

# Random Process Background

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Oct 27, 2023

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles,Jr. and B. Shi

# Outline

- 1 Open Sets and Classes
  - Open Set
  - Class
- 2 Borel Sets
  - Measurable Space
  - Topological Space
  - Borel Sets
- 3 Stochastic Process

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- 1 Open Sets and Classes
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# Open set examples

- The *circle* represents the set of points  $(x, y)$  satisfying  $x^2 + y^2 = r^2$ .  
the *circle* set is its **boundary set**
- The *disk* represents the set of points  $(x, y)$  satisfying  $x^2 + y^2 < r^2$ .  
The *disk* set is an **open set**
- the **union** of the *circle* and *disk* sets is a **closed set**.

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# Open set (1)

- an **open set** is a generalization of an **open interval** in the real line.
- a **metric space** is a **set** along with a **distance** defined between any two **points**
- in a **metric space**, an **open set** is a **set** that, along with every **point**  $P$ , contains all **points** that are *sufficiently near* to  $P$ 
  - all **points** whose **distance** to  $P$  is less than some value depending on  $P$

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

## Open set (2)

- More generally, an **open set** is a **member** of a given collection of **subsets** of a given set a **collection** that has the property of **containing**

- every union of its **members**
- every finite intersection of its members
- the **empty set**
- the **whole set** itself

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

## Open set (3)

- These conditions are very loose, and allow enormous flexibility in the choice of **open sets**.
- For example,
  - every **subset** can be **open** (the **discrete topology**)
  - no **subset** can be **open** (the **indiscrete topology**) except
    - the space itself and
    - the empty set

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)



## Open set (4)

- A **set** in which such a **collection** is given is called a **topological space**, and the **collection** is called a **topology**.
  - A **set** is a **collection** of distinct **objects**.
  - Given a **set**  $A$ , we say that  $a$  is an **element** of  $A$  if  $a$  is one of the distinct **objects** in  $A$ , and we write  $a \in A$  to denote this
  - Given two **sets**  $A$  and  $B$ , we say that  $A$  is a **subset** of  $B$  if every element of  $A$  is also an element of  $B$  write  $A \subseteq B$  to denote this.

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

## Open set (5) Open Balls

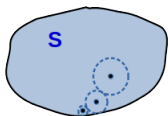
- An **open ball**  $B_r(\mathbf{a})$  in  $\mathbb{R}^n$   
centered at  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$  with radius  $r$   
is the set of all points  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$   
such that the distance between  $\mathbf{x}$  and  $\mathbf{a}$  is less than  $r$
- In  $\mathbb{R}^2$  an **open ball** is often called an **open disk**

We give these definitions in general, for when one is working in  $\mathbb{R}^n$   
since they are really not all that different to define in  $\mathbb{R}^n$  than in  $\mathbb{R}^2$

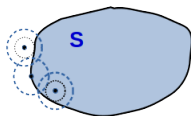
<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

## Open set (6) Interior points

- Suppose that  $S \subseteq \mathbb{R}^n$
- A point  $\mathbf{p} \in S$  is an **interior point** of  $S$  if there exists an **open ball**  $B_r(\mathbf{p}) \subseteq S$
- Intuitively,  $\mathbf{p}$  is an **interior point** of  $S$  if we can squeeze an entire **open ball** centered at  $\mathbf{p}$  within  $S$



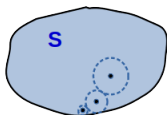
an interior point



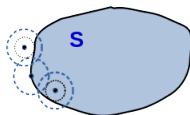
a boundary point

# Open set (7) Boundary points

- A point  $\mathbf{p} \in \mathbb{R}^n$  is a **boundary point** of  $S$  if all **open balls** centered at  $\mathbf{p}$  contain both **points** in  $S$  and **points** not in  $S$
- The **boundary** of  $S$  is the **set**  $\partial S$  that consists of all of the **boundary points** of  $S$ .



an interior point



a boundary point

## Open set (8) Open and Closed Sets

- A set  $O \subseteq \mathbb{R}^n$  is **open** if every point in  $O$  is an **interior point**.
- A set  $C \subseteq \mathbb{R}^n$  is **closed** if it contains all of its **boundary points**.

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

## Open set (9) Bounded and Unbounded

- A set  $S$  is **bounded** if there is an **open ball**  $B_M(0)$  such that

$$S \subseteq B.$$

intuitively, this means that we can enclose all of the **set**  $S$  within a large enough **ball** centered at the origin,  $B_M(0)$

- A **set** that is not **bounded** is called **unbounded**

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

# Family of sets (1)

- a **collection**  $F$  of **subsets** of a given **set**  $S$  is called a **family** of **subsets** of  $S$ , or a **family** of **sets** over  $S$ .
- More generally, a **collection** of any **sets** whatsoever is called a **family** of **sets**, **set family**, or a **set system**

[https://en.wikipedia.org/wiki/Family\\_of\\_sets](https://en.wikipedia.org/wiki/Family_of_sets)

## Family of sets (2)

- The term "**collection**" is used here because,
  - in some contexts,  
a **family** of **sets** may be allowed  
to contain repeated copies of any given **member**, and
  - in other contexts  
it may form a **proper class** rather than a **set**.

[https://en.wikipedia.org/wiki/Family\\_of\\_sets](https://en.wikipedia.org/wiki/Family_of_sets)



# Examples of family of sets (1)

- The **set** of all **subsets** of a given **set**  $S$  is called the **power set** of  $S$  and is denoted by  $\wp(S)$ .

The **power set**  $\wp(S)$  of a given **set**  $S$  is a **family** of **sets** over  $S$ .

- A **subset** of  $S$  having  $k$  elements is called a  **$k$ -subset** of  $S$ .

The  **$k$ -subset**  $S^{(k)}$  of a set  $S$  form a **family** of **sets**.

[https://en.wikipedia.org/wiki/Family\\_of\\_sets](https://en.wikipedia.org/wiki/Family_of_sets)

## Examples of family of sets (2)

- Let  $S = \{a, b, c, 1, 2\}$ .

An example of a **family** of **sets** over  $S$

(in the multiset sense) is given by  $F = \{A_1, A_2, A_3, A_4\}$ , where

$A_1 = \{a, b, c\}$ ,  $A_2 = \{1, 2\}$ ,  $A_3 = \{1, 2\}$ , and  $A_4 = \{a, b, 1\}$ .

[https://en.wikipedia.org/wiki/Family\\_of\\_sets](https://en.wikipedia.org/wiki/Family_of_sets)

# Filter in Set Theory (1-1)

- A **filter** on a **set** may be thought of as representing a "**collection** of *large subsets*", one intuitive example being the **neighborhood filter**.
- keep *large* grains excluding *small* impurities

[https://en.wikipedia.org/wiki/Filter\\_\(set\\_theory\)#filter\\_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)

# Filter in Set Theory (1-2)

- When you put a filter in your sink, the idea is that you filter out the *big* chunks of food, and let the water and the *smaller* chunks go through (which can, in principle, be washed through the pipes) .
- You filter out the *larger parts*.
- A filter filters out the *larger sets*.
- It is a way to say "these *sets* are '*large*'"

[https://en.wikipedia.org/wiki/Filter\\_\(set\\_theory\)#filter\\_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)

# Filter in Set Theory (1-3)

- a **filter** on a set  $X$  is a family  $\mathcal{B}$  of subsets such that:

- 1  $X \in \mathcal{B}$  and  $\emptyset \notin \mathcal{B}$
- 2 if  $A \in \mathcal{B}$  and  $B \in \mathcal{B}$ ,  
then  $A \cap B \in \mathcal{B}$
- 3 If  $A, B \subset X, A \in \mathcal{B}$ , and  $A \subset B$ ,  
then  $B \in \mathcal{B}$

[https://en.wikipedia.org/wiki/Filter\\_\(set\\_theory\)#filter\\_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)

## Filter in Set Theory (1-4)

- The set of "everything" is definitely *large*

$$X \in \mathcal{B}$$

- and "nothing" is definitely not;

$$\emptyset \notin \mathcal{B}$$

- if something is *larger* than a *large set*, then it is also *large*;

If  $A, B \subset X, A \in \mathcal{B}$ , and  $A \subset B$ , then  $B \in \mathcal{B}$

- and two *large sets intersect* on a *large set*.

