Characteristics of Multiple Random Variables

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

1 Joint Characteristic Functions

Joint Characteristic Function
two random variables

Definition

The joint characteristic function of two random variables X and Y is given by

$$\Phi_{X,Y}(\omega_1, \omega_2) = E\left[e^{j\omega_1 X + j\omega_2 Y}\right]$$

where$\omega_1$and $\omega_2$are real numbers. An equivalent form is

$$\Phi_{X,Y}(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x, y) e^{j\omega_1 x + j\omega_2 y} dxdy$$

Joint Characteristic Function and Fourier Transform

two random variables

Definition

the 2-dimension Fourier transform of $f_{X,Y}(x,y)$ when signs of $\omega_1$andω_2 are reversed

$$\Phi_{X,Y}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)e^{j\omega_1 x + j\omega_2 y}\,dxdy$$

the inverse Fourier transform

$$f_{X,Y}(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi_{X,Y}(\omega_1,\omega_2)e^{-j\omega_1 x - j\omega_2 y}\,d\omega_1 d\omega_2$$

Marginal Charateristic Functions

two random variables

Definition

Marginal characteristic functions are

$$\Phi_X(\omega_1) = \Phi_{X,Y}(\omega_1, 0) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) e^{j\omega_1 x} dx$$

$$\Phi_Y(\omega) = \Phi_{X,Y}(0, \omega_2) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) e^{j\omega_2 y} dy$$

Joint Moments
two random variables

Definition

Joint moments m_{nk} can be found from $\Phi_{X,Y}(\omega_1, \omega_2)$

$$m_{nk} = (-1)^{n+k} \frac{\partial^{n+k}}{\partial \omega_1^n \partial \omega_2^k} \Phi_{X,Y}(\omega_1, \omega_2)\Big|_{\omega_1=0,\omega_2=0}$$

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Joint Characteristic Function
Nrandom variables

Definition

The joint characteristic function of N random variables $X_1, X_2, \cdots, X_N$ is given by

$$\Phi_{X_1,\cdots,X_N}(\omega_1, \cdots, \omega_N) = E\left[e^{j\omega_1 X_1 + \cdots + j\omega_N X_N}\right]$$

Joint moments are obtained from

$$m_{n_1 \cdots n_N} = (-1)^R \frac{\partial^R}{\partial \omega_1^{n_1} \cdots \partial \omega_N^{n_N}} \Phi_{X_1,\cdots,X_N}(\omega_1, \cdots, \omega_2)\Big|_{all\ \omega_i=0}$$

$$R = n_1 + n_2 + \cdots + n_N$$