CLTI Differential Equations (3A)

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6/8/15

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Second Order ODEs

First Order ***Linear Equations***

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_1(x)y' + a_0(x)y = g(x)$$

Second Order ***Linear Equations***

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

Second Order ***Linear Equations*** ***with Constant Coefficients***

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = g(x)$$

$$a_2 y'' + a_1 y' + a_0 y = g(x)$$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c y = g(x)$$

$$a y'' + b y' + c y = g(x)$$

Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

try a solution $y = e^{mx}$

$$a\frac{d^2}{dx^2}\{e^{mx}\} + b\frac{d}{dx}\{e^{mx}\} + c\{e^{mx}\} = 0$$

$$a\{e^{mx}\}'' + b\{e^{mx}\}' + c\{e^{mx}\} = 0$$

$$a\{m^2e^{mx}\} + b\{me^{mx}\} + c\{e^{mx}\} = 0$$

$$a\{m^2e^{mx}\} + b\{me^{mx}\} + c\{e^{mx}\} = 0$$

$$(am^2 + bm + c)\cdot e^{mx} = 0$$

$$(am^2 + bm + c)\cdot e^{mx} = 0$$

auxiliary equation

$$(am^2 + bm + c) = 0$$

$$(am^2 + bm + c) = 0$$

Roots of the Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$a y'' + b y' + c y = 0$$

auxiliary equation

try a solution $y = e^{mx}$ → $(am^2 + bm + c) = 0$

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$	(A)	$b^2 - 4ac > 0$	Real, distinct m_1, m_2
$y_1 = e^{m_1 x} = y_2 = e^{m_2 x}$	(B)	$b^2 - 4ac = 0$	Real, equal m_1, m_2
$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$	(C)	$b^2 - 4ac < 0$	Conjugate complex m_1, m_2

Linear Combination of Solutions

DEQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

y_1

y_2

$$C_1 y_1 + C_2 y_2$$

$$a y_1'' + b y_1' + c y_1 = 0$$

$$a y_2'' + b y_2' + c y_2 = 0$$

$$a(y_1''+y_2'') + b(y_1'+y_2') + c(y_1+y_2) = 0$$

$$a(y_1+y_2)'' + b(y_1+y_2)' + c(y_1+y_2) = 0$$

$$y_3 = y_1 + y_2$$

$$y_4 = y_1 - y_2$$

$$y_5 = y_3 + 2y_4$$

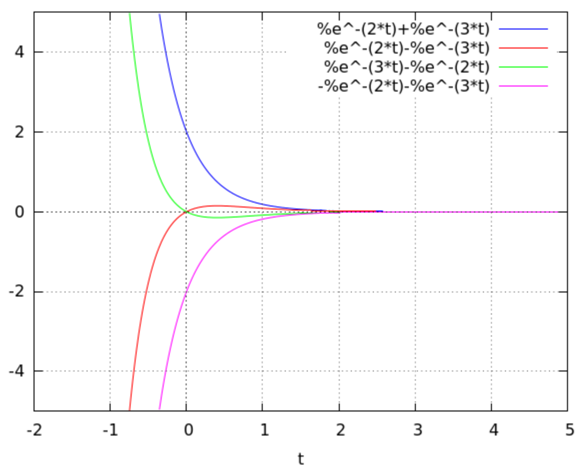
$$y_6 = y_3 - 2y_4$$

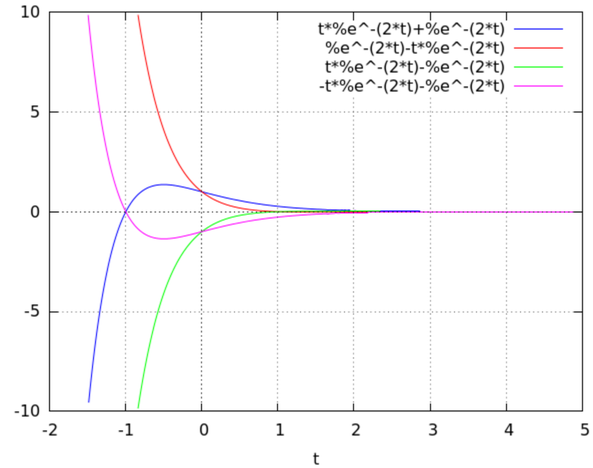
$$a(C_1y_1''+C_2y_2'') + b(C_1y_1'+C_2y_2') + c(C_1y_1+C_2y_2) = 0$$

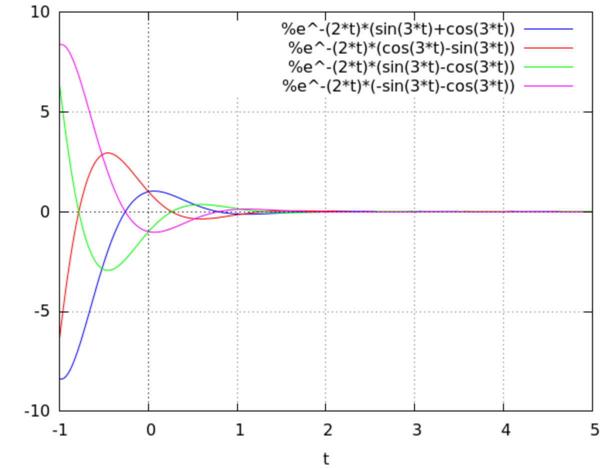
$$a(C_1y_1+C_2y_2)'' + b(C_1y_1+C_2y_2)' + c(C_1y_1+C_2y_2) = 0$$

Three Cases

overdamping	critical damping	underdamping







$$f(t) = C_1 e^{-2t} + C_2 e^{-3t}$$

$$g(t) = C_1 e^{-2t} + C_2 t e^{-2t}$$

$$h(t) = e^{-2t}(C_3 \cos(3t) + C_4 \sin(3t))$$

$b^2 - 4ac > 0$	Real, distinct m_1, m_2
$b^2 - 4ac = 0$	Real, equal m_1, m_2
$b^2 - 4ac < 0$	Conjugate complex m_1, m_2

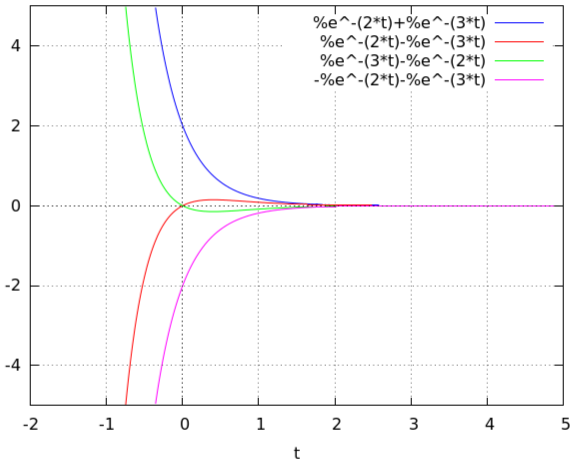
$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