If  $A \in \mathcal{B}$  and  $B \in \mathcal{B}$ , then  $A \cap B \in \mathcal{B}$

[https://en.wikipedia.org/wiki/Filter\\_\(set\\_theory\)#filter\\_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)



# Filter in Set Theory (1-5)

- you can think about this as
  - being **co-finite**,
  - or being of **measure 1** on the **unit interval**,
  - or having a **dense open subset** (again on the unit interval).
- These are examples of ways where a **set** can be thought of as "almost everything". and that is the idea behind a filter.

[https://en.wikipedia.org/wiki/Filter\\_\(set\\_theory\)#filter\\_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)

# Co-finite

- a **cofinite subset** of a set  $X$  is a subset  $A$  whose complement in  $X$  is a finite set.
- a subset  $A$  contains all but *finitely many* elements of  $X$
- If the complement is not finite, but is countable, then one says the set is **cocountable**.
- These arise naturally when generalizing structures on finite sets to infinite sets, particularly on infinite products, as in the product topology or direct sum.
- This use of the prefix "co" to describe a property possessed by a set's complement is *consistent* with its use in other terms such as "comeagre set".

<https://en.wikipedia.org/wiki/Cofiniteness>



# Unit interval

- the **unit interval** is the **closed interval**  $[0,1]$ , that is, the set of all real numbers that are greater than or equal to 0 and less than or equal to 1.
- It is often denoted  $I$  (capital letter I).
- In addition to its role in **real analysis**, the **unit interval** is used to study **homotopy theory** in the field of **topology**.
- the term "**unit interval**" is sometimes applied to the other shapes that an interval from 0 to 1 could take:  $(0,1]$ ,  $[0,1)$ , and  $(0,1)$ .
- However, the notation  $I$  is most commonly reserved for the **closed interval**  $[0,1]$ .

# Dense set

- In **topology**, a **subset**  $A$  of a topological space  $X$  is said to be **dense** in  $X$  if every **point** of  $X$  either belongs to  $A$  or else is arbitrarily "close" to a **member** of  $A$ 
  - for instance, the **rational numbers** are a **dense** subset of the **real numbers** because every **real number** either is a **rational number** or has a rational number arbitrarily close to it (see **Diophantine approximation**).
- Formally,  $A$  is **dense** in  $X$  if the *smallest* **closed subset** of  $X$  containing  $A$  is  $X$  itself.
- The **density** of a **topological space**  $X$  is the **least cardinality** of a **dense subset** of  $X$ .

[https://en.wikipedia.org/wiki/Dense\\_set](https://en.wikipedia.org/wiki/Dense_set)

# Filter in Set Theory (2-1)

- Let  $X = 1, 2, 3$   
Choose some element from  $X$  say  $F = 1, 1, 2, 1, 3, 1, 2, 3$
- Then every **intersection** of an element of  $F$  with another element in  $F$  is in  $F$  again.  
Examples:  $1 \cap 1, 2, 3 = 1$      $1, 2 \cap 1, 2, 3 = 1, 2$   
 $1, 3 \cap 1, 2, 3 = 1, 3$      $1, 2, 3 \cap 1, 2, 3 = 1, 2, 3$
- Also the original  $X = 1, 2, 3$  is also in  $F$ .  
Here  $F = 1, 1, 2, 1, 3, 1, 2, 3$  is called the **filter** on  $X = 1, 2, 3$

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

## Filter in Set Theory (2-2)

- Suppose we have the **collection**  $G = \{1, 1, 2, 1, 3, 2, 3, 1, 2, 3\}$
- Then we have  $1, 3 \cap 2, 3 = 3$  but 3 isn't in  $G$ .  
So this  $G$  is not called a **filter**.
- Now with  $F = \{1, 1, 2, 1, 3, 1, 2, 3\}$   
can we put as any other **element** in it  
so that after placing the **extra element** it is still a **filter**?  
Probably not in this case.  
So on  $X = \{1, 2, 3\}$ ,  $F = \{1, 1, 2, 1, 3, 1, 2, 3\}$  is an **Ultrafilter**.

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

# Filter in Set Theory (3-1)

- If we have started say with  $H = 1, 1, 2, 1, 2, 3$   
this is still a **filter** on  $X = 1, 2, 3$   
but we can still add  $1, 3$   
and it will still be classified as **filter**.
- So on  $X = 1, 2, 3$   
 $F = 1, 1, 2, 1, 3, 1, 2, 3$  is an **Ultrafilter**  
but  $H = 1, 1, 2, 1, 2, 3$  is a filter but not an **Ultrafilter**.

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

## Filter in Set Theory (3-2)

- Now suppose we have  $X = 1, 2, 3, 4$   
 Let  $F = 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$
- Every in intersection of element of  $F$  is in  $F$  again.  
 We have as examples  $1, 4 \cap 1, 4 = 1, 4$     $1, 4 \cap 1, 2, 4 = 1, 4$   
 $1, 4 \cap 1, 3, 4 = 1, 4$     $1, 2, 4 \cap 1, 2, 4 = 1, 2, 4$     $1, 2, 4 \cap 1, 3, 4 = 1, 4$   
 $1, 3, 4 \cap 1, 3, 4 = 1, 3, 4$     $1, 2, 3, 4 \cap 1, 2, 3, 4 = 1, 2, 3, 4$
- Also  $X = 1, 2, 3, 4$  is also in  $F = 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$   
 and the null element  $\emptyset =$  is not in  $F$ .

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

# Filter in Set Theory (3-3)

- We call  $F$  a filter but not an Ultrafilter on  $X = 1, 2, 3, 4$
- We can still add element in it and it will still be a filter for instance by adding the element 1 from  $X = 1, 2, 3, 4$  we can have the filter  $F = 1, 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$
- This is an Ultrafilter on  $X = 1, 2, 3, 4$  as we cannot add any further element from  $X = 1, 2, 3, 4$  that satisfies closures on intersection.

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

# Filter in Set Theory (4)

- There is another collection of sets taken from  $X=1,2,3,4$  Which is the powerset  $P=,1,2,3,4,1,2,1,3,1,4,2,3,2,4,3,4,1,2,3,1,2,4,1,3,4,2,3,4,1,2,3,4$  This contain the null element  $\emptyset=$  so we cannot call this as Ultrafilter. This is not a proper filter according to the article in Wikipedia. In the powerset every intersection of element is again in the powerset again but it contains the null element  $\emptyset=$  and isn't classified as proper filter.

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>



# Neighbourhood basis (1)

- A **neighbourhood basis** or **local basis** (or **neighbourhood base** or **local base**) for a **point**  $x$  is a **filter base** of the **neighbourhood filter**;
- this means that it is a **subset**  $\mathcal{B} \subseteq \mathcal{N}(x)$  such that for all  $V \in \mathcal{N}(x)$ , there exists some  $B \in \mathcal{B}$  such that  $B \subseteq V$ .  
That is, for any **neighbourhood**  $V$  we can find a **neighbourhood**  $B$  in the **neighbourhood basis** that is contained in  $V$ .

