

# Characteristics of Multiple Random Variables

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles, Jr. and B. Shi

# Outline

## 1 Joint Characteristic Functions

# Joint Characteristic Function

two random variables

## Definition

The joint characteristic function of two random variables  $X$  and  $Y$  is given by

$$\Phi_{X,Y}(\omega_1, \omega_2) = E \left[ e^{j\omega_1 X + j\omega_2 Y} \right]$$

where  $\omega_1$  and  $\omega_2$  are real numbers. An equivalent form is

$$\Phi_{X,Y}(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) e^{j\omega_1 x + j\omega_2 y} dx dy$$

# Joint Characteristic Function and Fourier Transform

two random variables

## Definition

the 2-dimension Fourier transform of  $f_{X,Y}(x,y)$  when signs of  $\omega_1$  and  $\omega_2$  are reversed

$$\Phi_{X,Y}(\omega_1, \omega_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) e^{j\omega_1 x + j\omega_2 y} dx dy$$

the inverse Fourier transform

$$f_{X,Y}(x,y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{X,Y}(\omega_1, \omega_2) e^{-j\omega_1 x - j\omega_2 y} d\omega_1 d\omega_2$$

# Marginal Characteristic Functions

two random variables

## Definition

Marginal characteristic functions are

$$\Phi_X(\omega_1) = \Phi_{X,Y}(\omega_1, 0) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) e^{j\omega_1 x} dx$$

$$\Phi_Y(\omega) = \Phi_{X,Y}(0, \omega_2) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) e^{j\omega_2 y} dy$$

# Joint Moments

two random variables

## Definition

Joint moments  $m_{nk}$  can be found from  $\Phi_{X,Y}(\omega_1, \omega_2)$

$$m_{nk} = (-1)^{n+k} \frac{\partial^{n+k}}{\partial \omega_1^n \partial \omega_2^k} \Phi_{X,Y}(\omega_1, \omega_2) \Big|_{\omega_1=0, \omega_2=0}$$

# Joint Characteristic Function

Random variables

## Definition

The joint characteristic function of  $N$  random variables  $X_1, X_2, \dots, X_N$  is given by

$$\Phi_{X_1, \dots, X_N}(\omega_1, \dots, \omega_N) = E \left[ e^{j\omega_1 X_1 + \dots + j\omega_N X_N} \right]$$

Joint moments are obtained from

$$m_{n_1 \dots n_N} = (-1)^R \frac{\partial^R}{\partial \omega_1^{n_1} \dots \partial \omega_N^{n_N}} \Phi_{X_1, \dots, X_N}(\omega_1, \dots, \omega_N) \Big|_{\text{all } \omega_i = 0}$$

$$R = n_1 + n_2 + \dots + n_N$$