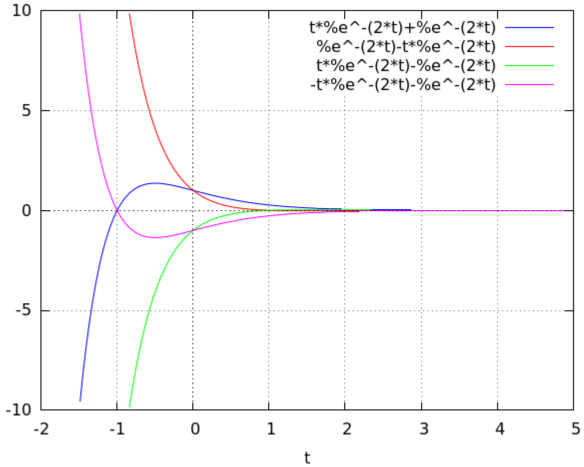
$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$$

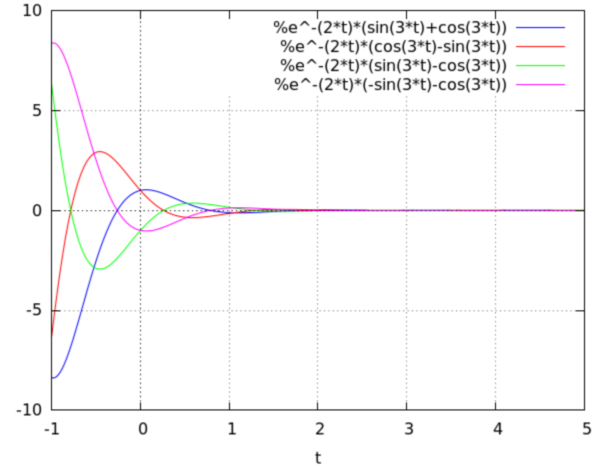
$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

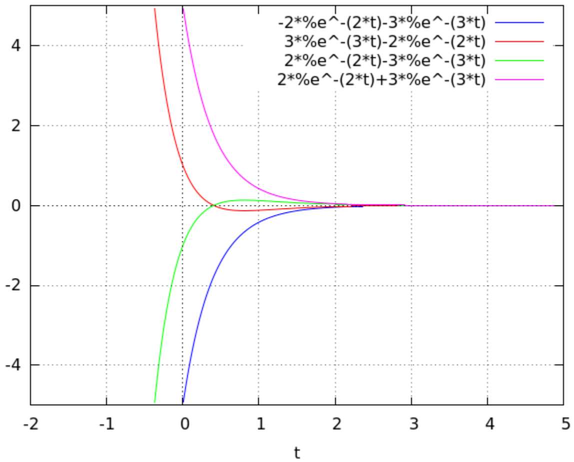
Three Cases

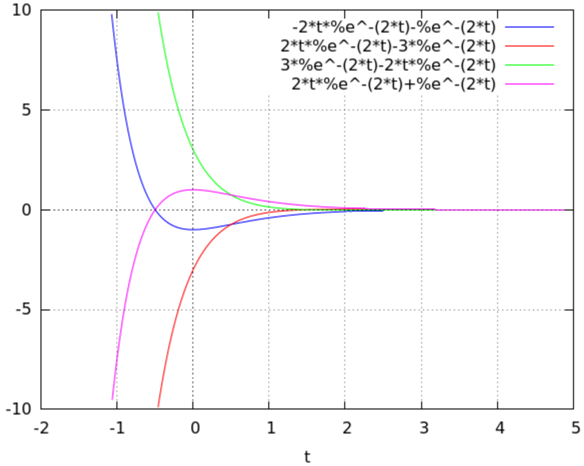
overdamping | critical damping | underdamping

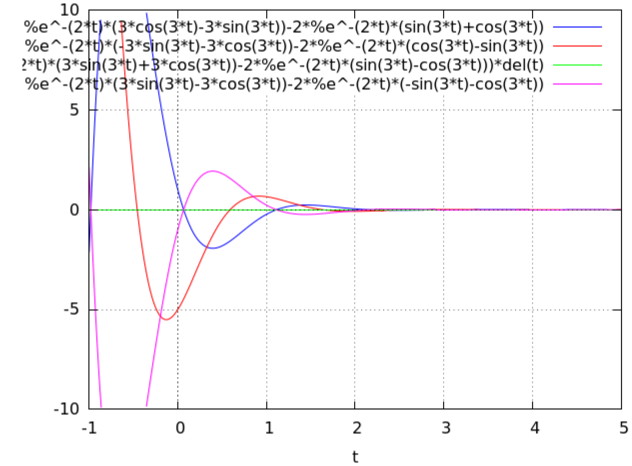












$f'(t) = (C_1 e^{-2t} + C_2 e^{-3t})'$ $\quad$ $g'(t) = (C_1 e^{-2t} + C_2 t\, e^{-2t})'$ $\quad$ $h'(t) = [e^{-2t}(C_3 \cos(3t) + C_4 \sin(3t))]'$

Three Cases

overdamping

$$f(t) = (C_1 e^{-2t} + C_2 e^{-3t})$$

$$f'(t) = (-2C_1 e^{-2t} - 3C_2 e^{-3t})$$

$$f(t) + f'(t) =$$

$$(-C_1 e^{-2t} - 2C_2 e^{-3t})$$

critical damping

$$g(t) = (C_1 e^{-2t} + C_2 t e^{-2t})$$

$$g'(t) = (-2C_1 e^{-2t} + C_2(-2t+1)e^{-2t})$$

$$g(t) + g'(t) =$$

$$(-C_1 e^{-2t} + C_2(-t+1)e^{-2t})$$

underdamping

$$h(t) = [e^{-2t}(C_3 \cos(3t) + C_4 \sin(3t))]$$

$$h'(t) = [e^{-2t}(-2C_3 \cos(3t) - 2C_4 \sin(3t))]$$
$$+[e^{-2t}(-3C_3 \sin(3t) + 3C_4 \cos(3t))]$$

$$= e^{-2t}[(-2C_3 + 3C_4)\cos(3t)]$$
$$+ e^{-2t}[(-3C_3 - 2C_4)\sin(3t)]$$

$$h(t) + h'(t) =$$

$$e^{-2t}[(-C_3 + 3C_4)\cos(3t)]$$
$$+ e^{-2t}[(-3C_3 - C_4)\sin(3t)]$$

Complex Exponential Conversion

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

$$m_1 = \left(-b + \sqrt{b^2 - 4ac}\right)/2a \Rightarrow m_1 = \left(-b + \sqrt{4ac - b^2}\, i\right)/2a = \alpha + i\beta$$

$$m_2 = \left(-b - \sqrt{b^2 - 4ac}\right)/2a \Rightarrow m_2 = \left(-b - \sqrt{4ac - b^2}\, i\right)/2a = \alpha - i\beta$$

$$y_1 = e^{m_1 x}$$

$$y_2 = e^{m_2 x}$$

$b^2 - 4ac < 0$ Conjugate complex m_1, m_2

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

Pick **two** homogeneous solution

$$y_1 = \left\{e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x}\right\}/2 = e^{\alpha x}\cos(\beta x) \qquad (C_1 = +1/2, \quad C_2 = +1/2)$$

$$y_2 = \left\{e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}\right\}/2i = e^{\alpha x}\sin(\beta x) \qquad (C_1 = +1/2i, \quad C_2 = -1/2i)$$

$$y = C_3 e^{\alpha x}\cos(\beta x) + C_4 e^{\alpha x}\sin(\beta x) = e^{\alpha x}\left(C_3\cos(\beta x) + C_4\sin(\beta x)\right)$$

Fundamental Set Examples (1)

Second Order EQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

y_1 $\quad e^{(\alpha+i\beta)x}$

y_2 $\quad e^{(\alpha-i\beta)x}$

$$y_3 = \frac{1}{2} y_1 + \frac{1}{2} y_2 \qquad \{e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x}\}/2 = e^{\alpha x}\cos(\beta x)$$

$$y_4 = \frac{1}{2i} y_1 - \frac{1}{2i} y_2 \qquad \{e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}\}/2i = e^{\alpha x}\sin(\beta x)$$

Fundamental Set Examples (2)

Second Order EQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$y_1 = y_3 + i\, y_4 \qquad e^{(\alpha+i\beta)x}$$

$$y_2 = y_3 - i\, y_4 \qquad e^{(\alpha-i\beta)x}$$

$$y_3 = e^{\alpha x}\cos(\beta x)$$

$$y_4 = e^{\alpha x}\sin(\beta x)$$

$$e^{\alpha x}[\cos(\beta x) + i\sin(\beta x)]$$

$$e^{\alpha x}[\cos(\beta x) - i\sin(\beta x)]$$

General Solution Examples

Second Order EQ

$$a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = 0$$

linearly independent → Fundamental Set of Solutions

$$\{y_1,\ y_2\} = \{e^{(\alpha+i\beta)x},\ e^{(\alpha-i\beta)x}\}$$

$$C_1\,y_1 + C_2\,y_2$$

$$C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$$

General Solution

linearly independent → Fundamental Set of Solutions

$$\{y_3,\ y_4\} = \{e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x)\}$$

$$c_3\,y_3 + c_3\,y_4$$

$$c_3 e^{\alpha x}\cos(\beta x) + c_4 e^{\alpha x}\sin(\beta x)$$
$$= e^{\alpha x}(c_3\cos(\beta x) + c_4\sin(\beta x))$$

General Solution

Finding a Particular Solution
- Undetermined Coefficients

Three Differential Equations

y_p **particular** solutions

EQ 1

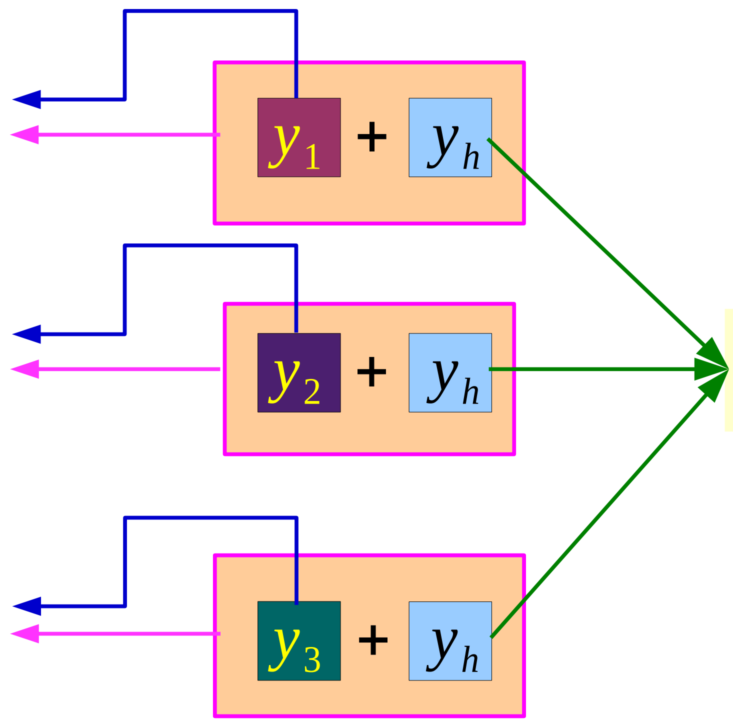
$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{d\,y(t)}{dt} + a_2 y(t) = x_1(x)$$

EQ 2

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{d\,y(t)}{dt} + a_2 y(t) = x_2(x)$$

EQ 3

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{d\,y(t)}{dt} + a_2 y(t) = x_3(x)$$



$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{d\,y(t)}{dt} + a_2 y(t) = 0$$

general solutions

y_h **homogeneous** solution

Particular Solutions

DEQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x) \longleftarrow y_p$$

particular solution
by a conjecture

(I) FORM Rule

(II) Multiplication Rule

When coefficients are constant

And

$g(x) =$

- ***A constant or*** .... k
- ***A polynomial or*** .... $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$
- ***An exponential function or*** .... $e^{\alpha x}$
- ***A sine and cosine functions or*** .... $\sin(\beta x) \quad \cos(\beta x)$
- ***Finite sum and products of the above functions*** .... $e^{\alpha x}\sin(\beta x) + x^2$

And

$$g(x) \neq \quad \ln x \quad \frac{1}{x} \quad \tan x \quad \sin^{-1} x$$

Form Rule

DEQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x) \longleftarrow y_p$$

When coefficients are constant

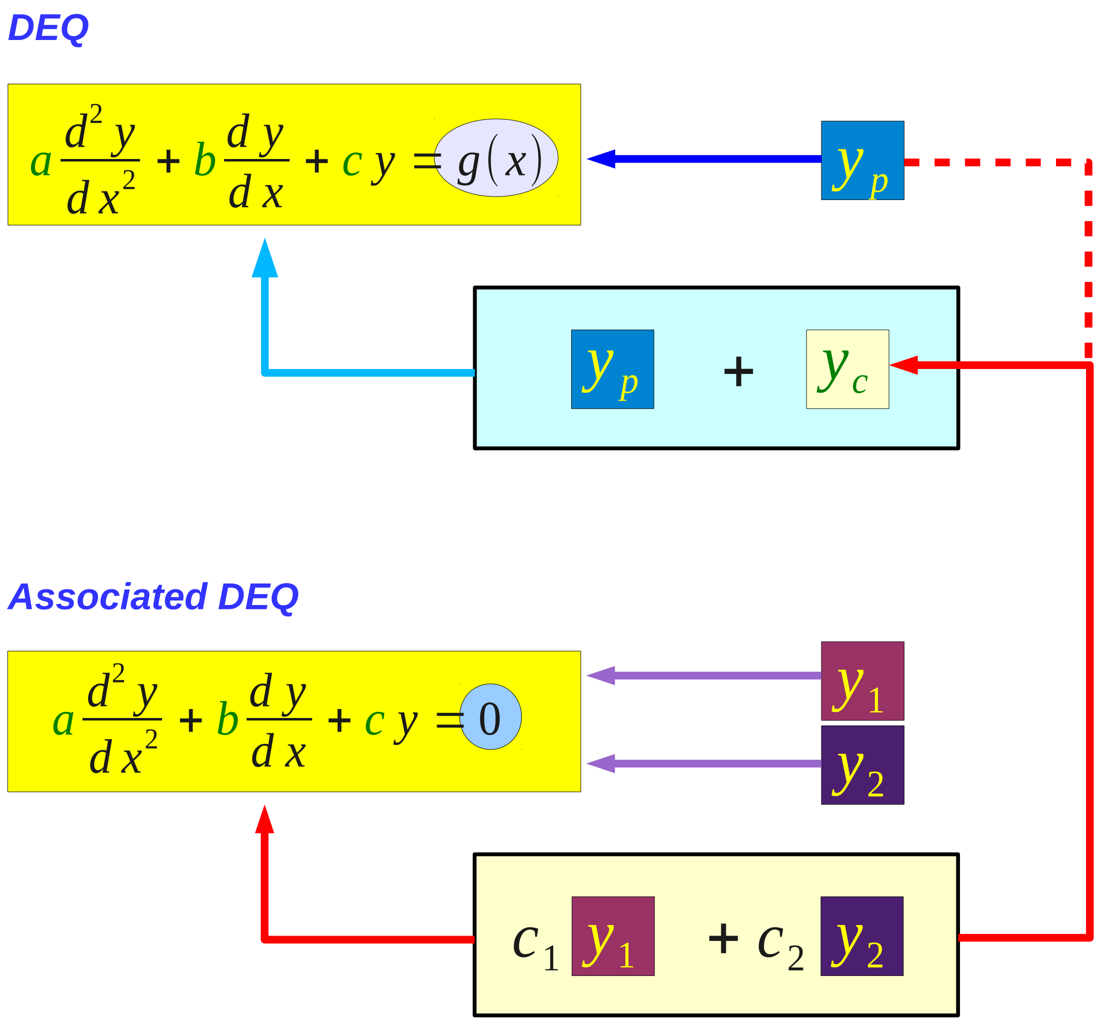
particular solution
by a conjecture

(I) FORM Rule

(II) Multiplication Rule

$$\begin{aligned}
g(x) &= 2 & y_p &= A \\
g(x) &= 3x+4 & y_p &= Ax+B \\
g(x) &= 5x^2 & y_p &= Ax^2+Bx+C \\
g(x) &= 6x^2-7 & y_p &= Ax^2+Bx+C \\
g(x) &= \sin 8x & y_p &= A\cos 8x+B\sin 8x \\
g(x) &= \cos 9x & y_p &= A\cos 9x+B\sin 9x \\
g(x) &= e^{10x} & y_p &= Ae^{10x} \\
g(x) &= xe^{11x} & y_p &= (Ax+B)e^{11x} \\
g(x) &= e^{11x}\sin 12x & y_p &= Ae^{11x}\sin 12x+Be^{11x}\cos 12x
\end{aligned}$$

Multiplication Rule



use $y_p = x^n y_1$ $y_p = x^n y_2$

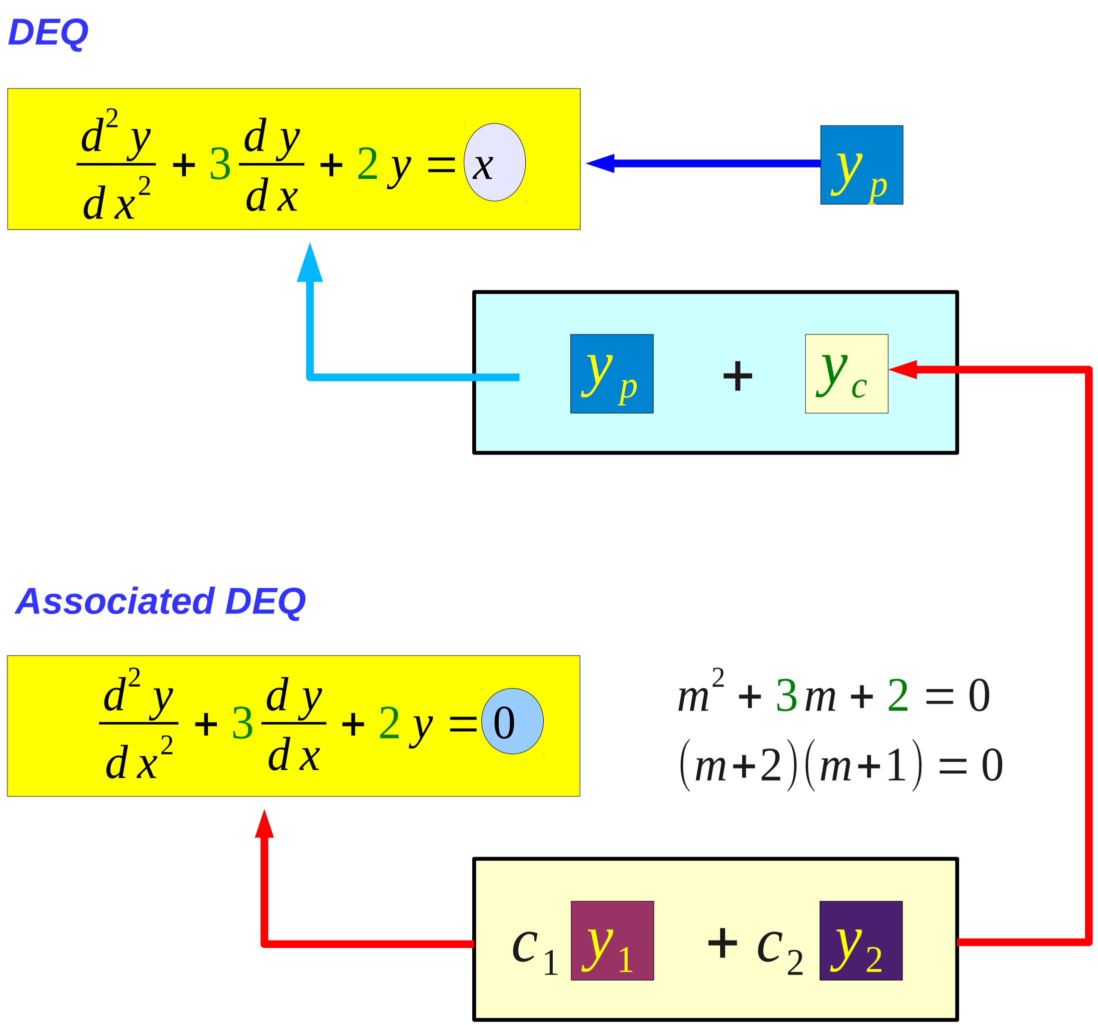
if $y_p = y_1$ $y_p = y_2$

Any y_p contains a term which is the same term in y_c

Use y_p multiplied by x^n

n is the smallest positive integer that eliminates the duplication

Example – Form Rule (1)



$$y_p = Ax + B$$
$$y_p' = A$$
$$y_p'' = 0$$

$$y_p'' + 3y_p' + 2y_p$$
$$= 3A + 2(Ax + B)$$
$$= 2Ax + 3A + 2B$$
$$= x$$

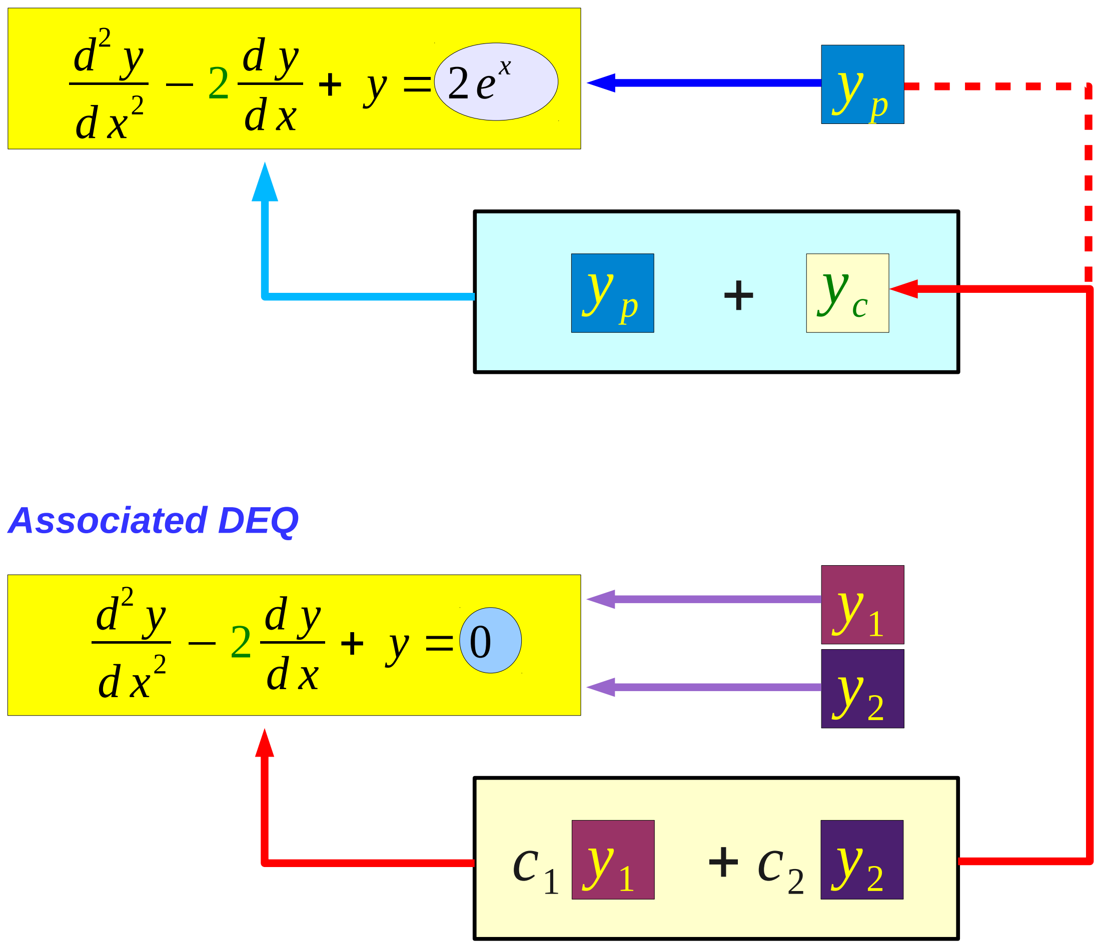
$$2A = 1 \qquad A = \tfrac{1}{2}$$
$$3A + 2B = 0 \qquad B = -\tfrac{3}{4}$$

$$y_p = \tfrac{1}{2}x - \tfrac{3}{4}$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + \tfrac{1}{2}x - \tfrac{3}{4}$$

Example – Multiplication Rule (2)

DEQ



Associated DEQ

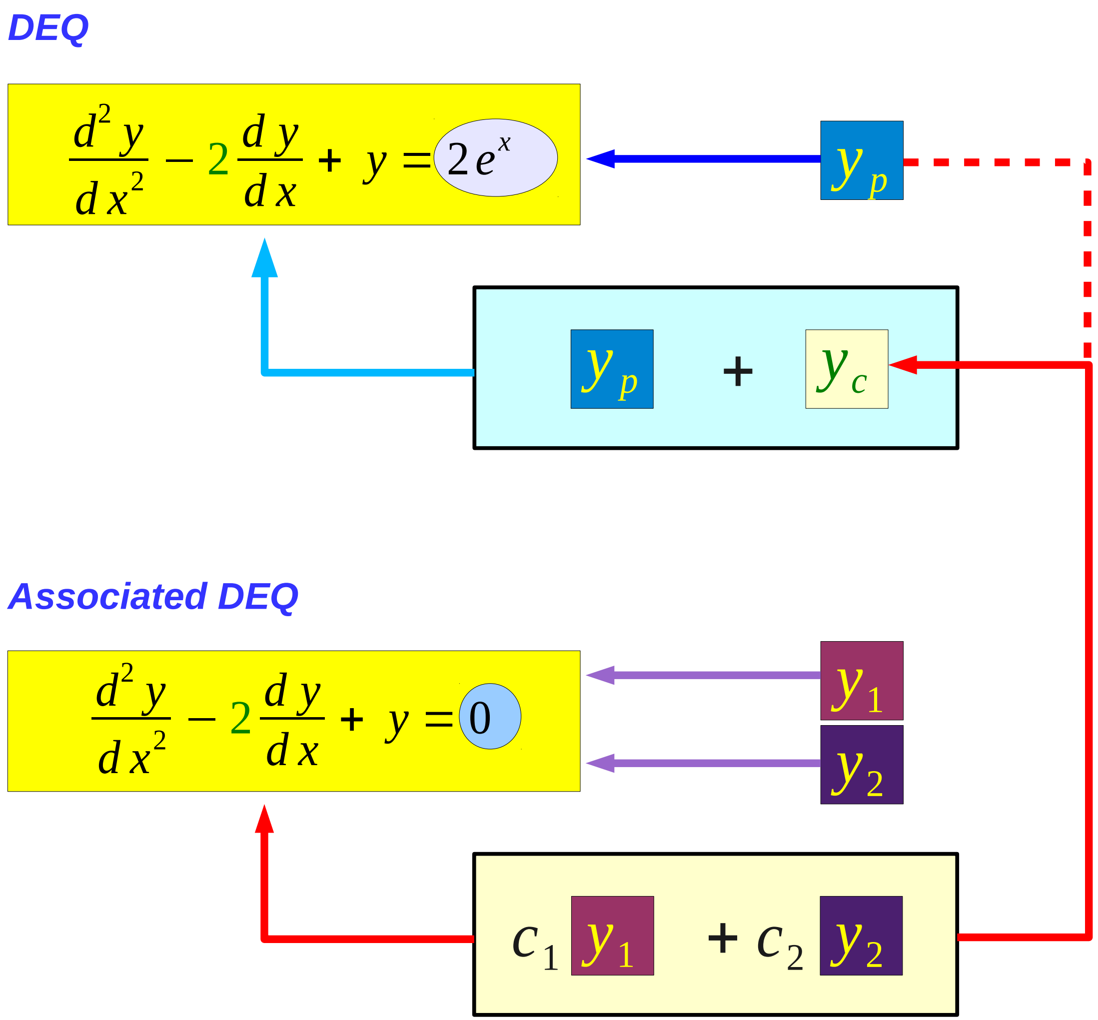
$$y'' - 2y' + y = 2e^x$$

$$y_p = \cancel{Ae^x} \Rightarrow \cancel{Axe^x} \Rightarrow Ax^2e^x$$

$$y_1 = e^x \qquad y_2 = xe^x$$

$$y'' - 2y' + y = 0$$

Example – Multiplication Rule (3)