[https://en.wikipedia.org/wiki/Neighbourhood\\_system#Neighbourhood\\_basis](https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis)

## Neighbourhood basis (2)

- Equivalently,  $\mathcal{B}$  is a local basis at  $x$  if and only if the neighbourhood filter  $\mathcal{N}$  can be recovered from  $\mathcal{B}$  in the sense that the following equality holds:

$$\mathcal{N}(x) = \{V \subseteq X : B \subseteq V \text{ for some } B \in \mathcal{B}\}$$

- A family  $\mathcal{B} \subseteq \mathcal{N}(x)$  is a neighbourhood basis for  $x$  if and only if  $\mathcal{B}$  is a cofinal subset of  $(\mathcal{N}(x), \supseteq)$  with respect to the partial order  $\supseteq$  (importantly, this partial order is the superset relation and not the subset relation).

[https://en.wikipedia.org/wiki/Neighbourhood\\_system#Neighbourhood\\_basis](https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis)

# A collection of sets around $x$

- In general, one refers to the family of **sets** containing 0, used to approximate 0, as a **neighborhood basis**;
- a **member** of this **neighborhood basis** is referred to as an **open set**.
- In fact, one may generalize these notions to an arbitrary set ( $X$ ); rather than just the **real numbers**.
- In this case, given a **point** ( $x$ ) of that **set** ( $X$ ), one may define a **collection** of **sets** "**around**" (that is, containing)  $x$ , used to approximate  $x$ .

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# Smaller sets containing $x$

- Of course, this **collection** would have to *satisfy* certain properties (known as **axioms**) for otherwise we may not have a *well-defined method* to measure **distance**.
- For example, every **point** in  $X$  should **approximate**  $x$  to some **degree** of **accuracy**.
- Thus  $X$  should be in this **family**.
- Once we begin to define "smaller" **sets** containing  $x$ , we tend to **approximate**  $x$  to a greater **degree** of **accuracy**.
- Bearing this in mind, one may define the remaining **axioms** that the **family** of **sets** about  $x$  is required to satisfy.

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# Open ball (1)

- a **ball** is the solid figure bounded by a **sphere**;  
it is also called a **solid sphere**.
  - a **closed ball**  
includes the *boundary points* that constitute the sphere
  - an **open ball**  
excludes them

[https://en.wikipedia.org/wiki/Ball\\_\(mathematics\)](https://en.wikipedia.org/wiki/Ball_(mathematics))

## Open ball (2)

- A **ball** in  $n$  dimensions is called a **hyperball** or **n-ball** and is bounded by a **hypersphere** or  $(n - 1)$ -**sphere**
- One may talk about **balls** in any **topological space**  $X$ , not necessarily induced by a **metric**.
- An  $n$ -**dimensional topological ball** of  $X$  is any **subset** of  $X$  which is **homeomorphic** to an **Euclidean n-ball**.

[https://en.wikipedia.org/wiki/Ball\\_\(mathematics\)](https://en.wikipedia.org/wiki/Ball_(mathematics))

# Outline

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  - Class
- 2 Borel Sets
  - Measurable Space
  - Topological Space
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- 3 Stochastic Process

# Class (1)

- a **class** is a **collection** of **sets**  
(or sometimes other **mathematical objects**)  
that can be unambiguously defined  
by a **property** that all its members share.
- **Classes** act as a way to have **set-like collections**  
while **differing** from **sets** so as to **avoid Russell's paradox**

[https://en.wikipedia.org/wiki/Class\\_\(set\\_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))



## Class (2)

- A class *that is not a set* is called a **proper class**, and
- a class *that is a set* is sometimes called a **small class**.
- the followings are **proper classes** in many formal systems
  - the **class** of all sets
  - the **class** of all ordinal numbers
  - the **class** of all cardinal numbers

[https://en.wikipedia.org/wiki/Class\\_\(set\\_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

## Class (3)

- consider "the **set** of all **sets** with **property**  $X$ ."
- especially when dealing with **categories**, since the **objects** of a **concrete category** are all **sets** with certain additional **structure**.
- However, **if** we're not *careful* about this we can get into serious *trouble* –

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

## Class (4)

- let  $X$  be the **set** of all **sets** which do not contain *themselves*
- Since  $X$  is a **set**, we can ask whether  $X$  is an element of *itself*.
- But then we run into a **paradox** – **if**  $X$  contains *itself* as an element, **then** it does not, and vice versa.

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

## Class (5)

- In order to avoid this **paradox**, we cannot consider the **collection** of all **sets** to be itself a **set**.
- This means we have to *throw out* the whole "the **set** of all **sets** with **property** X" construction. But we wanted that.
- So the way we get around it is to say that there's something called a **class**, which is like a **set** but not a **set**.

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

## Class (6)

- Then we can talk about "the class  $X$  of all sets with property  $Y$ ."
- Since  $X$  is not a set, it can't be an element of itself, and we're fine.
- Of course, if we need to talk about the collection of all classes, we need to create another name that goes another step back, and so forth.

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

# Class Examples (1)

- The **collection** of all **algebraic structures** of a given type will usually be a **proper class**.  
(a **class** *that is not a set* is called a **proper class**)
  - the **class** of all **groups**
  - the **class** of all **vector spaces**
  - and many others.
- Within set theory, many **collections** of **sets** turn out to be **proper classes**.

[https://en.wikipedia.org/wiki/Class\\_\(set\\_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

## Class Examples (2)

- One way to *prove* that a **class** is **proper** is to place it in **bijection** with the **class** of all ordinal numbers.
  - **Cardinal numbers** indicate an amount how many of something we have: one, two, three, four, five.
  - **Ordinal numbers** indicate position in a series: first, second, third, fourth, fifth.

[https://en.wikipedia.org/wiki/Class\\_\(set\\_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))  
<https://editarians.com/cardinals-ordinals/>

# Class Paradoxes (1)

- The **paradoxes** of naive set theory can be explained in terms of the *inconsistent tacit assumption* that "all **classes** are **sets**".
- These **paradoxes** do not arise with **classes** because there is no notion of **classes** containing **classes**.
- Otherwise, one could, for example, define a **class** of all **classes** that do not contain themselves, which would lead to a **Russell paradox** for classes.

[https://en.wikipedia.org/wiki/Class\\_\(set\\_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))



# Class Paradoxes (2)

- With a rigorous foundation, these **paradoxes** instead *suggest proofs* that certain **classes** are **proper** (i.e., that they are not **sets**).
  - **Russell's paradox** *suggests a proof* that the **class** of all **sets** which do not contain themselves is **proper**
  - the **Burali-Forti paradox** *suggests* that the **class** of all ordinal numbers is **proper**.

[https://en.wikipedia.org/wiki/Class\\_\(set\\_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

# Russell's Paradox (1)

- According to the unrestricted comprehension principle, for any sufficiently well-defined **property**, there is the **set** of **all** and only the **objects** that have that **property**.

[https://en.wikipedia.org/wiki/Russell%27s\\_paradox](https://en.wikipedia.org/wiki/Russell%27s_paradox)

## Russell's Paradox (2)

- Let  $R$  be the **set of all sets** ( $R = \{x \mid x \notin x\}$ )  
that are not members of themselves ( $R \notin R$ ).
  - *if*  $R$  is not a **member** of itself ( $R \notin R$ ),  
*then* its definition (the **set of all sets**) entails  
that it is a **member** of itself ( $R \in R$ );
  - yet, *if* it is a **member** of itself ( $R \in R$ ),  
*then* it is not a **member** of itself ( $R \notin R$ ),  
since it is the **set of all sets**  
that are not members of themselves ( $R \notin R$ )
- the resulting **contradiction** is **Russell's paradox**.
- Let  $R = \{x \mid x \notin x\}$ , then  $R \in R \iff R \notin R$

[https://en.wikipedia.org/wiki/Russell%27s\\_paradox](https://en.wikipedia.org/wiki/Russell%27s_paradox)

# Russell's Paradox (3)

- Most *sets* commonly encountered are not *members* of themselves.
- For example, consider the *set* of all squares in a plane.
- This *set* is not itself a square in the plane, thus it is not a *member* of itself.
- Let us call a *set* "**normal**" if it is not a *member* of itself, and "**abnormal**" if it is a *member* of itself.

