Random Processes

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Random Variable Definition

A random variable

a **function** over a **sample space** $S = \{s_1, s_2, s_3, ..., s_n\}$

$$s \rightarrow X(s)$$

$$x = X(s)$$

a **function** of a possible **outcome** s of an **experiment**

Random Variable Definition

A random variable

- a **random variable** : a capital letter X
- a particular value : a lowercase letter x
- a **sample space** $S = \{s_1, s_2, s_3, \ldots, s_n\}$
- an **outcome** (an element of S) : s

$s \rightarrow X(s)$

$x = X(s)$

$s \rightarrow x$

Understanding Random Variables (1)

- **random variables** are used to quantify outcomes of a random occurrence, and therefore, can take on many values.
- **random variables** are required to be measurable and are typically real numbers.

for example, the letter X may be designated
to represent the *sum* of the resulting numbers
after *three dice* are rolled.
therefore, X could be 3 (1 + 1+ 1), 18 (6 + 6 + 6),
or somewhere between 3 and 18

https://www.investopedia.com/terms/r/random-variable.asp

Understanding Random Variables (2)

- A **random variable** is different from an algebraic variable.
  - The variable in an algebraic equation is an unknown value that can be calculated.
  - The equation $10 + x = 13$ shows that we can calculate the specific value for x which is 3.
  - a **random variable** has a set of values, and any of those values *could be* the resulting outcome as seen in the example of the dice above.

https://www.investopedia.com/terms/r/random-variable.asp

Understanding Random Variables (3)

- A typical example of a random variable is the outcome of a coin toss.
    - Consider a probability distribution in which the outcomes of a random event are not equally likely to happen.
    - If the **random variable** Y is the number of heads we get from tossing two coins, then Y could be 0, 1, or 2. (no heads, one head, or both heads)

https://www.investopedia.com/terms/r/random-variable.asp

Understanding Random Variables (4)

- Let the **random variable** Y be the number of heads of tossing two coins
  - the two coins land in four different ways: TT, HT, TH, and HH.
  - the $P(Y = 0) = 1/4$ since we have one chance of getting no heads (i.e., two tails [TT] when the coins are tossed).
  - the $P(Y = 2) = 1/4$: the probability of getting two heads (HH)
  - getting one head has a likelihood of occurring twice: in HT and TH. In this case, $P(Y = 1) = 2/4 = 1/2$.

https://www.investopedia.com/terms/r/random-variable.asp

Formal definition of a random variable (1)

- A **random variable** X is a measurable function $X : \Omega \to E$ from a set of possible outcomes Ω to a measurable space E.
- The technical axiomatic definition requires Ω to be a sample space of a probability triple $(\Omega, \mathscr{F}, P)$
- A **random variable** is often denoted by capital roman letters such as X, Y, Z, T.

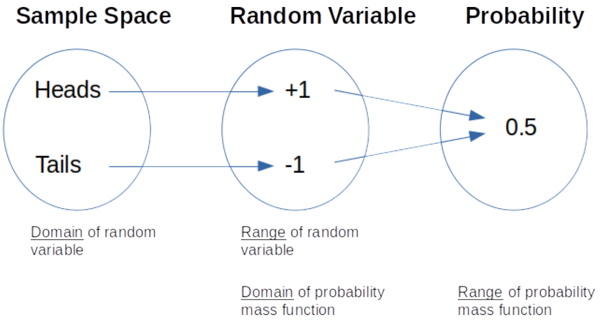
https://en.wikipedia.org/wiki/Random_variable

Formal definition of a random variable (2)

- a measurable function $X \colon \Omega \to E$
  from a set of possible outcomes Ω
  to a measurable space E.
- a sample space of a probability triple $(\Omega, \mathscr{F}, P)$
- The **probability** that X takes on a value
  in a measurable set $S \subseteq E$ is written as
  $P(X \in S) = P(\{\omega \in \Omega \mid X(\omega) \in S\})$

https://en.wikipedia.org/wiki/Random_variable

Random variable example



This graph shows how random variable is a function from all possible outcomes to real values.
It also shows how random variable is used for defining probability mass functions.

https://en.wikipedia.org/wiki/Random_variable

Probability Space (1)

a **probability space** or a **probability triple** $(\Omega, \mathscr{F}, P)$ is a mathematical construct that provides a formal model of a random process or "experiment".

- For example, one can define a **probability space** which models the throwing of a die

https://en.wikipedia.org/wiki/Probability_space

Probability Space (2)

A **probability space** consists of three elements $(\Omega, \mathscr{F}, P)$

- A **sample space**, Ω
  the set of all possible outcomes.
- An **event space**
  a set of events $\mathscr{F}$,
  an event is a set of outcomes in the sample space
- A **probability function**, P
  assigns each event in the event space a probability,
  which is a number between 0 and 1.

https://en.wikipedia.org/wiki/Probability_space

Probability Space (3)

In the example of the throw of a standard die,

- we would take the **sample space** to be $\{1,2,3,4,5,6\}$.
- For the **event space**, we could simply use the set of all subsets of the sample space,
  - simple events such as $\{5\}$ ("the die lands on 5"),
  - as well as complex events such as $\{2,4,6\}$ ("the die lands on an even number").
- Finally, for the **probability function**,

  we would map each event to the number of outcomes in that event divided by 6
  - so for example, $\{5\}$ would be mapped to $1/6$, and $\{2,4,6\}$ would be mapped to $3/6 = 1/2$.

https://en.wikipedia.org/wiki/Probability_space

Random Process (1)

A random process

a function of both **outcome** s and **time** t

$$X(t,s)$$

assigning a time function to every **outcome** s_i

$$s_i \rightarrow x(t,s_i)$$

Random Process (2)

A random process

the family of such time functions is called a random process

$$x(t, s_i) = X(t, s_i)$$

$$x(t, s) = X(t, s)$$

Random Process (3)

We have seen that a **random variable** X is
a rule which assigns a number
to every outcome e of an experiment.