$$y'' - 2y' + y = 6xe^x$$

$$y_p = Axe^x \Rightarrow Ax^2e^x \Rightarrow Ax^3e^x$$

$$LHS: 2Ae^x$$

$$y_1 = e^x \qquad y_2 = xe^x$$

$$y'' - 2y' + y = 0$$

Forced Response – variation of parameters

$$y'' + p\,y' + q\,y = f(x)$$
$$y(x_0) = 0$$
$$y'(x_0) = 0$$

Nonhomogeneous DEQ

Zero State Response

Zero Initial Conditions

Initially at rest

Green's function (Impulse Response)

$$y_p = u_1(x)\,y_1 + u_2(x)\,y_2 \quad = \int_{x_0}^{x} \left[\frac{y_1(t)\,y_2(x) - y_1(x)\,y_2(t)}{W(t)}\right] f(t)\,dt \quad = \int_{x_0}^{x} G(x,\ t)\,f(t)\,dt$$

$$y_p(x) = \int_{x_0}^{x} G(x,\ t)\,f(t)\,dt$$

$$y_p'(x) = G(x,\ x)\,f(x) + \int_{x_0}^{x} \frac{\partial}{\partial x}\left[G(x,\ t)\,f(t)\right] dt = \int_{x_0}^{x} \left[\frac{y_1(t)\,y_2'(x) - y_1'(x)\,y_2(t)}{W(t)}\right] f(t)\,dt$$

$$y_p(x_0) = \int_{x_0}^{x_0} G(x,\ t)\,f(t)\,dt = 0$$

$$y_p'(x_0) = \int_{x_0}^{x_0} \left[\frac{y_1(t)\,y_2'(x_0) - y_1'(x_0)\,y_2(t)}{W(t)}\right] f(t)\,dt = 0$$

Variation of Parameter [c → u(x)]

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

$$y_p'' + P(x)y_p' + Q(x)y_p = \frac{d}{dx}\begin{pmatrix} + u_1' y_1 \\ + u_2' y_2 \end{pmatrix} + P \begin{pmatrix} + u_1' y_1 \\ + u_2' y_2 \end{pmatrix} + \begin{pmatrix} + u_1' y_1' \\ + u_2' y_2' \end{pmatrix} = f(x)$$

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = f(x) \end{cases}$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$u_1'(x) = \frac{W_1}{W} = -\frac{y_2(x) f(x)}{W}$$

$$u_2'(x) = \frac{W_2}{W} = \frac{y_1(x) f(x)}{W}$$

Superposition

DEQ

$$\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + cy = 2x^2 + 3 + \cos 8x$$

$$(2x^2 + 3) + (\cos 8x)$$

$$\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad \leftarrow \quad y_c$$

$$\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + cy = 2x^2 + 3 \quad \leftarrow \quad y_{p1}$$

$$\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + cy = \cos 8x \quad \leftarrow \quad y_{p2}$$

$$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

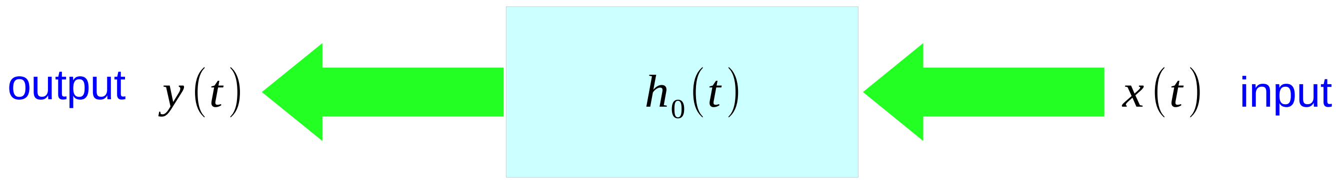
$$y_{p1} = Ax^2 + Bx + C$$

$$y_{p2} = E\cos 8x + F\sin 8x$$

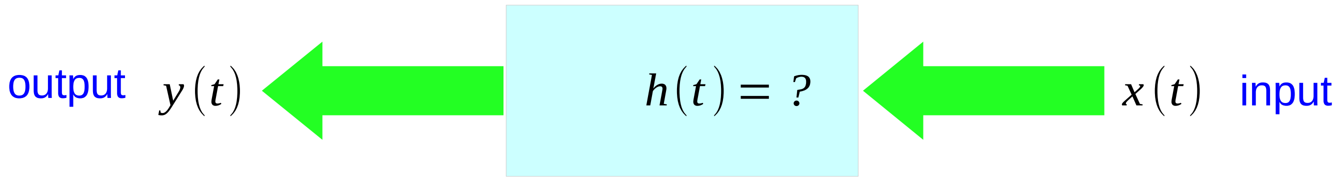
$$\frac{d^2}{dx^2}[y_c + y_{p1} + y_{p2}] + b\frac{d}{dx}[y_c + y_{p1} + y_{p2}] + c[y_c + y_{p1} + y_{p2}] = 2x^2 + 3 + \cos 8x$$

ODE's and Causal LTI Systems

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^N} + \cdots + a_1 \frac{d\, y(t)}{dt} + a_0 y(t) = x(t)$$



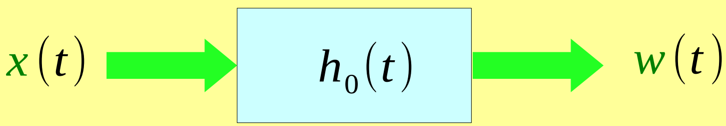
$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^N} + \cdots + a_1 \frac{d\, y(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{d\, x(t)}{dt} + b_0 x(t)$$



Base System & Derived System

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = x$$

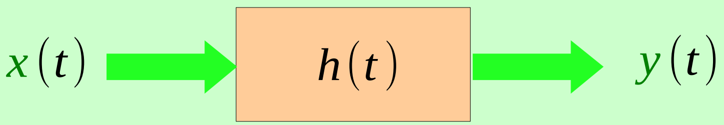
base system



notation: $y^{(N)} = \frac{d^N y}{dt^N} = \frac{d^N}{dt^N} y(t)$

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$

derived system



notation: $x^{(N)} = \frac{d^N x}{dt^N} = \frac{d^N}{dt^N} x(t)$

General System & Base System

general system

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} \quad \Rightarrow \quad b_0 h_0^{(N)}$$

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_1 x^{(N-1)} \quad \Rightarrow \quad b_1 h_0^{(N-1)}$$