[https://en.wikipedia.org/wiki/Russell%27s\\_paradox](https://en.wikipedia.org/wiki/Russell%27s_paradox)

# Russell's Paradox (4)

- Clearly every **set** must be either **normal** or **abnormal**.
- The **set** of squares in the plane is **normal**.
- In contrast, the **complementary set** that contains everything which is not a square in the plane is itself not a square in the plane, and so it is one of its own **members** and is therefore **abnormal**.

[https://en.wikipedia.org/wiki/Russell%27s\\_paradox](https://en.wikipedia.org/wiki/Russell%27s_paradox)

# Russell's Paradox (5)

- Now we consider the **set** of all **normal sets**,  $R$ , and try to determine whether  $R$  is **normal** or **abnormal**.
  - *If*  $R$  were **normal**, it would be contained in the **set** of all **normal sets** (itself), and therefore be **abnormal**;
  - on the other hand *if*  $R$  were **abnormal**, it would not be contained in the **set** of all **normal sets** (itself), and therefore be **normal**.
- This leads to the conclusion that  $R$  is neither **normal** nor **abnormal**: **Russell's paradox**.

[https://en.wikipedia.org/wiki/Russell%27s\\_paradox](https://en.wikipedia.org/wiki/Russell%27s_paradox)

# Outline

## 1 Open Sets and Classes

- Open Set
- Class

## 2 Borel Sets

- Measurable Space
- Topological Space
- Borel Sets

## 3 Stochastic Process

# Mathematical objects (1)

- a **mathematical object** is an **abstract concept** arising in mathematics.
- an **mathematical object** is anything that has been (or could be) **formally defined**, and with which one may do
  - **deductive reasoning**
  - **mathematical proofs**

[https://en.wikipedia.org/wiki/Mathematical\\_object](https://en.wikipedia.org/wiki/Mathematical_object)



## Mathematical objects (2)

- typically, a **mathematical object**
  - can be a value that can be assigned to a variable
  - therefore can be involved in formulas

[https://en.wikipedia.org/wiki/Mathematical\\_object](https://en.wikipedia.org/wiki/Mathematical_object)

## Mathematical objects (3)

- commonly encountered **mathematical objects** include
  - numbers
  - sets
  - functions
  - expressions
  - geometric objects
  - transformations of other mathematical objects
  - spaces

[https://en.wikipedia.org/wiki/Mathematical\\_object](https://en.wikipedia.org/wiki/Mathematical_object)

## Mathematical objects (4)

- **Mathematical objects** can be very *complex*;
  - for example, the followings are considered as **mathematical objects** in **proof theory**.
    - theorems
    - proofs
    - theories

[https://en.wikipedia.org/wiki/Mathematical\\_object](https://en.wikipedia.org/wiki/Mathematical_object)

# Structure (1)

- a **structure** is a **set** endowed with some *additional features* on the **set**
  - an *operation*
  - *relation*
  - *metric*
  - *topology*
- often, the *additional features* are attached or related to the **set**, so as to provide it with some *additional meaning* or *significance*.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

## Structure (2)

- A partial list of possible **structures** are
  - measures
  - algebraic structures (groups, fields, etc.)
  - topologies
  - metric structures (geometries)
  - orders
  - events
  - equivalence relations
  - differential structures
  - categories.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

# Space (1)

- A **space** consists of selected **mathematical objects** that are treated as **points**, and selected **relationships** between these **points**.
  - the **nature** of the **points** can vary widely:  
for example, the **points** can be
    - elements of a set
    - functions on another space
    - subspaces of another space
  - It is the **relationships** between **points** that define the **nature** of the **space**.

[https://en.wikipedia.org/wiki/Space\\_\(mathematics\)](https://en.wikipedia.org/wiki/Space_(mathematics))

## Space (2)

- *modern mathematics* uses many types of **spaces**, such as
  - Euclidean spaces
  - linear spaces
  - topological spaces
  - Hilbert spaces
  - probability spaces
- *modern mathematics* does not define the notion of **space** itself.

[https://en.wikipedia.org/wiki/Space\\_\(mathematics\)](https://en.wikipedia.org/wiki/Space_(mathematics))

## Space (3)

- a **space** is  
a **set** (or a **universe**) with some added **features**
- it is not always clear  
whether a given **mathematical object** should be considered  
as a **geometric space**, or an **algebraic structure**
- a general definition of **structure** embraces  
all common types of **space**

[https://en.wikipedia.org/wiki/Space\\_\(mathematics\)](https://en.wikipedia.org/wiki/Space_(mathematics))



# Mathematical space (1)

- A **mathematical space** is, informally, a **collection** of **mathematical objects** under consideration.
- The **universe** of **mathematical objects** within a **space** are *precisely defined entities* whose **rules** of *interaction* come baked into the **rules** of the **space**.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

## Mathematical space (2)

- A **space** differs from a **mathematical set** in several important ways:
  - A **mathematical set** is also a **collection** of **objects**
  - but these **objects** are being pulled from a **space** (or **universe**) of **objects** where the **rules** and **definitions** have already been agreed upon

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

## Mathematical space (3)

- A **space** differs from a **mathematical set** in several important ways:
  - a **mathematical set** has no **internal structure**,
  - a **space** usually has some **internal structure**.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

## Mathematical space (4)

- having some **internal structure** could mean a variety of things, but typically it involves
  - *interactions* and *relationships* between **elements** of the **space**
  - *rules* on how to *create* and *define* **new elements** of the **space**

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

# Measurable space (1)

- A measurable space is any space with a sigma-algebra which can then be equipped with a measure
  - collection of subsets of the space following certain rules with a way to assign sizes to those sets.

<https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have>

# Measurable space (2)

- Intuitively, certain **sets** belonging to a **measurable space** can be given a **size** in a *consistent way*.

*consistent way* means that certain **axioms** are met:

- the **empty set** is given a **size** of **zero**
- if a measurable set is **contained** inside another one, then its **size** is **less than** or **equal to** the size of the **containing set**
- the size of a **disjoint union** of sets is the **sum** of the individual sets' **sizes**

<https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have>

# The set of all real numbers

- In the [set](#) of all [real numbers](#), one has the natural [Euclidean metric](#); that is, a function which *measures* the [distance](#) between two [real numbers](#):  $d(x, y) = |x - y|$ .

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# All points close to a real number $x$

- Therefore, given a **real number**  $x$ , one can speak of the **set** of all **points close** to that **real number**  $x$ ; that is, **within**  $\varepsilon$  of  $x$ .
- In essence, **points within**  $\varepsilon$  of  $x$  **approximate**  $x$  to an **accuracy** of **degree**  $\varepsilon$ .
- Note that  $\varepsilon > 0$  always, but as  $\varepsilon$  becomes *smaller* and *smaller*, one obtains **points** that **approximate**  $x$  to a *higher* and *higher* **degree** of **accuracy**.

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)



# The points within $\varepsilon$ of $x$

- For example, if  $x = 0$  and  $\varepsilon = 1$ , the **points** within  $\varepsilon$  of  $x$  are precisely the **points** of the interval  $(-1, 1)$ ;
- However, with  $\varepsilon = 0.5$ , the **points** within  $\varepsilon$  of  $x$  are precisely the **points** of  $(-0.5, 0.5)$ .
- Clearly, these **points** approximate  $x$  to a *greater degree* of **accuracy** than when  $\varepsilon = 1$ .

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

## without a concrete Euclidean metric

- The previous examples shows, for the case  $x = 0$ , that one may approximate  $x$  to *higher and higher* degrees of accuracy by defining  $\varepsilon$  to be *smaller and smaller*.
- In particular, sets of the form  $(-\varepsilon, \varepsilon)$  give us a lot of information about **points close** to  $x = 0$ .
- Thus, rather than speaking of a concrete Euclidean metric, one may use **sets** to describe **points close** to  $x$ .

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# Different collections of sets containing 0

- This innovative idea has far-reaching consequences; in particular, by defining

different collections of sets containing 0  
(distinct from the sets  $(-\varepsilon, \varepsilon)$ ),  
one may find different results  
regarding the distance  
between 0 and other real numbers.

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# A set for measuring distance

- For example, if we were to define  $R$  as the *only* such set for "*measuring distance*", all points are close to 0
- since there is only one possible degree of accuracy one may achieve in approximating 0: being a member of  $R$ .

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# The measure as a binary condition

- Thus, we find that in some sense, every real number is **distance** 0 away from 0.
- It may help in this case to think of the **measure** as being a **binary condition**:
  - all things in  $\mathbf{R}$  are equally close to 0,
  - while any item that is not in  $\mathbf{R}$  is not close to 0.

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# Probability space

- A **probability space** is simply a **measurable space** equipped with a **probability measure**.
- A **probability measure** has the special property of giving the entire space a size of **1**.
  - this then implies that the **size** of any disjoint union of sets (the sum of the **sizes** of the sets) in the **probability space** is less than or equal to 1

<https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have>

# Euclidean space definition (1)

- A **subset**  $U$  of the **Euclidean n-space**  $\mathbb{R}^n$  is open  
if, for every **point**  $x$  in  $U$ ,  
there exists a positive **real number**  $\varepsilon$   
(depending on  $x$ )  
**such that** any **point** in  $\mathbb{R}^n$   
whose **Euclidean distance** from  $x$  is smaller than  $\varepsilon$   
belongs to  $U$

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

## Euclidean space definition (2)

- Equivalently, a subset  $U$  of  $\mathbb{R}^n$  is open if every point in  $U$  is the center of an open ball contained in  $U$
- An example of a subset of  $\mathbb{R}$  that is not open is the closed interval  $[0, 1]$ , since neither  $0 - \varepsilon$  nor  $1 + \varepsilon$  belongs to  $[0, 1]$  for any  $\varepsilon > 0$ , no matter how small.

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)



# Metric space definition (1)

- A **subset**  $U$  of a **metric space**  $(M, d)$  is called **open**  
if, for any **point**  $x$  in  $U$ , there exists a **real number**  $\varepsilon > 0$   
**such that** any **point**  $y \in M$  satisfying  $d(x, y) < \varepsilon$  belongs to  $U$ .
- Equivalently,  $U$  is **open**  
if every **point** in  $U$   
has a **neighborhood** contained in  $U$ .
- This generalizes the **Euclidean space** example,  
since **Euclidean space** with the **Euclidean distance**  
is a **metric space**.

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

## Metric space definition (2)

- formally, a **metric space** is an **ordered pair**  $(M, d)$  where  $M$  is a **set** and  $d$  is a **metric** on  $M$ , i.e., a **function**

$$d : M \times M \rightarrow \mathbb{R}$$

satisfying the following **axioms** for all points  $x, y, z \in M$ :

- $d(x, x) = 0$ .
- if  $x \neq y$ , then  $d(x, y) > 0$ .
- $d(x, y) = d(y, x)$ .
- $d(x, z) \leq d(x, y) + d(y, z)$ .

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# Metric space definition (3)

- satisfying the following **axioms** for all points  $x, y, z \in M$ :
  - the distance from a point *to itself* is zero:
  - (**Positivity**) the **distance** between two distinct points is always **positive**:
  - (**Symmetry**) the **distance** from  $x$  to  $y$  is always the same as the **distance** from  $y$  to  $x$ :
  - (**Triangle inequality**) you can arrive at  $z$  from  $x$  by taking a detour through  $y$ , but this will not make your journey any faster than the shortest path.
- If the **metric**  $d$  is unambiguous, one often refers by abuse of notation to "the **metric space**  $M$ ".

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# Outline

- 1 Open Sets and Classes
  - Open Set
  - Class
- 2 Borel Sets
  - Measurable Space
  - Topological Space
  - Borel Sets
- 3 Stochastic Process

# Topology (1)

- **topology**  
from the Greek words  
τόπος, 'place, location',  
and λόγος, 'study'

<https://en.wikipedia.org/wiki/Topology>

## Topology (2)

- **topology** is concerned with the properties of a **geometric object** that are preserved
  - under **continuous deformations** such as
    - stretching
    - twisting
    - crumpling
    - bending
  - that is, without
    - closing holes
    - opening holes
    - tearing
    - gluing
    - passing through itself

<https://en.wikipedia.org/wiki/Topology>

# Topological space (1)

- a **topological space** is, roughly speaking,  
a **geometrical space**  
in which **closeness** is defined  
but cannot necessarily be **measured**  
by a **numeric distance**.

[https://en.wikipedia.org/wiki/Borel\\_set](https://en.wikipedia.org/wiki/Borel_set)

# Topological space (2)

- More specifically, a **topological space** is
  - a set whose elements are called points,
  - along with an additional structure called a topology,
- which can be defined as
  - a set of neighbourhoods for each point
  - that satisfy some axioms  
formalizing the concept of closeness.

[https://en.wikipedia.org/wiki/Borel\\_set](https://en.wikipedia.org/wiki/Borel_set)



## Topological space (3)

- There are several *equivalent definitions* of a **topology**, the most commonly used of which is the *definition* through *open sets*, which is easier than the others to manipulate.

[https://en.wikipedia.org/wiki/Borel\\_set](https://en.wikipedia.org/wiki/Borel_set)

# Topological space (4)

- A **topological space** is the most **general type** of a **mathematical space** that allows for the definition of
  - **limits**
  - **continuity**
  - **connectedness**
- Although very **general**, the concept of **topological spaces** is **fundamental**, and used in virtually every branch of modern mathematics.
- The study of **topological spaces** in their own right is called **point-set topology** or **general topology**.

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

# Topological space (5)

- Common types of **topological spaces** include
  - **Euclidean spaces** : a **set** of **points** satisfying certain **relationships**, expressible in terms of **distance** and **angles**.
  - **metric spaces** : a **set** together with a notion of **distance** between **points**. The **distance** is measured by a function called a **metric** or **distance function**.
  - **manifolds** : a topological space that *locally* resembles **Euclidean space** near each point. More precisely, an **n-manifold** is a topological space with the property that each **point** has a **neighborhood** that is **homeomorphic** to an **open subset** of **n-dimensional Euclidean space**.

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

# Discrete Topology

- a **discrete space** is a **topological space**,  
in which the **points** form a **discontinuous sequence**,  
meaning they are isolated from each other in a certain sense.
- The **discrete topology** is  
the finest **topology** that can be given on a **set**.
  - every **subset** is **open**
  - every **singleton subset** is an **open set**

[https://en.wikipedia.org/wiki/Discrete\\_space](https://en.wikipedia.org/wiki/Discrete_space)

# Singleton

- a **singleton**, also known as a **unit set** or **one-point set**, is a **set** with exactly one element.
- for example, the **set**  $\{0\}$  is a **singleton** whose single element is 0

[https://en.wikipedia.org/wiki/Discrete\\_space](https://en.wikipedia.org/wiki/Discrete_space)

# Indiscrete Space (1)

- a **topological space** with the **trivial topology** is one where the only **open sets** are the **empty set** and the **entire space**.
- Such spaces are commonly called **indiscrete**, **anti-discrete**, **concrete** or **codiscrete**.
  - every **subset** can be **open** (the **discrete topology**), or
  - no **subset** can be **open** (the **indiscrete topology**) except the space itself and the empty set .