The **random variable** is a function $X(e)$
that maps the set of experiment outcomes
to the set of numbers.

A **random process** is a rule
that maps every outcome e of an experiment
to a function $X(t, e)$.

https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf

Random Process (4)

- A **random process** is usually conceived of as a function of time, but there is no reason to not consider **random processes** that are functions of other independent variables, such as spatial coordinates.
- The function $X(u, v, e)$ would be a function whose value depended on the location (u, v) and the outcome e, and could be used in representing random variations in an image

https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf

Random Process (5)

- The domain of e is the set of outcomes of the experiment.
- We assume that a probability distribution is known for this set.
- The domain of t is a set, T , of real numbers. If T is the real axis then $X(t, e)$ is a continuous-time **random process**, and if T is the set of integers then $X(t, e)$ is a discrete-time **random process**

https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf

Ensemble of time functions

Time functions

A random process $X(t, s)$ represents
a family or ensemble of time functions

$X(t, s)$ represents

- a single time function $x(t, s)$
- when t is a variable and s is fixed at an outcome

$x(t, s)$ represents

- a sample function,
- an ensemble member,
- a realization of the process

Short-form notation for time functions

The short-form notation $x(t)$

to represent a specific waveform of a random process $X(t)$
for a given **outcome** s_i

$$x(t) = x(t, s)$$

$$X(t) = X(t, s)$$

Random Process Example

Example

$$X(t, s_1) = x_1(t) \qquad s_1 \longrightarrow x_1(t)$$
$$X(t, s_2) = x_2(t) \qquad s_2 \longrightarrow x_2(t)$$
$$\ldots \qquad \ldots$$
$$X(t, s_n) = x_n(t) \qquad s_n \longrightarrow x_n(t)$$

$S = \{s_1, s_2, s_3, \ldots, s_n\}$ a sample space

$X(t) = \{x_1(t), x_2(t), x_3(t), \ldots, x_n(t)\}$ a random process

Random variables with time

a **random process** $X(t,s)$ represents a single time function when t is a variable and s is fixed at an outcome

a random process $X(t,s)$ represents a single **random variable** when both t and s are fixed at a time and an outcome, respectively

$$X_i = X(t_i, s) = X(t_i)$$ ***random variable***

$$X(t,s) = X(t)$$ ***random process***

An alphabet

the alphabet of $X(t)$

the set of its possible values

- the values of **time** t for which a random process is defined
- the **alphabet** of the random variable $X = X(t)$ at time t

Classification of Random Processes

(1) Types of time and alphabet

- the values of **time** t for which a random process is defined
  - continuous time
  - discrete time
- the **alphabet** of the random variable $X = X(t)$ at time t
  - continuous alphabet
  - discrete alphabet

Classification of Random Processes

(2) types of the random variable $X(t)$ and the time t

- a continuous **alphabet** continuous **time** random process
  - $X(t)$ has continuous values and t has continuous values
- a discrete **alphabet** continuous **time** random process
  - $X(t)$ has discrete values and t has continuous values
- a continuous **alphabet** discrete **time** random process
  - $X(t)$ has continuous values and t has discrete values
- a discrete **alphabet** discrete **time** random process
  - $X(t)$ has discrete values and t has discrete values

Deterministic and Non-deterministic Random Processes

- A process is **non-deterministic**
  if future values of any sample function
  cannot be predicted exactly from observed past values
- A process is **deterministic**
  if future values of any sample function
  can be predicted from observed past values

Deterministic Random Process Example (1)

$$X(t) = A\cos(\omega_0 t + \Theta)$$

A, Θ, or ω_0(or all) can be random variables.

a sample function corresponds to the above equation with particular values of these random variables.

$$x_i(t) = A_i\cos(\omega_{0,i} t + \Theta_i)$$

Deterministic Random Process Example (2)

$$x_i(t) = A_i \cos(\omega_{0,i} t + \Theta_i)$$

the knowledge of the sample function
prior to any time instance fully allows
the prediction of the sample function's future values
because all the necessary information is known

$$x_i(t) \quad t \leq 0 \qquad \Longrightarrow \qquad x_i(t) \quad t > 0$$

Functions and variables of a random process $X(t, \theta)$ (1)

$X(t, \theta)$	a family of functions, an ensemble
$X(t, \theta_k)$	a single time function selected by the outcome θ_k
$X(t_1, \theta)$	a random variable at the time $t = t_1$
$X(t_1, \theta_k)$	a number at the time $t = t_1$, of the outcome θ_k

https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf

Functions and variables of a random process $X(t, \theta)$ (2)

- $X(t, \theta)$ is a family of functions. Imagine a giant strip chart recording in which each pen is identified with a different θ. This family of functions is traditionally called an ensemble.
- A single function $X(t, \theta_k)$ is selected by the outcome θ_k. This is just a time function that we could call $X_k(t)$. Different outcomes give us different time functions.
- If t is fixed, say $t = t_1$, then $X(t_1, \theta)$ is a random variable. Its value depends on the outcome θ.
- If both t_1 and θ_k are given then $X(t_1, \theta_k)$ is just a number.

https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf