Second Order ODE's (2A)

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Homogeneous Linear Equations with constant coefficients

Types of First Order ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

Separable Equations

$$\frac{dy}{dx} = g_1(x) g_2(y)$$

$$y' = g_1(x) g_2(y)$$

$$y = f(x)$$

Linear Equations

$$a_1(x)\frac{dy}{dx} + a_0(x)\, y = g(x)$$

$$a_1(x)\, y' + a_0(x)\, y = g(x)$$

$$y = f(x)$$

Exact Equations

$$M(x, y)dx + N(x, y)dy = 0$$

$$\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = 0$$

$$z = f(x, y)$$

Second Order ODEs

***First Order** Linear Equations*

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_1(x)y' + a_0(x)y = g(x)$$

***Second Order** Linear Equations*

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

Second Order** Linear Equations **with Constant Coefficients

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = g(x)$$

$$a_2 y'' + a_1 y' + a_0 y = g(x)$$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c\,y = g(x)$$

$$a\,y'' + b\,y' + c\,y = g(x)$$

Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c\,y = 0$$

$$a\,y'' + b\,y' + c\,y = 0$$

try a solution $y = e^{mx}$

$$a \frac{d^2}{dx^2}\{e^{mx}\} + b \frac{d}{dx}\{e^{mx}\} + c\{e^{mx}\} = 0$$

$$a\{e^{mx}\}'' + b\{e^{mx}\}' + c\{e^{mx}\} = 0$$

$$a\{m^2 e^{mx}\} + b\{m e^{mx}\} + c\{e^{mx}\} = 0$$

$$a\{m^2 e^{mx}\} + b\{m e^{mx}\} + c\{e^{mx}\} = 0$$

$$(a m^2 + b m + c) \cdot e^{mx} = 0$$

$$(a m^2 + b m + c) \cdot e^{mx} = 0$$

auxiliary equation

$$(a m^2 + b m + c) = 0$$

$$(a m^2 + b m + c) = 0$$

Roots of the Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$a\,y'' + b\,y' + c\,y = 0$$

auxiliary equation

try a solution $y = e^{mx}$ → $(am^2 + bm + c) = 0$

$$m_1 = \left(-b + \sqrt{b^2 - 4ac}\right)/2a$$

$$m_2 = \left(-b - \sqrt{b^2 - 4ac}\right)/2a$$

$y_1 = e^{m_1 x}\,,\quad y_2 = e^{m_2 x}$	(A)	$b^2 - 4ac > 0$	Real, distinct m_1, m_2
$y_1 = e^{m_1 x} = y_2 = e^{m_2 x}$	(B)	$b^2 - 4ac = 0$	Real, equal m_1, m_2
$y_1 = e^{m_1 x}\,,\quad y_2 = e^{m_2 x}$	(C)	$b^2 - 4ac < 0$	Conjugate complex m_1, m_2

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4/30/15

Linear Combination of Solutions

DEQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

y_1

y_2

$$C_1 y_1 + C_2 y_2$$

$$a y_1'' + b y_1' + c y_1 = 0$$

$$a y_2'' + b y_2' + c y_2 = 0$$

$$a(y_1'' + y_2'') + b(y_1' + y_2') + c(y_1 + y_2) = 0$$

$$a(y_1 + y_2)'' + b(y_1 + y_2)' + c(y_1 + y_2) = 0$$

$$y_3 = y_1 + y_2$$

$$y_4 = y_1 - y_2$$

$$y_5 = y_3 + 2y_4$$

$$y_6 = y_3 - 2y_4$$

$$a(C_1 y_1'' + C_2 y_2'') + b(C_1 y_1' + C_2 y_2') + c(C_1 y_1 + C_2 y_2) = 0$$

$$a(C_1 y_1 + C_2 y_2)'' + b(C_1 y_1 + C_2 y_2)' + c(C_1 y_1 + C_2 y_2) = 0$$

Solutions of 2nd Order ODEs

DEQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

y_1, y_2

$$C_1 y_1 + C_2 y_2$$

$$\begin{cases} y_1 = e^{m_1 x} \\ y_2 = e^{m_2 x} \end{cases} \quad (D > 0) \qquad \begin{cases} y_1 = e^{m_1 x} \end{cases} \quad (D = 0) \qquad \begin{cases} y_1 = e^{m_1 x} \\ y_2 = e^{m_2 x} \end{cases} \quad (D < 0)$$

$$\begin{cases} y = C_1 e^{m_1 x} + C_2 e^{m_2 x} & (D > 0) \\ y = C_1 e^{m_1 x} \quad ? & (D = 0) \\ y = C_1 e^{m_1 x} + C_2 e^{m_2 x} & (D < 0) \end{cases}$$

auxiliary equation

$$(am^2 + bm + c) = 0$$

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

(A) Real Distinct Roots Case

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

try a solution $y = e^{mx}$ → $(am^2 + bm + c) = 0$ auxiliary equation

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$
$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

$$y_1 = e^{m_1 x}$$
$$y_2 = e^{m_2 x}$$

$b^2 - 4ac > 0$	Real, distinct m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$
$b^2 - 4ac = 0$	Real, equal m_1, m_2	$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$
$b^2 - 4ac < 0$	Conjugate complex m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$

(B) Repeated Real Roots Case

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

try a solution $y = e^{mx}$ → $(am^2 + bm + c) = 0$ auxiliary equation

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$
$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

→ $b^2 - 4ac = 0$ →

$$m_1 = -b/2a$$
$$m_2 = -b/2a$$

$$e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a}x}$$

$b^2 - 4ac > 0$	Real, distinct m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$
$b^2 - 4ac = 0$	Real, equal m_1, m_2	$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$
$b^2 - 4ac < 0$	Conjugate complex m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$

(C) Complex Roots of the Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

try a solution $y = e^{mx}$ → $(am^2 + bm + c) = 0$ auxiliary equation

$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$ → $m_1 = (-b + \sqrt{4ac - b^2}\, i)/2a$ $\quad y_1 = e^{m_1 x}$

$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$ → $m_2 = (-b - \sqrt{4ac - b^2}\, i)/2a$ $\quad y_2 = e^{m_2 x}$

$b^2 - 4ac > 0$ Real, distinct m_1, m_2 $\quad y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$b^2 - 4ac = 0$ Real, equal m_1, m_2 $\quad y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$

$b^2 - 4ac < 0$ Conjugate complex m_1, m_2 $\quad y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$

Fundamental Set Examples (1)

Second Order EQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

y_1 $\quad e^{(\alpha+i\beta)x}$

y_2 $\quad e^{(\alpha-i\beta)x}$

$$y_3 = \frac{1}{2}y_1 + \frac{1}{2}y_2 \qquad \{e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x}\}/2 = e^{\alpha x}\cos(\beta x)$$

$$y_4 = \frac{1}{2i}y_1 - \frac{1}{2i}y_2 \qquad \{e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}\}/2i = e^{\alpha x}\sin(\beta x)$$

Fundamental Set Examples (2)

Second Order EQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$y_1 = y_3 + i\,y_4 \qquad e^{(\alpha+i\beta)x}$$

$$y_2 = y_3 - i\,y_4 \qquad e^{(\alpha-i\beta)x}$$

$$y_3 = e^{\alpha x}\cos(\beta x)$$

$$y_4 = e^{\alpha x}\sin(\beta x)$$

$$e^{\alpha x}[\cos(\beta x) + i\sin(\beta x)]$$

$$e^{\alpha x}[\cos(\beta x) - i\sin(\beta x)]$$

General Solution Examples

Second Order EQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

linearly independent

Fundamental Set of Solutions

$$\{y_1,\ \ y_2\} = \{e^{(\alpha+i\beta)x},\ \ e^{(\alpha-i\beta)x}\}$$

$$C_1\,y_1 + C_2\,y_2$$

$$C_1e^{(\alpha+i\beta)x} + C_2e^{(\alpha-i\beta)x}$$

General Solution

linearly independent

Fundamental Set of Solutions

$$\{y_3,\ \ y_4\} = \{e^{\alpha x}\cos(\beta x),\ \ e^{\alpha x}\sin(\beta x)\}$$

$$c_3\,y_3 + c_3\,y_4$$

$$c_3e^{\alpha x}\cos(\beta x) + c_4e^{\alpha x}\sin(\beta x)$$
$$= e^{\alpha x}(c_3\cos(\beta x) + c_4\sin(\beta x))$$

