *large* sets *intersect* on a *large* set.

  If $A \in \mathscr{B}$ and $B \in \mathscr{B}$, then $A \cap B \in \mathscr{B}$

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Filter in Set Theory (1-5)

- you can think about this as
  - being **co-finite**,
  - or being of **measure 1** on the **unit interval**,
  - or having a **dense open subset** (again on the unit interval).

- These are examples of ways

  where a set can be thought of as "almost everything".

  and that is the idea behind a filter.

  https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

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Co-finite

- a **cofinite** subset of a set X is
  a subset A whose complement in X is a finite set.

- a subset A contains all <u>but</u> *finitely many* elements of X

- If the complement is <u>not</u> finite, <u>but</u> is countable,
  then one says the set is **cocountable**.

- These arise naturally when <u>generalizing</u> structures
  on finite sets to infinite sets,
  particularly on infinite products,
  as in the product topology or direct sum.

- This use of the prefix "**co**" to describe a property
  possessed by a set's complement
  is *consistent* with its use in other terms such as "comeagre set".

  https://en.wikipedia.org/wiki/Cofiniteness

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Unit interval

- the **unit interval** is the closed interval [0,1],
  that is, the set of all real numbers
  that are greater than or equal to 0
  and less than or equal to 1.

- It is often denoted **I** (capital letter I).

- In addition to its role in real analysis,
  the **unit interval** is used to study homotopy theory
  in the field of topology.

- the term "**unit interval**" is sometimes applied
  to the other shapes that an interval from 0 to 1 could take:
  (0,1], [0,1), and (0,1).

- However, the notation **I** is most commonly reserved
  for the closed interval [0,1].

https://en.wikipedia.org/wiki/Unit_interval

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Dense set

- In topology, a subset A of a topological space X is said to be **dense** in X if *every* point of X either belongs to A or else is arbitrarily "close" to a member of A
    - for instance, the rational numbers are a **dense** subset of the real numbers because every real number either is a rational number or has a rational number arbitrarily close to it (see **Diophantine approximation**).

- Formally, A is **dense** in X if the *smallest* closed subset of X containing A is X itself.

- The **density** of a topological space X is the least cardinality of a **dense subset** of X.

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Proper Subset

- a set A is a subset of a set B
  if all elements of A are also elements of B;

- B is then a superset of A.

- It is possible for A and B to be equal;

- if they are unequal, then A is a proper subset of B.

- The relationship of one set being a subset of another
  is called inclusion (or sometimes containment).

- A is a subset of B may also be expressed
  as B includes (or contains) A or A is included (or contained) in B.

- A k-subset is a subset with k elements.

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Proper Filter (1-1)

- Fix a partially ordered set (poset) P.

- Intuitively, a filter F is a subset of P
  whose members are elements large enough
  to satisfy some *criterion*.

- For instance, if $x \in P$, then the set of elements above x is a filter,
  called the principal filter at x.

  https://en.wikipedia.org/wiki/Filter_(mathematics)

Proper Filter (1-2)

- If x and y are incomparable elements of P,

  then neither the principal filter at x nor y
  is contained in the other

  - two elements x and y of a set P are said to be **comparable**

    with respect to a binary relation $\leq$

    if at least one of $x \leq y$ or $y \leq x$ is true.
    They are called **incomparable** if they are not **comparable**.

  - Hasse diagram of the natural numbers,

    partially ordered by "$x \leq y$ if x divides y".

    The numbers 4 and 6 are **incomparable**,
    since neither divides the other.

    https://en.wikipedia.org/wiki/Filter_(mathematics)
    https://en.wikipedia.org/wiki/Comparability

Proper Filter (1-3)

- Similarly, a filter on a set S contains
  those subsets that are sufficiently large
  to contain some given *thing*.

- For example, if S is the *real line* and $x \in S$,
  then the family of sets including x *in their interior* is
  a filter, called the neighborhood filter at x.

- The *thing* in this case is *slightly larger* than x,
  but it still does not contain any other specific point of the line.

  https://en.wikipedia.org/wiki/Filter_(mathematics)

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Proper Filter (2)

- The above considerations motivate

  the upward closure requirement in the definition below:
  "*large enough*" objects can always be made *larger*.