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = \cdots$$

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_N x \quad \Rightarrow \quad b_N h_0$$

derivatives of the impulse response of the **base system**

$$h = b_0 h_0^{(N)} + b_1 h_0^{(N-1)} + \cdots + b_{N-1} h_0^{(1)} + b_N h_0$$

h(*t*) = Impulse response of the **general system**

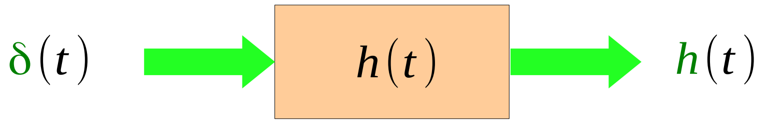
base system

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = x \quad \Rightarrow \quad h_0$$

$h_0(t)$ = Impulse response of the **base system**

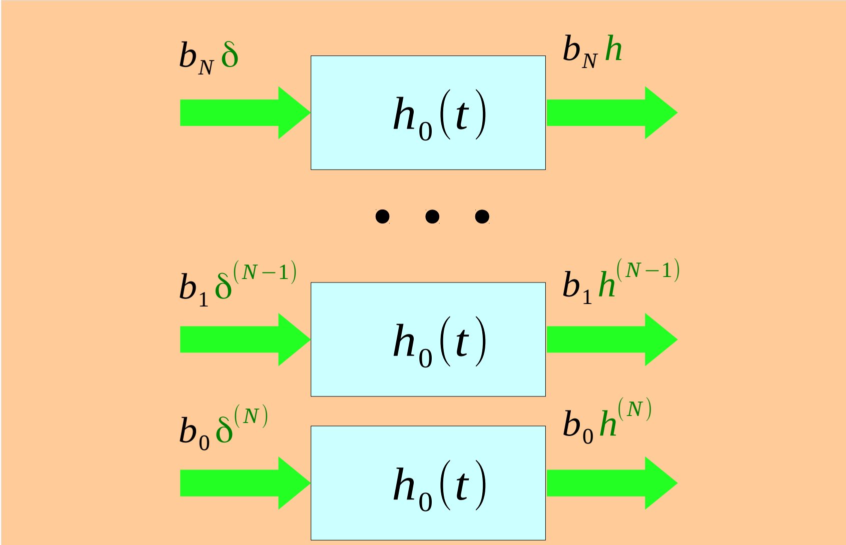
Superposition of derivatives of a delta function

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 \delta^{(N)} + b_1 \delta^{(N-1)} + \cdots + b_{N-1} \delta^{(1)} + b_N \delta$$



$$h(t) = b_0 h_0^{(N)} + b_1 h_0^{(N-1)} + \cdots + b_{N-1} h_0^{(1)} + b_N h_0 = P(D) h_0(t)$$

superposition of the derivatives of delta function



$$h_0^{(N)} + a_1 h_0^{(N-1)} + \cdots + a_N h_0 = b_N \delta$$

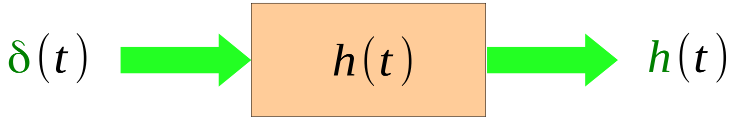
$$\bullet \bullet \bullet \qquad \bullet \bullet \bullet \qquad \bullet \bullet \bullet$$

$$h_0^{(2N-1)} + a_1 h_0^{(2N-2)} + \cdots + a_N h_0^{(N-1)} = b_{N-1} \delta^{(N-1)}$$

$$h_0^{(2N)} + a_1 h_0^{(2N-1)} + \cdots + a_N h_0^{(N)} = b_0 \delta^{(N)}$$

General System: Impulse Response

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$



$$h = b_0 h_0^{(N)} + b_1 h_0^{(N-1)} + \cdots + b_{N-1} h_0^{(1)} + b_N h_0 = P(D) h_0(t)$$

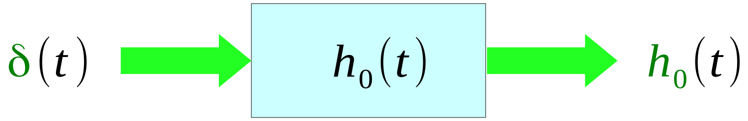
$$Q(D) = (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)$$

$$P(D) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)$$

$$Q(D) h_0(t) = \delta(t)$$

$$Q(D) P(D) h_0(t) = P(D) \delta(t)$$

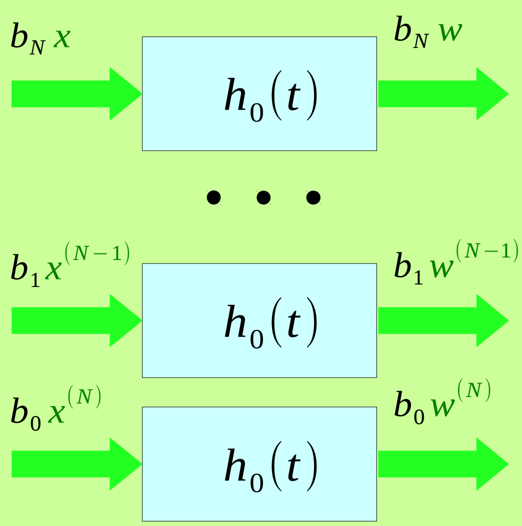
$$P(D) h_0(t) \Rightarrow h(t)$$



Superposition of derivatives of an input

general system **base system**

$$y(t) = b_0 w^{(N)} + b_1 w^{(N-1)} + \cdots + b_{N-1} w^{(1)} + b_N w = P(D)w(t)$$



$$w^{(N)} + a_1 w^{(N-1)} + \cdots + a_N w = b_N x$$

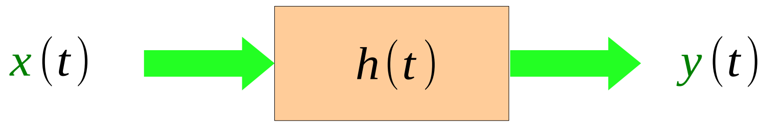
$$\bullet \bullet \bullet \qquad \bullet \bullet \bullet \qquad \bullet \bullet \bullet$$

$$w^{(2N-1)} + a_1 w^{(2N-2)} + \cdots + a_N w^{(N-1)} = b_{N-1} x^{(N-1)}$$

$$w^{(2N)} + a_1 w^{(2N-1)} + \cdots + a_N w^{(N)} = b_0 x^{(N)}$$

General System: Output

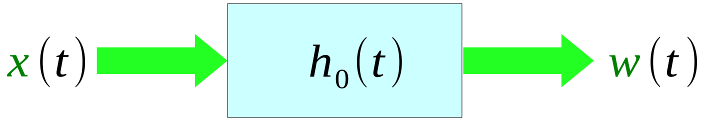
$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$



$$y(t) = b_0 w^{(N)} + b_1 w^{(N-1)} + \cdots + b_{N-1} w^{(1)} + b_N w = P(D) w(t)$$