[https://en.wikipedia.org/wiki/Discrete\\_space](https://en.wikipedia.org/wiki/Discrete_space)

# Indiscrete Space (2)

- Intuitively, this has the consequence that all **points** of the **space** are "**lumped together**" and cannot be **distinguished** by topological means (not topologically **distinguishable** points)
- Every **indiscrete space** is a **pseudometric space** in which the **distance** between any two **points** is zero.

[https://en.wikipedia.org/wiki/Discrete\\_space](https://en.wikipedia.org/wiki/Discrete_space)

# $T_0$ Space

- a **topological space**  $X$  is a  $T_0$  **space** or **if** for every **pair** of distinct points of  $X$ , at least one of them has a neighborhood not containing the other.
- In a  $T_0$  **space**, all **points** are topologically distinguishable.
- This condition, called the  $T_0$  **condition**, is the weakest of the **separation axioms**.
- Nearly all topological spaces *normally* studied in mathematics are  $T_0$  **space**.

[https://en.wikipedia.org/wiki/Kolmogorov\\_space](https://en.wikipedia.org/wiki/Kolmogorov_space)



# Topologically distinguishable points

- Intuitively, an **open set** provides a *method* to *distinguish* two **points**.
- two points in a **topological space**, there exists an **open set**
  - containing one point but
  - not containing the other (distinct) point
  - the two points are **topologically distinguishable**.

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# Topologically distinguishable points

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[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# Metric spaces

- In this manner, one may speak of whether two **points**, or more generally two **subsets**, of a **topological space** are "**near**" without concretely defining a **distance**.
- Therefore, **topological spaces** may be seen as a generalization of **spaces** equipped with a notion of **distance**, which are called **metric spaces**.

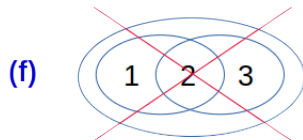
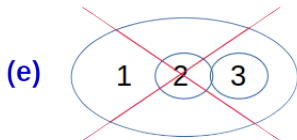
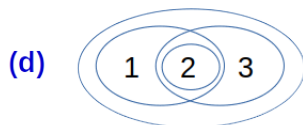
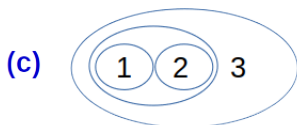
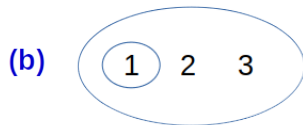
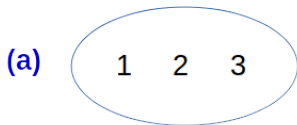
[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# Examples of topology (1)

- Let  $\tau$  be denoted with the circles, here are four examples **(a)**, **(b)**, **(c)**, **(d)**, and two non-examples **(e)**, **(f)** of topologies on the three-point set  $\{1, 2, 3\}$ .
- **(e)** is not a topology because the union of  $\{2\}$  and  $\{3\}$  [i.e.  $\{2, 3\}$ ] is missing;
- **(f)** is not a topology because the intersection of  $\{1, 2\}$  and  $\{2, 3\}$  [i.e.  $\{2\}$ ], is missing.

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

# Examples of topology (2)



Every union of  $(c)$ 

$(c)$  is a topology  $\{\{\}, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$   
**every union of  $(c)$**

$\cup$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{2\}$	$\{2\}$	$\{1,2\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

Every intersection of  $(c)$ 

$(c)$  is a topology  $\{\{\}, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$   
every intersection of  $(c)$

$\cap$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$
$\{1\}$	$\{\}$	$\{1\}$	$\{\}$	$\{1\}$	$\{1\}$
$\{2\}$	$\{\}$	$\{\}$	$\{2\}$	$\{2\}$	$\{2\}$
$\{1,2\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2\}$
$\{1,2,3\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

## Every union of (f)

(f) is not a topology  $\{\{\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$   
every union of (f)

$\cup$	$\{\}$	$\{1, 2\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$\{\}$	$\{\}$	$\{1, 2\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$\{1, 2\}$	$\{1, 2\}$	$\{1, 2\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$
$\{2, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)



## Every intersection of (f)

(f) is not a topology  $\{\{\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$   
every intersection of (f)

$\cap$	$\{\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$
$\{1,2\}$	$\{\}$	$\{1,2\}$	$\{2\}$	$\{1,2\}$
$\{2,3\}$	$\{\}$	$\{2\}$	$\{2,3\}$	$\{2,3\}$
$\{1,2,3\}$	$\{\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

## Examples of topology (3)

- Given  $X = \{1, 2, 3, 4\}$ ,  
the *trivial* or *indiscrete topology* on  $X$  is  
the family  $\tau = \{\{\}, \{1, 2, 3, 4\}\} = \{\emptyset, X\}$   
consisting of only the two subsets of  $X$   
required by the axioms  
forms a topology of  $X$ .

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

# Examples of topology (4)

- Given  $X = \{1, 2, 3, 4\}$ ,  
the family  $\tau = \{\{\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$   
 $= \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, X\}$   
of six **subsets** of  $X$  forms another **topology** of  $X$ .

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

## Examples of topology (5)

- Given  $X = \{1, 2, 3, 4\}$ ,  
the *discrete topology* on  $X$  is  
the *power set* of  $X$ , which is the family  $\tau = \wp(X)$   
consisting of *all possible subsets* of  $X$ .  
the family

$$\begin{aligned}\tau = & \{\{\}, \{1\}, \{2\}, \{3\}, \{4\} \\ & \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \\ & \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}\end{aligned}$$

- In this case the topological space  $(X, \tau)$   
is called a *discrete space*.

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

## Examples of topology (6)

- Given  $X = \mathbb{Z}$ , the set of integers, the family  $\tau$  of all finite subsets of the integers plus  $\mathbb{Z}$  itself is not a topology, because (for example) the union of all finite sets not containing zero is not finite but is also not all of  $\mathbb{Z}$ , and so it cannot be in  $\tau$ .

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

# Definition via Open Sets (1)

- A **topology**  $\tau$  on a **set**  $X$  is a **set** of **subsets** of  $X$  with the *properties* below.
  - a **topology**  $\tau$  on a **set**  $X$  : a **set** of **subsets** of  $X$
  - **members** of  $\tau$  : **subsets** of  $X$
- each **member** of  $\tau$  is called an **open set**.
- $X$  together with  $\tau$  is called a **topological space**

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

## Definition via Open Sets (2)

- topology  $\tau$  : a set of subsets of  $X$  has the *properties* below
  - $X \in \tau$  and  $\emptyset \in \tau$
  - any union of sets in  $\tau$  belong to  $\tau$  :  
any union of subsets of  $X$  belong to  $\tau$  :  
if  $\{U_i : i \in I\} \subseteq \tau$  then

$$\bigcup_{i \in I} U_i \in \tau$$

- any finite intersection of sets in  $\tau$  belong to  $\tau$   
any finite intersection of subsets of  $X$  belong to  $\tau$  :  
if  $U_1, \dots, U_n \in \tau$  then

$$U_1 \cap \dots \cap U_n \in \tau$$

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

## Definition via Open Sets (3)

- Infinite intersections of open sets need not be open.
- For example, the intersection of all intervals of the form  $(-1/n, 1/n)$ , where  $n$  is a positive integer, is the set  $\{0\}$  which is not open in the real line.
- A metric space is a topological space, whose topology consists of the collection of all subsets that are unions of open balls.
- There are, however, topological spaces that are not metric spaces.

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)



# Definition via Open Sets (4)

- A **topology** on a set  $X$  may be defined as a **collection**  $\tau$  of **subsets** of  $X$ , called **open sets** and satisfying the following **axioms**:
  - The **empty set** and  $X$  itself belong to  $\tau$  .
  - any arbitrary (**finite** or **infinite**) **union** of members of  $\tau$  belongs to  $\tau$  .
  - the **intersection** of any **finite** number of members of  $\tau$  belongs to  $\tau$  .

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

## Definition via Open Sets (5)

- As this definition of a topology is the most commonly used, the set  $\tau$  of the **open sets** is commonly called a **topology** on  $X$ .
- A **subset**  $C \subseteq X$  is said to be **closed** in  $(X, \tau)$  if its complement  $X \setminus C$  is an **open set**.

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

# Definition via Neighborhoods (1)

- This axiomatization is due to Felix Hausdorff.
- Let  $X$  be a **set**;
- the **elements** of  $X$  are usually called **points**, though they can be any mathematical object.
- We allow  $X$  to be **empty**.

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

## Definition via Neighborhoods (2)

- Let  $\mathcal{N}$  be a **function** assigning to each  $x$  (**point**) in  $X$  a non-empty **collection**  $\mathcal{N}(x)$  of **subsets** of  $X$ .
- The **elements** of  $\mathcal{N}(x)$  will be called **neighbourhoods** of  $x$  with respect to  $\mathcal{N}$  (or, simply, **neighbourhoods** of  $x$ ).
- The **function**  $\mathcal{N}$  is called a neighbourhood topology if *the axioms* below are satisfied; and
- then  $X$  with  $\mathcal{N}$  is called a **topological space**.

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

## Definition via Neighborhoods (3)

- If  $N$  is a neighbourhood of  $x$  (i.e.,  $N \in \mathcal{N}(x)$ ), then  $x \in N$ .  
In other words, each point belongs to every one of its neighbourhoods.
- If  $N$  is a subset of  $X$  and includes a neighbourhood of  $x$ , then  $N$  is a neighbourhood of  $x$ . I.e., every superset of a neighbourhood of a point  $x \in X$  is again a neighbourhood of  $x$ .
- The intersection of two neighbourhoods of  $x$  is a neighbourhood of  $x$ .
- Any neighbourhood  $\mathcal{N}$  of  $x$  includes a neighbourhood  $\mathcal{M}$  of  $x$  such that  $\mathcal{M}$  is a neighbourhood of each point of  $M$ .

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

## Definition via Neighborhoods (4)

- The first three axioms for neighbourhoods have a clear meaning.
- The fourth axiom has a very important use in the structure of the theory, that of linking together the neighbourhoods of different points of  $X$ .
- A standard example of such a system of neighbourhoods is for the real line  $\mathbb{R}$ , where a subset  $N$  of  $\mathbb{R}$  is defined to be a neighbourhood of a real number  $x$  if it includes an open interval containing  $x$ .

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

# Definition via Neighborhoods (5)

- Given such a **structure**, a **subset**  $U$  of  $X$  is defined to be **open** if  $U$  is a **neighbourhood** of all **points** in  $U$ .
- The **open sets** then satisfy the **axioms** given below.
- Conversely, when given the **open sets** of a **topological space**, the **neighbourhoods** satisfying the above **axioms** can be recovered by defining  $N$  to be a **neighbourhood** of  $x$  if  $N$  includes an open set  $U$  such that  $x \in U$ .

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

# Definitions via Closed Sets (1)

- Using **de Morgan's laws**, the above axioms defining **open sets** become axioms defining **closed sets**:
- The **empty set** and  $X$  are **closed**.
  - The **intersection** of any **collection** of **closed sets** is also **closed**.
  - The **union** of any finite number of **closed sets** is also **closed**.
- Using these **axioms**, another way to define a **topological space** is as a set  $X$  together with a **collection**  $\tau$  of **closed subsets** of  $X$ . Thus the **sets** in the **topology**  $\tau$  are the **closed sets**, and their complements in  $X$  are the **open sets**.

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)



# Homeomorphism (1)

- a **homeomorphism**

(from Greek ὁμοιος (homoios) 'similar, same',

and μορφή (morphē) 'shape, form',

named by Henri Poincaré), **topological isomorphism**,

or **bicontinuous function** is

a **bijjective** and **continuous** function

between topological spaces

that has a **continuous inverse** function.

<https://en.wikipedia.org/wiki/Homeomorphism>

# Homeomorphism (2)

- **Homeomorphisms** are the **isomorphisms** in the category of **topological spaces** – the **mappings** that **preserve** all the **topological properties** of a given space.
- Two **spaces** with a **homeomorphism** between them are called **homeomorphic**, and from a topological viewpoint they are the same.

<https://en.wikipedia.org/wiki/Homeomorphism>

## Homeomorphism (3)

- Very roughly speaking,  
a **topological space** is a **geometric object**,  
and the **homeomorphism** is  
a *continuous* **stretching** and **bending**  
of the object into a *new* **shape**.

<https://en.wikipedia.org/wiki/Homeomorphism>

# Homeomorphism (4)

- Thus, a *square* and a *circle* are **homeomorphic** to each other, but a *sphere* and a *torus* are not.
- However, this description can be misleading.
- Some *continuous deformations* are not **homeomorphisms**, such as the *deformation* of a *line* into a *point*.
- Some **homeomorphisms** are not *continuous deformations*, such as the homeomorphism between a *trefoil knot* and a *circle*.

<https://en.wikipedia.org/wiki/Homeomorphism>

# Outline

- 1 Open Sets and Classes
  - Open Set
  - Class
- 2 **Borel Sets**
  - Measurable Space
  - Topological Space
  - **Borel Sets**
- 3 Stochastic Process

# Sigma algebra (1)

- We term the **structures** which allow us to use **measure** to be **sigma algebras**
- the only requirements for **sigma algebras** (on a set  $X$ ) are:
  - the  $\{\}$  and  $X$  are in the **set**.
  - if  $A$  is in the **set**, *complement*( $A$ ) is in the **set**.
  - for any **sets**  $E_i$  in the set,  
 $\bigcup_i E_i$  is in the **set** (for countable  $i$ ).

<https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7>

## Sigma algebra (2)

- The most intuitive way to think about a **sigma algebra** is that it is the kind of **structure** we can do **probability** on.
  - for example, we can assign ratios of areas and length, so the **measure** on such a **set**  $X$  tells something about the **probability** of its **subsets**.
  - we can find the **probability** of **subsets**  $A$  and  $B$  because we know their ratios with respect to a **set**  $X$  ;
  - we also know that
    - (the measure of) their **complements** are defined, and
    - their **unions** and **intersections** are defined,
    - so we know how to find the **probability** of things in this set  $X$ .

<https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7>

## Sigma algebra (3)

- The **sigma algebra** which contains the **standard topology** on  $\mathbb{R}$  (that is, *all open sets* on  $\mathbb{R}$ ) is called the **Borel Sigma Algebra**, and the elements of this **set** are called **Borel sets**.
- What this gives us, is the set of **sets** on which outer measure gives our list of dreams. That is, if we take a **Borel set** and we check that length follows translation, additivity, and interval length, it will always hold.

<https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7>



# Sigma algebra (4)

- The **set** of Lebesgue measurable sets is the **set** of **Borel sets**, along with (union) all the sets which differ from a Borel set by a **set of measure 0**.
- More intuitively, it is all the sets we can normally measure, plus a bunch of stuff that doesn't affect our ideas of area or volume (think about the **border** of the circle above).

<https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7>

# Borel Sets (1-1)

- a **Borel set** is any **set** in a **topological space** that can be formed from **open sets** (or, equivalently, from **closed sets**) through the operations of
  - countable union,
  - countable intersection, and
  - relative complement.

[https://en.wikipedia.org/wiki/Borel\\_set](https://en.wikipedia.org/wiki/Borel_set)

## Borel Sets (1-2)

- For a **topological space  $X$** , the collection of all Borel sets on  $X$  forms a  $\sigma$ -algebra, known as the **Borel algebra** or **Borel  $\sigma$ -algebra**.
- The **Borel algebra on  $X$**  is the smallest  **$\sigma$ -algebra** containing all open sets (or, equivalently, all closed sets).

