LU Factorization

Young Won Lim
10/14/2015

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Gauss-Jordan Elimination

Forward Phase – Gaussian Elimination

$$\left(\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

Backward Phase – Guass-Jordan Elimination

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & 0 & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

Getting Upper Triangular Matrix

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \cdot E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot E_1 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \left[\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right]$$

$$E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \cdot E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right]$$

Upper Triangular

$$E_6 \, E_5 \, E_4 \, E_3 \, E_2 \, E_1 \, A = U$$

Getting the original matrix

$$
E_4^{-1} \cdot E_5^{-1} \cdot E_6^{-1} \cdot U:\quad
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \bullet
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \bullet
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet
\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right]
$$

$$
E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1}:\quad
\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet
\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bullet
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}
\qquad
\left[\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right]
$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1} U$$

E_i and E_i^{-1} examples (1)

```
(%i1) E1 : matrix([1/2, 0, 0], [0, 1, 0], [0, 0, 1]);
```

(%o1) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

```
(%i2) E2 : matrix([1, 0, 0], [3, 1, 0], [0, 0, 1]);
```

(%o2) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

```
(%i3) E3 : matrix([1, 0, 0], [0, 1, 0], [2, 0, 1]);
```

(%o3) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

```
(%i7) E1_1 : invert(E1);
```

(%o7) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

```
(%i8) E2_1 : invert(E2);
```

(%o8) $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

```
(%i9) E3_1 : invert(E3);
```

(%o9) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$$E_6\,E_5\,E_4\,E_3\,E_2\,E_1\,A = U$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}U$$

E_i and E_i^{-1} examples (1)

```
(%i4) E4 : matrix([1, 0, 0], [0, 2, 0], [0, 0, 1]);
```

$$(\%o4)\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
(%i5) E5 : matrix([1, 0, 0], [0, 1, 0], [0, -2, 1]);
```

$$(\%o5)\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

```
(%i6) E6 : matrix([1, 0, 0], [0, 1, 0], [0, 0, -1]);
```

$$(\%o6)\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

```
(%i10) E4_1 : invert(E4);
```

$$(\%o10)\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

```
(%i11) E5_1 : invert(E5);
```

$$(\%o11)\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

```
(%i12) E6_1 : invert(E6);
```

$$(\%o12)\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$E_6\, E_5\, E_4\, E_3\, E_2\, E_1\, A = U$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1} U$$

$A = LU$ examples

```
(%i13) L_1 : E6.E5.E4.E3.E2.E1;
```

$$(\%o13)\quad \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 3 & 2 & 0 \\ 5 & 4 & -1 \end{bmatrix}$$

```
(%i14) L : invert(L_1);
```

$$(\%o14)\quad \begin{bmatrix} 2 & 0 & 0 \\ -3 & \frac{1}{2} & 0 \\ -2 & 2 & -1 \end{bmatrix}$$

```
(%i16) A : matrix([2, 1, -1], [-3, -1, 2], [-2, 1, 2]);
```

$$(\%o16)\quad \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix}$$

```
(%i18) L_1.A;
```

$$(\%o18)\quad \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

```
(%i22) U : L_1.A;
```

$$(\%o22)\quad \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

```
(%i23) L.U;
```

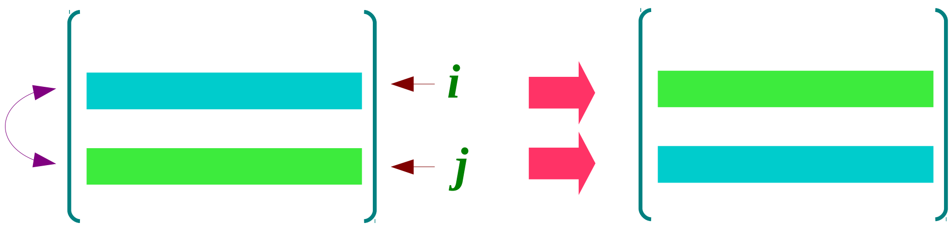
$$(\%o23)\quad \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix}$$

$$E_6\, E_5\, E_4\, E_3\, E_2\, E_1\, A = U$$

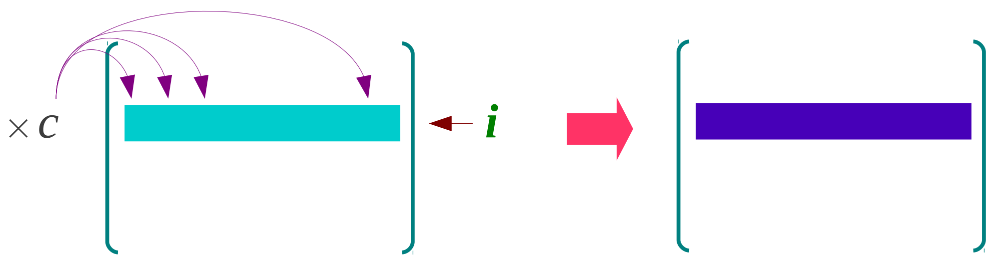
$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1} U$$

Elementary Matrices

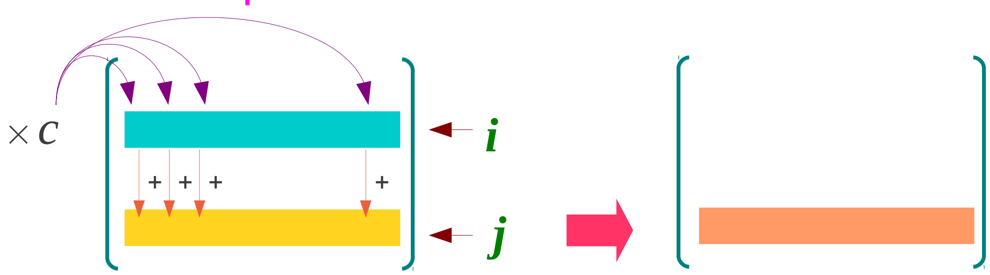
Interchange two rows

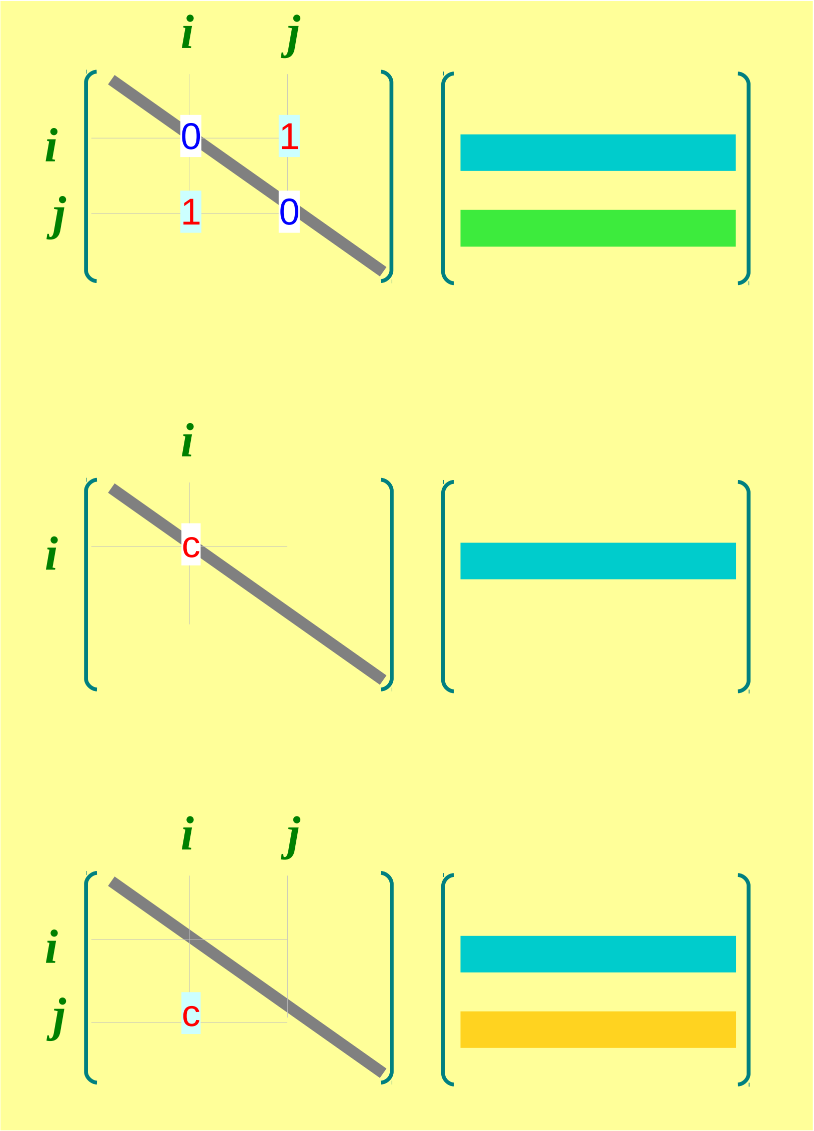


Multiply a row by a nonzero constant

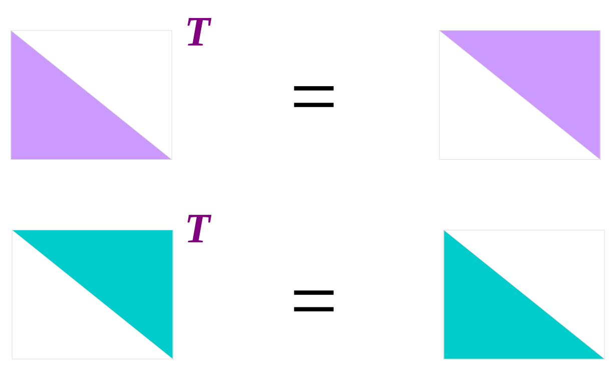


Add a multiple of one row to another

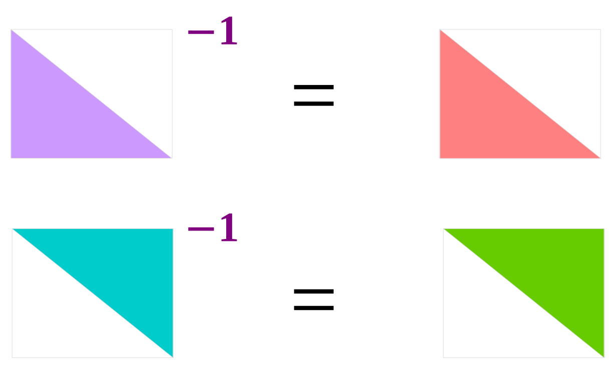




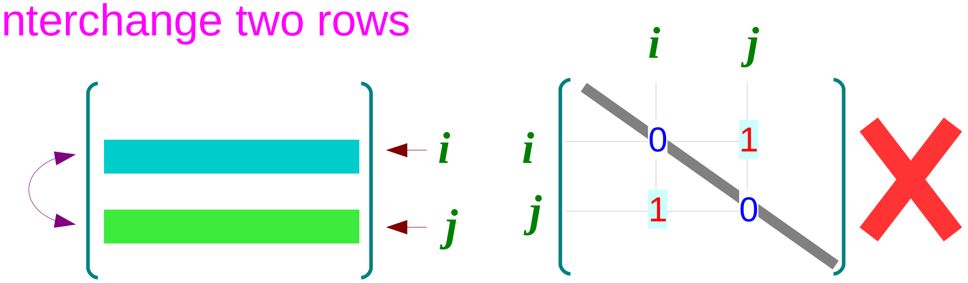
Triangular and Elementary Matrices



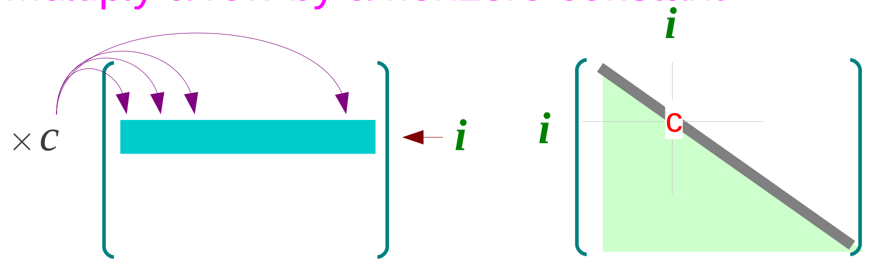
$\prod a_{ii} \neq 0 \Rightarrow$ Invertible



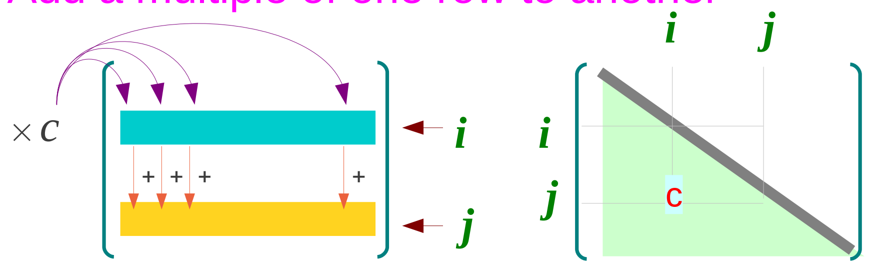
Interchange two rows



Multiply a row by a nonzero constant

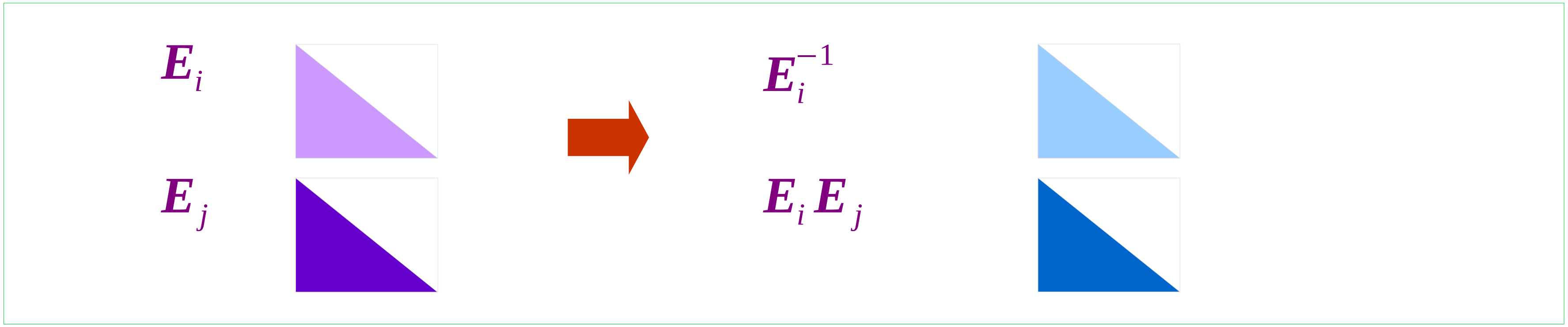