$$Q(D) = (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)$$

$$P(D) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)$$



$$Q(D) w(t) = x(t)$$

$$Q(D) P(D) w(t) = P(D) x(t)$$

$$P(D) w(t) \Rightarrow y(t)$$

ODE's and Causal LTI Systems

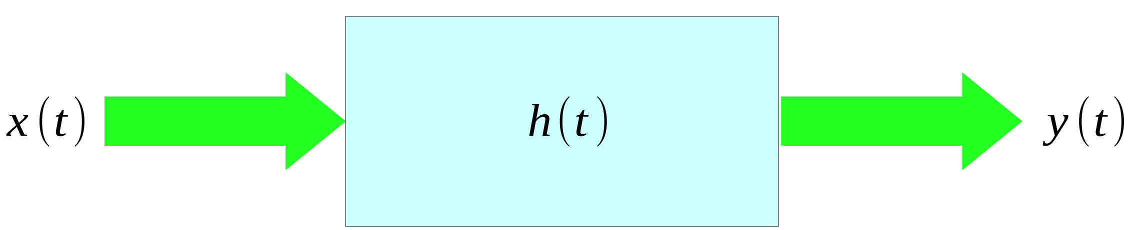
$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^N} + \cdots + a_1 \frac{d\,y(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{d\,x(t)}{dt} + b_0 x(t)$$

N: the highest order of derivatives of the output y(t) (LHS)

M: the highest order of derivatives of the input x(t) (RHS)

N < M : (M-N) differentiator – magnify high frequency components of noise (seldom used)

N > M : (N-M) Integrator



Different Indexing Schemes

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^N} + \cdots + a_1 \frac{d\, y(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{d\, x(t)}{dt} + b_0 x(t)$$

$[N > M]$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\, x(t)}{dt} + b_M x(t)$$

$[N = M]$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_0 \frac{d^N x(t)}{dt^N} + b_1 \frac{d^{N-1} x(t)}{dt^{N-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t)$$

$[N > M]$ $\quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{M-1} D + b_M) x(t)$

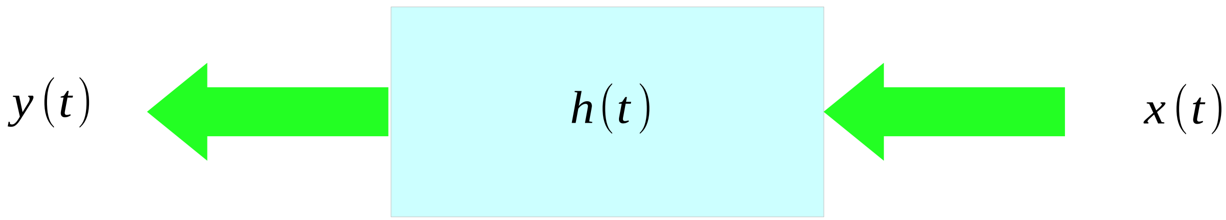
$[N = M]$ $\quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N) x(t)$

ODE Solutions and System Responses

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_0 \frac{d^N x(t)}{dt^N} + b_1 \frac{d^{N-1} x(t)}{dt^{N-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

$$Q(D)\, y(t) = P(D)\, x(t)$$

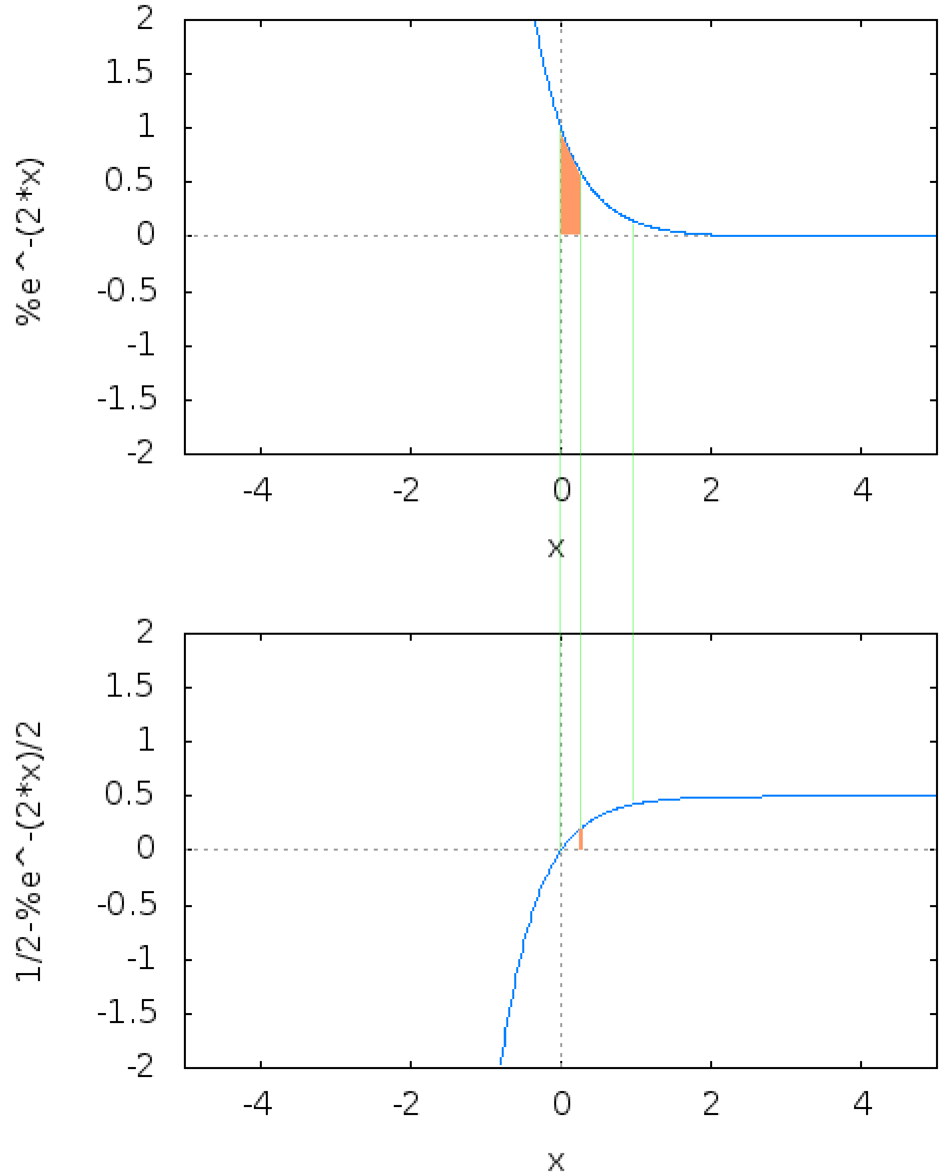


$$y(t) = f(t, x(t))$$

- Zero Input Response
- Zero State Response (Convolution with h(t))

- Natural Response (Homogeneous Solution)
- Forced Response (Particular Solution)

Converting Initial Conditions



$$y_1(t) = e^{-2t}$$

$$y_2(x) = \int_0^x e^{-2t}\,dt \quad = 1 - \frac{1}{2}e^{-2x}$$

References

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