[https://en.wikipedia.org/wiki/Borel\\_set](https://en.wikipedia.org/wiki/Borel_set)

## Borel Sets (1-3)

- **Borel sets** are important in measure theory, since any measure defined on the open sets of a space, or on the closed sets of a space, must also be defined on all Borel sets of that space.
- Any measure defined on the Borel sets is called a **Borel measure**.
- **Borel sets** and the associated **Borel hierarchy** also play a fundamental role in descriptive set theory.

[https://en.wikipedia.org/wiki/Borel\\_set](https://en.wikipedia.org/wiki/Borel_set)

# Borel Sets (2)

- **Borel sets** are those obtained from intervals by means of the operations allowed in a  **$\sigma$ -algebra**. So we may construct them in a (transfinite) "sequence" of steps:
- ... And again and again.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

## Borel Sets (3-1)

1. Start with **finite unions** of **closed-open intervals**.  
These sets are completely **elementary**, and they form an **algebra**.
2. **Adjoin countable unions** and **intersections** of elementary sets.  
What you get already includes **open sets** and **closed sets**, **intersections** of an open set and a closed set, and so on.  
Thus you obtain an **algebra**, that is still not a  **$\sigma$ -algebra**.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

## Borel Sets (3)

3. Again, **adjoin countable unions** and **intersections** to 2.  
Observe that you get a strictly larger class, since a **countable intersection** of **countable unions** of intervals is not necessarily included in 2.  
Explicit examples of sets in 3 but not in 2 include  $F_\sigma$  sets, like, say, the set of *rational numbers*.
4. And do the same again.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

# Borel Sets (4-1)

- And even after a sequence of steps we are not yet finished. Take, say, a countable union of a set constructed at step 1, a set constructed at step 2, and so on. This union may very well not have been constructed at any step yet. By axioms of  $\sigma$ -algebra, you should include it as well - if you want, as step  $\infty$

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>



## Borel Sets (4-2)

- (or, technically, the first infinite ordinal, if you know what that means).
- And then continue in the same way until you reach the first uncountable ordinal. And only then will you finally obtain the generated  $\sigma$ -algebra.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

# Stochastic Process (1)

In probability theory and related fields, a **stochastic** (/stou'kæstɪk/) or **random** process is a mathematical object usually defined as a family of **random variables**.

The word stochastic in English was originally used as an adjective with the definition "pertaining to **conjecturing**", and stemming from a Greek word meaning "to aim at a mark, guess", and the Oxford English Dictionary gives the year 1662 as its earliest occurrence.

From Ancient Greek στοχαστικός (stokhastikós), from στοχάζομαι (stokhá-zomai, "aim at a target, guess"), from στόχος (stókhos, "an aim, a guess").

<https://en.wikipedia.org/wiki/Stochastic>  
<https://en.wiktionary.org/wiki/stochastic>

## Stochastic Process (2)

The definition of a **stochastic process** varies, but a **stochastic process** is *traditionally* defined as a collection of **random variables** indexed by some set.

The terms **random process** and **stochastic process** are considered synonyms and are used interchangeably, without the **index set** being precisely specified.

Both "**collection**", or "**family**" are used while instead of "**index set**", sometimes the terms "**parameter set**" or "**parameter space**" are used.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

## Stochastic Process (3)

The term **random function** is also used to refer to a **stochastic** or **random process**, though sometimes it is only used when the stochastic process takes real values.

This term is also used when the **index sets** are **mathematical spaces** other than the **real line**,

while the terms **stochastic process** and **random process** are usually used when the **index set** is interpreted as time,

and other terms are used such as **random field** when the **index set** is  $n$ -dimensional **Euclidean space**  $\mathbb{R}^n$  or a manifold

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# Stochastic Process (4)

A **stochastic process** can be denoted, by  $\{X(t)\}_{t \in T}$ ,  $\{X_t\}_{t \in T}$ ,  $\{X(t)\}$ ,  $\{X_t\}$  or simply as  $X$  or  $X(t)$ , although  $X(t)$  is regarded as an abuse of function notation.

For example,  $X(t)$  or  $X_t$  are used to refer to the **random variable** with the **index**  $t$ , and not the entire **stochastic process**.

If the **index set** is  $T = [0, \infty)$ , then one can write, for example,  $(X_t, t \geq 0)$  to denote the **stochastic process**.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# Stochastic Process Definition (1)

A **stochastic process** is defined as a collection of **random variables** defined on a common **probability space**  $(\Omega, \mathcal{F}, P)$ ,

- $\Omega$  is a **sample space**,
- $\mathcal{F}$  is a  $\sigma$ -**algebra**,
- $P$  is a **probability measure**;
- the **random variables**, indexed by some set  $T$ ,
- all take values in the same **mathematical space**  $S$ , which must be **measurable** with respect to some  $\sigma$ -algebra  $\Sigma$

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# Stochastic Process Definition (2)

In other words, for a given **probability space**  $(\Omega, \mathcal{F}, P)$  and a **measurable space**  $(S, \Sigma)$ , a **stochastic process** is a **collection** of  $S$ -valued **random variables**, which can be written as:

$$\{X(t) : t \in T\}.$$

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# Stochastic Process Definition (3)

Historically, in many problems from the natural sciences a point  $t \in T$  had the meaning of time, so  $X(t)$  is a **random variable** representing a value observed at time  $t$ .

A **stochastic process** can also be written as  $\{X(t, \omega) : t \in T\}$  to reflect that it is actually a function of two variables,  $t \in T$  and  $\omega \in \Omega$ .

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)



## Stochastic Process Definition (4)

There are other ways to consider a stochastic process, with the above definition being considered the traditional one.

For example, a stochastic process can be interpreted or defined as a  $S^T$ -valued **random variable**, where  $S^T$  is the space of all the possible functions from the set  $T$  into the space  $S$ .

However this alternative definition as a "**function-valued random variable**" in general requires additional regularity assumptions to be **well-defined**.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

## Index set (1)

The set  $T$  is called the **index set** or **parameter set** of the **stochastic process**.

Often this set is some subset of the real line, such as the natural numbers or an interval, giving the set  $T$  the interpretation of time.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

## Index set (2)

In addition to these sets, the index set  $T$  can be another set with a **total order** or a more general set, such as the Cartesian plane  $R^2$  or  $n$ -dimensional **Euclidean space**, where an element  $t \in T$  can represent a point in space.

That said, many results and theorems are only possible for **stochastic processes** with a **totally ordered index set**.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# State space

The **mathematical space**  $S$  of a **stochastic process** is called its **state space**.

This mathematical space can be defined using integers, real lines,  $n$ -dimensional Euclidean spaces, complex planes, or more abstract mathematical spaces.

The **state space** is defined using elements that reflect the different values that the **stochastic process** can take.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# Sample function (1)

A **sample function** is a single outcome of a **stochastic process**, so it is formed by taking a single possible value of each **random variable** of the **stochastic process**.

More precisely, if  $\{X(t, \omega) : t \in T\}$  is a **stochastic process**, then for any point  $\omega \in \Omega$ , the mapping  $X(\cdot, \omega) : T \rightarrow S$ , is called a **sample function**, a **realization**, or, particularly when  $T$  is interpreted as time, a **sample path** of the **stochastic process**  $\{X(t, \omega) : t \in T\}$ .

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

## Sample function (2)

This means that for a fixed  $\omega \in \Omega$  ,  
there exists a **sample function**  
that maps the **index set**  $T$  to the **state space**  $S$ .

Other names for a **sample function** of a **stochastic process**  
include **trajectory**, **path function** or **path**

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