Add a multiple of one row to another



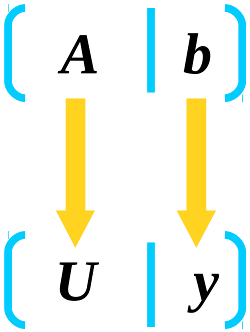
For all lower triangular E_i's

If every elementary matrix involved in the
Forward Phase – Gaussian Elimination
is a lower triangular matrix



Can get $\boldsymbol{L}$ from the $\boldsymbol{E}_i$'s

If every elementary matrix involved in the
Forward Phase – Gaussian Elimination
is a lower triangular matrix



$$E_7\,E_6\,E_5\,E_4\,E_3\,E_2\,E_1\,A = U$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}U = L\,U$$

$$E_7\,E_6\,E_5\,E_4\,E_3\,E_2\,E_1 = L^{-1}$$

$$E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1} = L$$

If every elementary matrix involved in the
Forward Phase – Gaussian Elimination

$$A\mathbf{x} = \mathbf{b} \quad \longrightarrow \quad U\mathbf{x} = \mathbf{y}$$

$$\left[\, A \mid \mathbf{b} \,\right] \quad \longrightarrow \quad \left[\, U \mid \mathbf{y} \,\right]$$

$$L^{-1}A\mathbf{x} = L^{-1}\mathbf{b}$$

$$U\mathbf{x} = \mathbf{y}$$

$$E_7 E_6 E_5 E_4 E_3 E_2 E_1 = L^{-1}$$

$$E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1} = L$$

LU Decomposition

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$$ If A is LU factorizable

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{L}\boldsymbol{U}\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{L}\boldsymbol{y} = \boldsymbol{b}$$ First, find y

$$\boldsymbol{U}\boldsymbol{x} = \boldsymbol{y}$$ Next, find x

$$\boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1}\boldsymbol{E}_3^{-1}\boldsymbol{E}_4^{-1}\boldsymbol{E}_5^{-1}\boldsymbol{E}_6^{-1}\boldsymbol{E}_7^{-1} = \boldsymbol{L}$$

Pulse

References

[1] http://en.wikipedia.org/
[2] Anton & Busby, “Contemporary Linear Algebra”
[3] Anton & Rorres, “Elementary Linear Algebra”