General Solution

Reduction of Orders

Finding another solution y_2 from the known y_1

Second Order EQ

$$a \frac{d^2 y}{d x^2} + b \frac{d y}{d x} + c y = 0$$

$y_1 = f(x)$ ***known solution***

$y_2 = u(x) f(x)$ ***another solution to be found***

We know one solution $\quad y_1(x) = e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a} x}$

Suppose the other solution $\quad y_2(x) = u(x) y_1(x) = u(x) e^{m_1 x}$

Condition for $y_2(t)$ to be a solution → Find u(x)

Conditions for y_2 to be another solution

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$a y'' + b y' + c y = 0$$

$$\begin{cases} y_2 = u y_1 \\ y_2' = u' y_1 + u y_1' \\ y_2'' = u'' y_1 + 2u' y_1' + u y_1'' \end{cases}$$

$$a y_2'' + b y_2' + c y_2 = 0 \quad \Rightarrow \quad a[u'' y_1 + 2u' y_1' + \underline{u y_1''}] + b[u' y_1 + \underline{u y_1'}] + \underline{c u y_1} = 0$$

$$a y_1'' + b y_1' + c y_1 = 0 \quad \Rightarrow \quad \underline{u[a y_1'' + b y_1' + c y_1]} + a[u'' y_1 + 2u' y_1'] + b[u' y_1] = 0$$

Condition for $y_2(t)$ to be a solution

$$y_2(x) = u(x) y_1(x) \quad \Rightarrow \quad a u'' y_1 + u'[2a y_1' + b y_1] = 0$$

Reduction of Order

We know one solution $y_1(x)$

Suppose the other solution $y_2(x) = u(x) y_1(x)$

$$a\frac{d^2 y}{d x^2} + b\frac{d y}{d x} + c\, y = 0$$

$$a\, y'' + b\, y' + c\, y = 0$$

$$a\, y''_2 + b\, y'_2 + c\, y_2 = 0 \quad \Rightarrow \quad a u'' y_1 + u'[2 a\, y_1' + b\, y_1] = 0$$

$$w(x) = u'(x) \quad \Rightarrow \quad a w' y_1 + w[2 a\, y_1' + b\, y_1] = 0$$

$$u = c_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx + c_2 \quad \Rightarrow \quad y_2 = c_1 y_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx + c_2 y_1 \qquad (c_1 = 1,\ c_2 = 0)$$

$$\Rightarrow \quad y_2 = y_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx$$

General Solutions for the repeated roots case

$$y_2 = y_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx$$

$$m_1 = \left(-b + \sqrt{b^2 - 4ac}\right)/2a$$
$$m_2 = \left(-b - \sqrt{b^2 - 4ac}\right)/2a$$

$\Rightarrow$ $b^2 - 4ac = 0$ $\Rightarrow$

$$m_1 = -b/2a$$
$$m_2 = -b/2a$$

$$e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a}x}$$

$$y_1(x) = e^{-\frac{b}{2a}x}$$

$$y_1^2 = e^{-\frac{b}{a}x}$$

$$y_2 = e^{-\frac{b}{2a}x} \int \frac{e^{-(b/a)x}}{e^{-(b/a)x}} dx = e^{-\frac{b}{2a}x} \int 1\, dx$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$\Rightarrow$

$$y_1(x) = e^{-\frac{b}{2a}x}$$

$$y_2(x) = x e^{-\frac{b}{2a}x}$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