- To understand the other two conditions,

  reverse the roles and instead consider F
  as a "*locating scheme*" to find x.

- In this interpretation, one searches in some space X,
  and expects F to describe those subsets of X that contain the goal.

- The goal must be located somewhere;
  thus the empty set $\varnothing$ can never be in F.

- And if two subsets both contain the goal,
  then should "zoom in" to their common region.

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Proper Filter (3)

- An ultrafilter describes a "perfect locating scheme"
  where each scheme component gives new information
  (either "look here" or "look elsewhere").

- Compactness is the property that "every search is fruitful,"
  or, to put it another way,
  "every locating scheme ends in a search result."

- A common use for a filter is to define properties
  that are satisfied by "generic" elements of some topological space.

- This application generalizes the "locating scheme"
  to find points that might be hard to write down explicitly.

  https://en.wikipedia.org/wiki/Filter_(mathematics)

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Neighborhood Filter (1-1)

- Let X be a set;

- the elements of X are usually called points

- We allow X to be empty.

- Let $\mathcal{N}$ be a function

  assigning to each x (point) in X
  a non-empty collection $\mathcal{N}(x)$ of subsets of X.

- The elements of $\mathcal{N}(x)$ will be called

  neighbourhoods of x with respect to $\mathcal{N}$
  (or, simply, neighbourhoods of x).

  https://en.wikipedia.org/wiki/Topological_space

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Neighborhood Filter (1-2)

- Let X be a set;
- $\mathcal{N}$: a function assigning to each point x in X
- $\mathcal{N}(x)$: a non-empty collection of subsets of X.

- The elements of $\mathcal{N}(x)$
    - subsets of X
    - neighbourhoods of x with respect to $\mathcal{N}$

    https://en.wikipedia.org/wiki/Topological_space

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Neighborhood Filter (1-3)

- The function $\mathscr{N}$ is called
  a neighbourhood topology
  if *some axioms* are satisfied;

- then X with $\mathscr{N}$ is called
  a topological space – $(X, \mathscr{N})$

  https://en.wikipedia.org/wiki/Topological_space

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Neighborhood Filter (1-4)

- If $(X, \mathscr{T})$ is a topological space and $p \in X$,

  a neighbourhood of p is a subset V of X,
  in which $p \in U \subseteq V$, and U is open.

- We say that V is a $\mathscr{T}-$ neighbourhood of $x \in X$
  or that V is a neighborhood of x

- The set of all neighbourhoods of $x \in X$, denoted $\mathscr{N}_{\mathscr{X}}$
  is called the neighbourhood filter of x

  https://math.stackexchange.com/questions/799732/neighborhood-vs-

  neighborhood-filter

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Neighborhood Filter (1-4)

- An example of Neighborhood Filters on a Topological space.
- Let $X = \{a, b, c\}$ and let $\mathscr{T} = \{\varnothing, \{a\}, \{b\}, \{b, c\}, \{a, b\}, X\}$
- Let

  $\mathscr{N}_a = \{\{a\}, \{a, b\}, \{a, c\}, X\}$

  $\mathscr{N}_b = \{\{b\}, \{a, b\}, \{b, c\}, X\}$

  $\mathscr{N}_c = \{\{b, c\}, X\}.$

- In this example a, c is a neighborhood of a but not of c.
- Thus a set does not have to be a neighborhood of all of its points.

  https://math.stackexchange.com/questions/799732/neighborhood-vs-

  neighborhood-filter

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Neighborhood Filter (2)

- One can specify a topology in more than four different ways.

  1. The standard definition specifies the open sets, what we usually call a "topology."

  2. to specify the close sets - this is of course only a *trivial difference*.

  3. to specify a closure operation on subsets of your space

  4. to specify a neighborhood filter for every point satisfying the natural axiom that every neighborhood of x is a neighborhood of every point of one of its subsets

     - So in this sense neighborhood filters tell you everything they possibly could about a topological space.

  https://math.stackexchange.com/questions/799732/neighborhood-vs-

  neighborhood-filter

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Neighborhood Filter (3-1)

- Probably the best way to think about the neighborhood filter of x:
  is that it contains all information regarding convergence to x.

- In the first topological spaces one encounters,
  convergence is usually of sequences.

- But this isn't enough to describe the topology in arbitrary spaces,
  for instance the infinite-dimensional spaces of functional analysis.

- It becomes important to speak of convergence of nets, or of filters.

  https://math.stackexchange.com/questions/799732/neighborhood-vs-

  neighborhood-filter

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Neighborhood Filter (3-2)

- A filter on X is just a <u>nontrivial</u> subset of the powerset of X
  closed under <u>finite</u> intersection and superset,
  and a filter converges to a point x if and only if it contains
  the neighborhood filter of x.

- In contrast to the case with sequences,
  this is <u>enough</u> to specify a topology:
  in fact it's <u>enough</u> to describe how ultrafilters,
  that is, maximal filters, converge.

- So in this sense the neighborhood filter encapsulates the viewpoint
  that topology <u>generalizes</u> the study of convergent sequences.

    https://math.stackexchange.com/questions/799732/neighborhood-vs-

    neighborhood-filter

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Neighborhood Filter (4-1)

- in a sense the neighborhood filter describes

  the smallest neighborhood of a point
  - except that there is no smallest neighborhood!

- That's true, at least, in many of the most interesting spaces,

  and is the main reason to worry about a whole filter of neighborhoods

  - if there were a smallest neighborhood then in any hypothesis

  requiring something to hold on a sufficiently small neighborhood of x
  we could just pick the smallest neighborhood.

  https://math.stackexchange.com/questions/799732/neighborhood-vs-

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Neighborhood Filter (4-2)

- But the <u>smallest</u> neighborhood of a point must be contained
  in the intersection of all its neighborhoods,
  and in, say, a Hausdorff space the intersection of all neighborhoods of x is
  x,
  which is <u>not</u> a neighborhood of x when x is <u>not</u> isolated.

- So the filter functions as a <u>virtual</u> <u>smallest</u> neighborhood of x:
  it doesn't converge to a neighborhood of x,
  so we can't think about its limit, but functionally we do just that.

    https://math.stackexchange.com/questions/799732/neighborhood-vs-

    neighborhood-filter

Ultrafilter (1)

- an **ultrafilter** on a given partially ordered set (or "poset") P is
  a certain subset of P, namely a maximal filter on P;
  that is, a proper filter on P that cannot be enlarged
  to a bigger proper filter on P.

- If X is an arbitrary set, its power set $\mathscr{P}(X)$,
  ordered by set inclusion,
  is always a Boolean algebra and hence a poset,
  and ultrafilters on $\mathscr{P}(X)$ are usually called
  ultrafilter on the set X.

https://en.wikipedia.org/wiki/Ultrafilter

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Ultrafilter (2)

- In order theory, an **ultrafilter** is

  a subset of a partially ordered set
  that is maximal among all proper filters.

- This implies that any filter that properly contains an ultrafilter
  has to be equal to the whole poset.

  https://en.wikipedia.org/wiki/Ultrafilter

Ultrafilter (3)

- An ultrafilter on a set X may be considered as
  a finitely additive measure on X.

- In this view, every subset of X is
  either considered "*almost everything*" (has measure 1) or
  "*almost nothing*" (has measure 0),
  depending on whether it belongs to the given ultrafilter or not

  https://en.wikipedia.org/wiki/Ultrafilter

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Ultrafilter (4)

- Formally, if P is a set, partially ordered by $\leq$ then

- a subset $F \subseteq P$ is called a filter on P

  if F is nonempty, for every $x, y \in F$,

  there exists some element $z \in F$

  such that $z \leq x$ and $z \leq y$,

  and for every $x \in F$ and $y \in P$,

  $x \leq y$ implies that y is in F too;

- a proper subset U of P is called

  an ultrafilter on P if U is a filter on P,

  and there is no proper filter F on P

  that properly extends U

  (that is, such that U is a proper subset of F).

  https://en.wikipedia.org/wiki/Ultrafilter

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Filter Examples (1)

- Let $X = 1, 2, 3$
  Choose some element from X say $F = 1, 1, 2, 1, 3, 1, 2, 3$

- Then every intersection of an element of F

  with another element in F is in F again.

  Examples: $1 \bigcap 1, 2, 3 = 1$ $1, 2 \bigcap 1, 2, 3 = 1, 2$
     $1, 3 \bigcap 1, 2, 3 = 1, 3$ $1, 2, 3 \bigcap 1, 2, 3 = 1, 2, 3$

- Also the original $X = 1, 2, 3$ is also in F.
  Here $F = 1, 1, 2, 1, 3, 1, 2, 3$ is called the filter on $X = 1, 2, 3$

  https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-

  theory

Filter Examples (2)

- .Suppose we have the collection $G = 1, 1, 2, 1, 3, 2, 3, 1, 2, 3$

- Then we have $1, 3 \bigcap 2, 3 = 3$ but 3 isn't in G.
  So this G is not called a filter.

- Now with $F = 1, 1, 2, 1, 3, 1, 2, 3$

  can we put as any other element in it

  so that after placing the extra element it is still a filter?

  Probably not in this case.

  So on $X = 1, 2, 3$, $F = 1, 1, 2, 1, 3, 1, 2, 3$ is an Ultrafilter.

Filter Examples (3)

- If we have started say with $H = 1, 1, 2, 1, 2, 3$

  this is still a filter on $X = 1, 2, 3$

  but we can still add $1, 3$
  and it will still be classified as filter.

- So on $X = 1, 2, 3$

  $F = 1, 1, 2, 1, 3, 1, 2, 3$ is an Ultrafilter
  but $H = 1, 1, 2, 1, 2, 3$ is a filter but not an Ultrafilter.

  https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-

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Filter Examples (4)

- Now suppose we have $X = 1, 2, 3, 4$
  Let $F = 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$

- Every in intersection of element of F is in F again.
  We have as examples $1, 4 \cap 1, 4 = 1, 4$ $1, 4 \cap 1, 2, 4 = 1, 4$
  $1, 4 \cap 1, 3, 4 = 1, 4$ $1, 2, 4 \cap 1, 2, 4 = 1, 2, 4$ $1, 2, 4 \cap 1, 3, 4 = 1, 4$
  $1, 3, 4 \cap 1, 3, 4 = 1, 3, 4$ $1, 2, 3, 4 \cap 1, 2, 3, 4 = 1, 2, 3, 4$

- Also $X = 1, 2, 3, 4$ is also in $F = 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$
  and the null element $\varnothing =$ is not in F.

  https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-

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Filter Examples (5)

- We call F a filter but not an Ultrafilter on $X = 1, 2, 3, 4$

- We can still <u>add</u> element in it and it will still be a filter

  for instance by adding the element 1 from $X = 1, 2, 3, 4$
  we can have the filter $F = 1, 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$

- This is an Ultrafilter on $X = 1, 2, 3, 4$

  as we cannot add any further element from $X = 1, 2, 3, 4$
  that satisfies closures on intersection.

    https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-

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Filter Examples (6)

- There is another collection of sets taken from $X = 1, 2, 3, 4$
- which is the powerset
  $P =, 1, 2, 3, 4, 1, 2, 1, 3, 1, 4, 2, 3, 2, 4, 3, 4, 1, 2, 3, 1, 2, 4, 1, 3, 4, 2, 3, 4, 1, 2, 3, 4$
- This contain the null element $\varnothing =$ so we <u>cannot</u> call this as Ultrafilter.
- This is <u>not</u> a proper filter according to the article in Wikipedia.
- In the powerset every intersection of element is again in the powerset again but it contains t
- he null element $\varnothing =$ and isn't classified as proper filter.

  https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-

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Filter Examples (7)

- There is another collection of sets taken from $X = 1, 2, 3, 4$
- which is the powerset
  $P =, 1, 2, 3, 4, 1, 2, 1, 3, 1, 4, 2, 3, 2, 4, 3, 4, 1, 2, 3, 1, 2, 4, 1, 3, 4, 2, 3, 4, 1, 2, 3, 4$
- This contain the null element $\varnothing =$ so we <u>cannot</u> call this as Ultrafilter.
- This is <u>not</u> a proper filter according to the article in Wikipedia.
- In the powerset every intersection of element is again in the powerset again but it contains t
- he null element $\varnothing =$ and isn't classified as proper filter.

  https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

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Topology

- **topology**
  from the Greek words
  τόπος, 'place, location',
  and λόγος, 'study'

  https://en.wikipedia.org/wiki/Topology

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Topology (2)

- **topology** is concerned with

  the *properties* of a geometric object that are *preserved*

  - under continuous
    deformations such as
    - stretching
    - twisting
    - crumpling
    - bending

  - that is, *without*
    - closing holes
    - opening holes
    - tearing
    - gluing
    - passing through itself

  https://en.wikipedia.org/wiki/Topology

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Topological space (1)

- a **topological space** is, roughly speaking,

  a geometrical space
  in which closeness is defined

  but cannot necessarily be measured
  by a numeric distance.

  https://en.wikipedia.org/wiki/Borel_set

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Topological space (2)

- More specifically, a **topological space** is
  - a set whose elements are called points,
  - along with an additional structure called a topology,

- which can be defined as
  - a set of neighbourhoods for each point
  - that satisfy some axioms
    formalizing the concept of closeness.

  https://en.wikipedia.org/wiki/Borel_set

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Topological space (3)

- There are several *equivalent* definitions of a **topology**,

  the most commonly used of which is
  the definition through open sets,

  which is <u>easier</u> than the others to manipulate.

  https://en.wikipedia.org/wiki/Borel_set

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Topological space (4)

- A **topological space** is
  the most general type of a mathematical space
  that allows for the definition of

  - limits
  - continuity
  - connectedness

- Although very general,

  the concept of **topological spaces** is fundamental,
  and used in virtually every branch of modern mathematics.

- The study of **topological spaces** in their own right
  is called point-set topology or general topology.

  https://en.wikipedia.org/wiki/Topological_space

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Topological space (5)

- Common types of **topological spaces** include

  - Euclidean spaces : a set of points satisfying certain relationships, expressible in terms of distance and angles.

  - metric spaces : a set together with a notion of distance between points. The distance is measured by a function called a metric or distance function.

  - manifolds : a topological space that *locally* resembles Euclidean space near each point. More precisely, an n-manifold is a topological space with the property that each point has a neighborhood that is homeomorphic to an open subset of n-dimensional Euclidean space.

  https://en.wikipedia.org/wiki/Topological_space

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Discrete Topology

- a **discrete space** is a topological space,

  in which the points form a discontinuous sequence,
  meaning they are isolated from each other in a certain sense.

- The **discrete topology** is
  the finest topology that can be given on a set.

  - every subset is open
  - every singleton subset is an open set

  https://en.wikipedia.org/wiki/Discrete_space

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Singletone

- a **singleton**, also known as a **unit set** or **one-point set**, is a set with exactly one element.
- for example, the set $\{0\}$ is a **singleton** whose single element is 0

  https://en.wikipedia.org/wiki/Discrete_space

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Indiscrete Space (1)

- a **topological space** <u>with</u> the **trivial topology**
  is one where <u>the only</u> open sets
  are the empty set and the entire space.

- Such spaces are commonly called **indiscrete**,
  anti-discrete, **concrete** or **codiscrete**.

  - <u>every</u> subset can be open (the **discrete topology**), or

  - <u>no</u> subset can be open (the **indiscrete topology**) except
    the space itself and the empty set .

    https://en.wikipedia.org/wiki/Discrete_space

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Indiscrete Space (2)

- Intuitively, this has the consequence that
  all points of the space are "lumped together"
  and cannot be distinguished by topological means
  (not topologically distinguishable points)

- Every **indiscrete space** is a **pseudometric space**
  in which the distance between any two points is zero.

  https://en.wikipedia.org/wiki/Discrete_space

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T_0 Space

- a topological space X is a T_0 **space** or
  if for every pair of distinct points of X,
  at least one of them has a neighborhood
  not containing the other.

- In a T_0 **space**, all points are topologically distinguishable.

- This condition, called the T_0 **condition**,
  is the weakest of the separation axioms.

- Nearly all topological spaces *normally*
  studied in mathematics are T_0 **space**.

  https://en.wikipedia.org/wiki/Kolmogorov_space

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Topologically distinguishable points

- Intuitively, an open set provides a *method* to *distinguish* two points.

- two points in a topological space,
  there exists an open set
    - containing one point but
    - not containing the other (distinct) point
    - the two points are **topologically distinguishable**.

https://en.wikipedia.org/wiki/Open_set

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Topologically distinguishable points

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  https://en.wikipedia.org/wiki/Open_set

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Metric spaces

- In this manner, one may speak of
  whether two points, or more generally two subsets,
  of a topological space are "**near**"
  without concretely defining a distance.

- Therefore, **topological spaces** may be seen as a
  generalization of **spaces** equipped with a notion of distance,
  which are called **metric spaces**.

  https://en.wikipedia.org/wiki/Open_set

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Why called a discrete topology? (1)

- The discrete topology is the finest topology—it cannot be subdivided further. If you think of the elements of the set as indivisible "discrete" atoms, each one appears as a singleton set. You can effectively "see" the individual points in the topology itself.

- Contrast this with the indiscrete topology, consisting only of X itself and $\varnothing$. This topology obscures everything about how many points were in the original set. It fully agglomerates the points of the set together.

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Why called a discrete topology? (2)

- Revisiting this point, it's sometimes helpful to think of topologies as obscuring or blurring together the underlying points of the set. Topologies are all about nearness relations: points in an open set are in the vicinity of one another. If there are two points that never appear alone in an open set, those points are topologically indistinguishable. From the perspective of the topology, they are so close as to be identical.

- It is therefore remarkable that the discrete topology has no indistinguishable points. The discrete topology is the topology that obscures nothing about the underlying set. Each point in the set is clearly highlighted and distinguishable and recoverable as an open singleton set in the topology.

  https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-

  called-a-discrete-topology

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Why called a discrete topology? (3)

- If you think of topologies that can arise from metrics, the discrete topology arises from metrics such as d(x,y)={01x=yx≠y. This metric "shatters" the points X, isolating each one within its own unit ball. In such a space, the only convergent sequences are the ones that are eventually constant; you can't find points arbitrarily close to any other points. Because points are isolated in this way, it makes sense to call the space "discrete".

  https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-called-a-discrete-topology

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Why called a discrete topology? (4)

- Every function from a discrete space is automatically continuous. I'd argue that for this reason, the discrete topology is the one that best "represents" X

- in topological space. Indeed, in many ways the nature of a set is characterized by its functions, and the nature of a topological space is characterized by its continuous functions.

- So, note that if T is any topological space, there's a natural bijective correspondence between functions f:X→set(T) and continuous morphisms g:discrete(X)→T. For every function on X, you can find a continuous function on discrete(X), and given any continuous function on discrete(X), you can uniquely recover a function on X

- .

- The discrete topology best represents the structure of the set X which, as you say, is discretized into individual points.

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Why called a discrete topology? (5)

- Throughout abstract algebra, isomorphisms describe which structures are "the same". A topological isomorphism (a homeomorphism) between two topologies says that they are essentially the same topology. An isomorphism of sets is just a bijection; it says that the sets contain the same number of elements.

- Continuing the discussion of functions above, two discrete topologies are topologically isomorphic (homeomorphic) if and only if their underlying sets are isomorphic as sets (bijective). Put casually, this means that the discrete-topology-creating process maintains the similarity and differences between the underlying sets: discrete topologies are the same if and only if their underlying sets are.

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-

called-a-discrete-topology

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Why called a discrete topology? (6)

- This is all the more important when we realize that sets are the same when they have the same number of points. Hence discrete topologies are the same when (and only when) their underlying sets have "discrete points" in the same quantity. You can count the points in a discrete topology through isomorphisms, and the discrete topology is the only topology for which this is possible.

-

    https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-

    called-a-discrete-topology

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Examples of topoloy (1)

- Let τ be denoted with the circles,
  here are four examples **(a)**, **(b)**, **(c)**, **(d)**, and
  two non-examples **(e)**, **(f)** of topologies
  on the three-point set $\{1,2,3\}$.

