Solution 2

1. We use Newtons's second law for Rotation which is $\Sigma\tau = I\alpha$ for the derivation.

$$\Sigma\tau_1 = I_1\alpha_1 \tag{1}$$

We note that there are three torques applied on the particle 1; gravitational torque, controlling torque, and spring torque.
Gravitational torque is

$$\tau_g = \vec{l} \times \vec{F_g} = -l\sin\theta_1 m_1 g \tag{2}$$

Controlling torque is

$$\tau_c = \vec{l} \times \vec{F_c} = l\cos\theta_1 u_1 \tag{3}$$

Spring torque is

$$\tau_s = \vec{a} \times \vec{F_s} = -a\cos\theta_1 ka(\sin\theta_1 - \sin\theta_2) \tag{4}$$

Moment of inertia for the point mass is

$$I_1 = m_1 l^2 \tag{5}$$

Angular acceleration for the point mass is

$$\alpha_1 = \ddot{\theta}_1 \tag{6}$$

We set up the equation of motion for the first particle.

$$m_1 l^2 \ddot{\theta}_1 \ = \tau_a + \tau_g + \tau_c = -a\cos\theta_1 ka(\sin\theta_1 - \sin\theta_2) - l\sin\theta_1 m_1 g + l\cos\theta_1 u_1 \tag{7}$$

Apply the small-angle approximation which states that $\sin x \approx x$ and $\cos x \approx 1$ to simplify equation as

$$m_1 l^2 \ddot{\theta}_1 \ = -ka^2(\theta_1 - \theta_2) - l\theta_1 m_1 g + lu_1 \tag{8}$$

We use the same techniques to derive the equation of motion for second particle.
Gravitational torque is

$$\tau_g = \vec{l} \times \vec{F_g} = -l\sin\theta_2 m_2 g \tag{9}$$

Controlling torque is

$$\tau_c = \vec{l} \times \vec{F_c} = l\cos\theta_2 u_2 \tag{10}$$

Spring torque is

$$\tau_s = \vec{a} \times \vec{F_s} = -a\cos\theta_2 ka(\sin\theta_2 - \sin\theta_1) \tag{11}$$

Moment of inertia for the point mass is

$$I_2 = m_2 l^2 \tag{12}$$

Angular acceleration for the point mass is

$$\alpha_2 = \ddot{\theta}_2 \tag{13}$$

We set up the equation of motion for the second particle.

$$m_2 l^2 \ddot{\theta}_2 \ = \tau_a + \tau_g + \tau_c = -a\cos\theta_2 ka(\sin\theta_2 - \sin\theta_1) - l\sin\theta_2 m_2 g + l\cos\theta_2 u_2 \tag{14}$$

Apply the small-angle approximation which states that $\sin x \approx x$ and $\cos x \approx 1$ to simplify equation as

$$m_2 l^2 \ddot{\theta}_2 \ = -ka^2(\theta_2 - \theta_1) - l\theta_2 m_2 g + lu_2 \tag{15}$$

2. We write $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ in matrix form.

$$\begin{bmatrix} \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \end{bmatrix} \begin{bmatrix} u_1 l \\ u_2 l \end{bmatrix} \tag{16}$$

We divide the both side of $m_1 l^2 \ddot{\theta}_1 \;= -ka^2(\theta_1 - \theta_2) - l\theta_1 m_1 g + lu_1$ by $m_1 l^2$ to get

$$\ddot{\theta}_1 \;= -\frac{ka^2}{m_1 l^2}(\theta_1 - \theta_2) - \frac{1}{l}\theta_1 g + lu_1 = -\theta_1\left(\frac{ka^2}{m_1 l^2} + \frac{g}{l}\right) + \theta_2 \frac{ka^2}{m_1 l^2} + lu_1 \tag{17}$$

Similarly, we divide the both side of $m_2 l^2 \ddot{\theta}_2 \;= -ka^2(\theta_2 - \theta_1) - l\theta_2 m_2 g + lu_2$ by $m_2 l^2$ to get

$$\ddot{\theta}_2 \;= -\frac{ka^2}{m_2 l^2}(\theta_2 - \theta_1) - \frac{1}{l}\theta_2 g + lu_2 = \theta_1 \frac{ka^2}{m_2 l^2} - \theta_2\left(\frac{ka^2}{m_2 l^2} + \frac{g}{l}\right) + lu_2 \tag{18}$$

We can easily determine that the all the components on row 1 and row 3 of matrix A and matrix B are null, so we write that

$$\begin{bmatrix} \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & 0 & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ B_{21} & B_{22} \\ 0 & 0 \\ B_{41} & B_{42} \end{bmatrix} \begin{bmatrix} u_1 l \\ u_2 l \end{bmatrix} \tag{19}$$

Now we plug in the expressions of $\ddot{\theta}_1$ and $\ddot{\theta}_2$ into the matrix.

$$\begin{bmatrix} \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{ka^2}{m_1 l^2} - \frac{g}{l} & 0 & \frac{ka^2}{m_1 l^2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{ka^2}{m_2 l^2} & 0 & -\frac{ka^2}{m_2 l^2} - \frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 l \\ u_2 l \end{bmatrix} \tag{20}$$