General Solutions

- Homogeneous Equation

- Non-homogeneous Equation

General Solution – Homogeneous Equations

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

auxiliary equation

$$(am^2 + bm + c) = 0$$

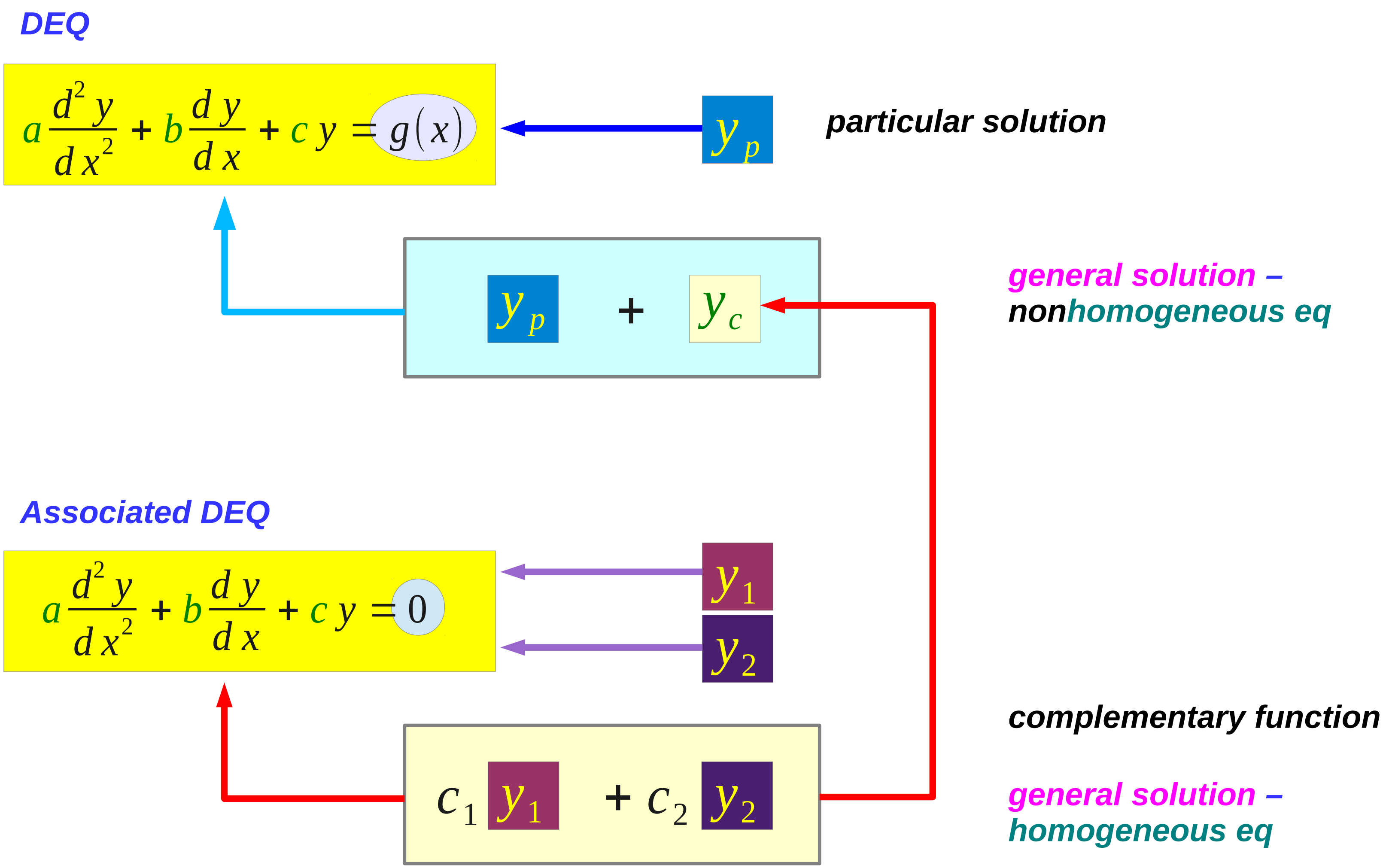
$$m_1 = \left(-b + \sqrt{b^2 - 4ac}\right)/2a$$

$$m_2 = \left(-b - \sqrt{b^2 - 4ac}\right)/2a$$

(A)	$b^2 - 4ac > 0$	Real, distinct m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$
(B)	$b^2 - 4ac = 0$	Real, equal m_1, m_2	$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$
(C)	$b^2 - 4ac < 0$	Conjugate complex m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$

$$= e^{\alpha x}\left(C_3 \cos(\beta x) + C_4 \sin(\beta x)\right)$$

Complementary Function



y_c and y_p

DEQ

$$a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = g(x) \quad \leftarrow \quad y_p$$