- **(e)** is <u>not</u> a topology
  because the union of $\{2\}$ and $\{3\}$
  [i.e. $\{2,3\}$] is missing;

- **(f)** is not a topology
  because the intersection of $\{1,2\}$ and $\{2,3\}$
  [i.e. $\{2\}$], is missing.

  https://en.wikipedia.org/wiki/Topological_space

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Examples of topoloy (2)

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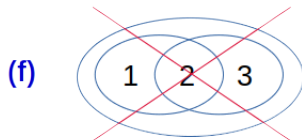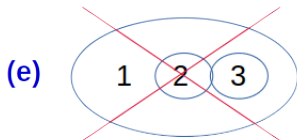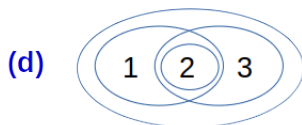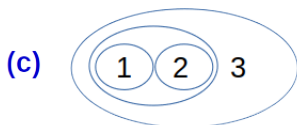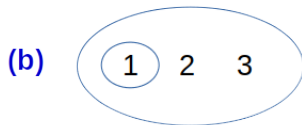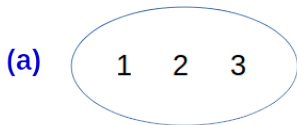
Every union of (c)

(c) is a topology $\{\{\}, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$

every union of (c)

$\cup$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{2\}$	$\{2\}$	$\{1,2\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$

https://en.wikipedia.org/wiki/Topological_space

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Every intersection of (c)

(c) is a topology $\{\{\}, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$
every intersection of (c)

$\cap$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$
$\{1\}$	$\{\}$	$\{1\}$	$\{\}$	$\{1\}$	$\{1\}$
$\{2\}$	$\{\}$	$\{\}$	$\{2\}$	$\{2\}$	$\{2\}$
$\{1,2\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2\}$
$\{1,2,3\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$

https://en.wikipedia.org/wiki/Topological_space

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Every union of (f)

(f) is <u>not</u> a topology $\{\{\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$

every union of (f)

∪	{}	{1,2}	{2,3}	{1,2,3}
{}	{}	{1,2}	{2,3}	{1,2,3}
{1,2}	{1,2}	{1,2}	{1,2,3}	{1,2,3}
{2,3}	{2,3}	{1,2,3}	{2,3}	{1,2,3}
{1,2,3}	{1,2,3}	{1,2,3}	{1,2,3}	{1,2,3}

https://en.wikipedia.org/wiki/Topological_space

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Every intersection of (f)

(f) is <u>not</u> a topology $\{\{\},\{1,2\},\{2,3\},\{1,2,3\}\}$
every intersection of (f)

∩	{}	{1,2}	{2,3}	{1,2,3}
{}	{}	{}	{}	{}
{1,2}	{}	{1,2}	{2}	{1,2}
{2,3}	{}	{2}	{2,3}	{2,3}
{1,2,3}	{}	{1,2}	{2,3}	{1,2,3}

https://en.wikipedia.org/wiki/Topological_space

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Examples of topoloy (3)

- Given $X = \{1, 2, 3, 4\}$,
  the *trivial* or *indiscrete* topology on X is
  the family $\tau = \{\{\}, \{1, 2, 3, 4\}\} = \{\varnothing, X\}$
  consisting of only the two subsets of X
  required by the axioms
  forms a topology of X.

  https://en.wikipedia.org/wiki/Topological_space

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Examples of topoloy (4)

- Given $X = \{1, 2, 3, 4\}$,
  the family $\tau = \{\{\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$
  $= \{\varnothing, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, X\}$
  of six subsets of X forms another topology of X.

  https://en.wikipedia.org/wiki/Topological_space

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Examples of topoloy (5)

- Given $X = \{1, 2, 3, 4\}$,

  the *discrete* topology on X is

  the power set of X, which is the family $\tau = \wp(X)$

  consisting of *all possible* subsets of X.

  the family

$$\tau = \quad \{\{\}, \{1\}, \{2\}, \{3\}, \{4\}$$
$$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\},$$
$$\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}$$

  - In this case the topological space (X, τ)
    is called a *discrete* space.

https://en.wikipedia.org/wiki/Topological_space

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Examples of topoloy (6)

- Given $X = \mathbb{Z}$, the set of integers,
  the family τ of
  all finite subsets of the integers
  plus $\mathbb{Z}$ itself
  is <u>not</u> a topology,
  because (for example)
  the <u>union</u> of *all finite* sets
  <u>not</u> containing <u>zero</u> is <u>not</u> finite
  <u>but</u> is also <u>not</u> all of $\mathbb{Z}$,
  and so it cannot be in τ .

  https://en.wikipedia.org/wiki/Topological_space