particular solution

$$y_c + y_p$$

general solution – nonhomogeneous eq

$$a\frac{d^2 y_c}{dx^2} + b\frac{dy_c}{dx} + c\,y_c \Rightarrow 0$$

many such complementary functions

c_i **many possible coefficients**

$$a\frac{d^2 y_p}{dx^2} + b\frac{dy_p}{dx} + c\,y_p \Rightarrow g(x)$$

only one particular functions

coefficients can be determined

Finding a Particular Solution
- Undetermined Coefficients

Particular Solutions

DEQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x)$$ ← y_p

particular solution
by a conjecture

(I) FORM Rule

(II) Multiplication Rule

When coefficients are constant

And

$g(x) =$
- ***A constant*** *or* ------ k
- ***A polynomial*** *or* ------ $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$
- ***An exponential function*** *or* ------ $e^{\alpha x}$
- ***A sine and cosine functions*** *or* ------ $\sin(\beta x)$ $\cos(\beta x)$
- ***Finite sum and products of the above functions*** ------ $e^{\alpha x}\sin(\beta x) + x^2$

And

$$g(x) \neq \quad \ln x \quad \frac{1}{x} \quad \tan x \quad \sin^{-1} x$$

Form Rule

DEQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x)$$

y_p

When coefficients are constant

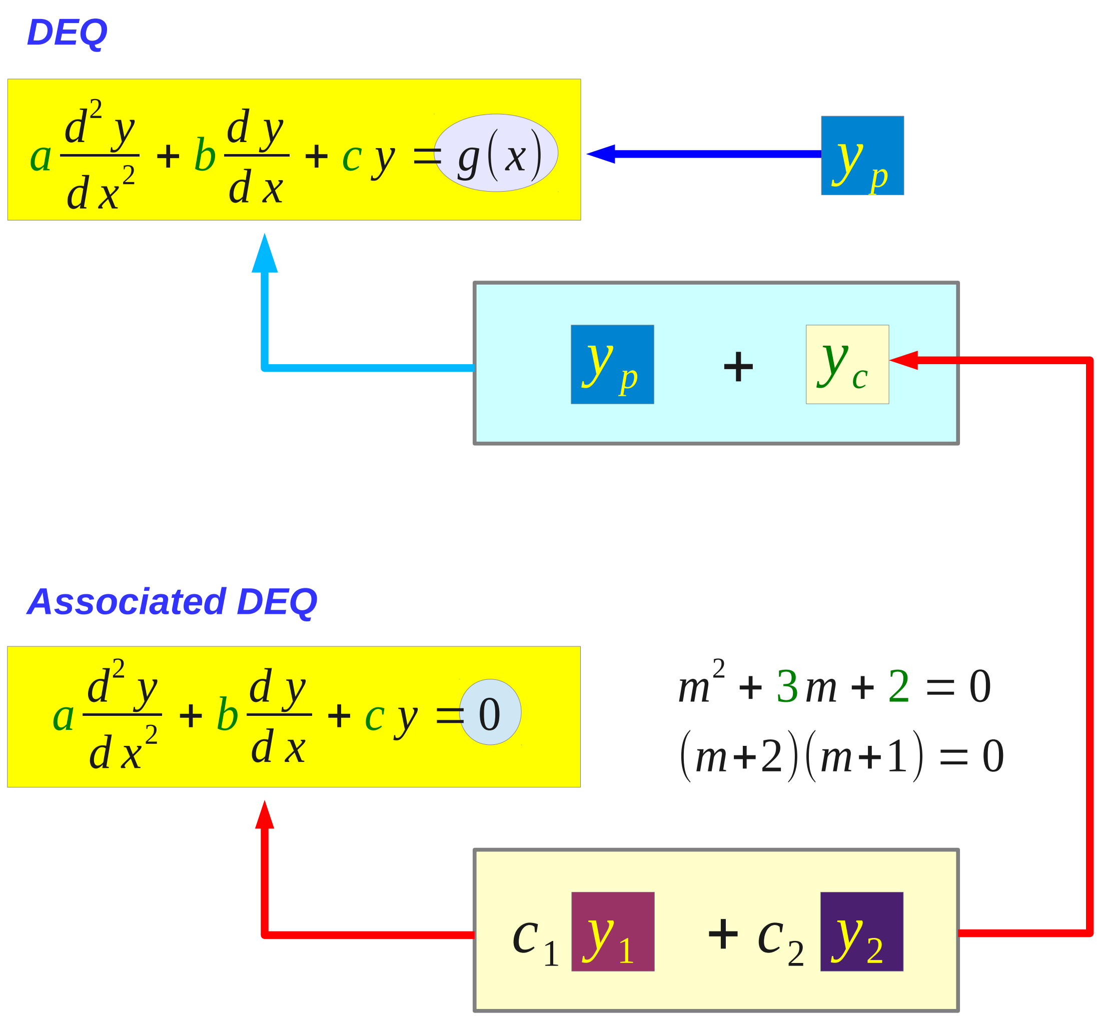
particular solution
by a conjecture

(I) FORM Rule

(II) Multiplication Rule

$$
\begin{aligned}
g(x) &= 2 & y_p &= A \\
g(x) &= 3x+4 & y_p &= Ax+B \\
g(x) &= 6x^2-7 & y_p &= Ax^2+Bx+C \\
g(x) &= \sin 8x & y_p &= A\cos 8x + B\sin 8x \\
g(x) &= \cos 9x & y_p &= A\cos 9x + B\sin 9x \\
g(x) &= e^{10x} & y_p &= Ae^{10x} \\
g(x) &= xe^{11x} & y_p &= (Ax+B)e^{11x} \\
g(x) &= e^{11x}\sin 12x & y_p &= Ae^{11x}\sin 12x + Be^{11x}\cos 12x \\
g(x) &= 5x\sin(3x) & y_p &= (Ax+B)\cos(3x) + (Cx+D)\sin(3x)
\end{aligned}
$$

Form Rule Example



$$m^2 + 3m + 2 = 0$$
$$(m+2)(m+1) = 0$$

$$y_p = Ax + B$$
$$y_p' = A$$
$$y_p'' = 0$$

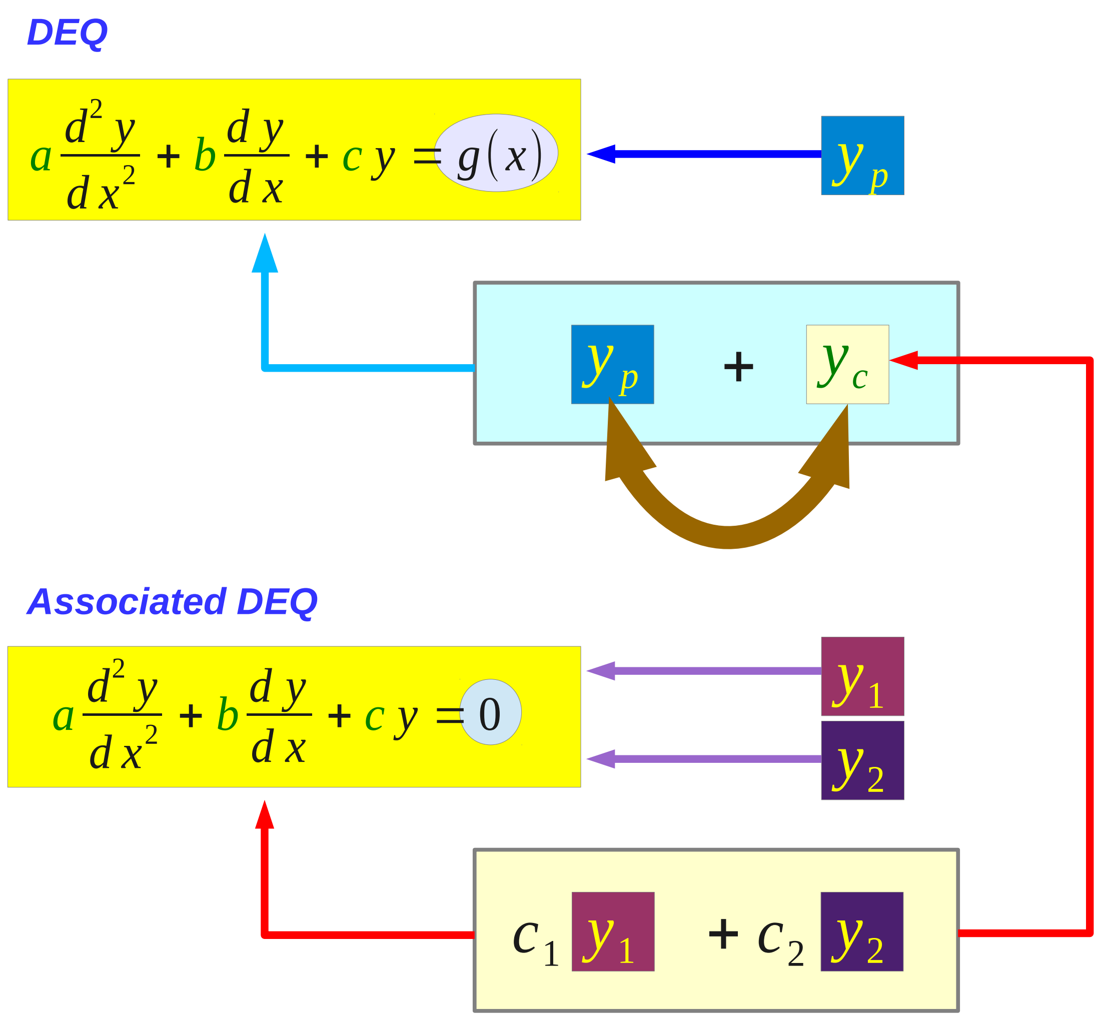
$$y_p'' + 3y_p' + 2y_p$$
$$= 3A + 2(Ax + B)$$
$$= 2Ax + 3A + 2B$$
$$= x$$

$$2A = 1 \qquad A = \tfrac{1}{2}$$
$$3A + 2B = 0 \qquad B = -\tfrac{3}{4}$$

$$y_p = \tfrac{1}{2}x - \tfrac{3}{4}$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + \tfrac{1}{2}x - \tfrac{3}{4}$$

Multiplication Rule



use $y_p = x^n y_1$ $\quad$ $y_p = x^n y_2$

if $y_p = y_1$ $\quad$ $y_p = y_2$

When y_p contains a term which is the same term in y_c

Use y_p multiplied by x^n

n is the smallest positive integer that eliminates the duplication

Multiplication Rule Example (1)

$$y'' - 2y' + y = 2e^x$$

$$y_p = Ae^x \Rightarrow Axe^x \Rightarrow Ax^2e^x$$

$$y_1 = e^x \qquad y_2 = xe^x$$

$$y'' - 2y' + y = 0$$

$$y'' - 2y' + y = 6xe^x$$

$$y_p = Axe^x \Rightarrow Ax^2e^x \Rightarrow Ax^3e^x$$

$$2Ae^x \neq 6xe^x$$

$$y_1 = e^x \qquad y_2 = xe^x$$

$$y'' - 2y' + y = 0$$

$$y_p = x(Ax + B)e^x \quad \rightarrow \quad Bxe^x$$

$$y_p = x^2(Ax + B)e^x$$

Multiplication Rule Example (2)

$$y'+4y = e^{x}\sin(2t) + 2t\cos(2t)$$

$$y_p(t) = e^{x}(A\cos(2t)+B\sin(2t)) + (Ct+D)\cos(2t) + (Et+F)\sin(2t)$$ ✗

$$y_p(t) = e^{x}(A\cos(2t)+B\sin(2t)) + \mathbf{t}(Ct+D)\cos(2t) + \mathbf{t}(Et+F)\sin(2t)$$

$$y_h(t) = c_1 e^{+i2t}+c_2 e^{-i2t}$$
$$= (c_3\cos(2t)+c_4\sin(2t))$$

$$y''+5y'+6y = t^2 e^{-3t}$$

$$y_p(t) = (At^2+Bt+C)e^{-3t}$$ ✗

$$y_p(t) = \mathbf{t}(At^2+Bt+C)e^{-3t}$$

$$y_h = c_1 e^{-2t}+c_2 e^{-3t}$$

Superposition (1)

DEQ

$$\frac{d^2 y}{d x^2} + b\frac{d y}{d x} + c\,y = 2x^2 + 3 + \cos 8x$$

$$(2x^2 + 3) + (\cos 8x)$$

$$\frac{d^2 y}{d x^2} + b\frac{d y}{d x} + c\,y = 0 \quad \longleftarrow \quad y_c$$

$$\frac{d^2 y}{d x^2} + b\frac{d y}{d x} + c\,y = 2x^2 + 3 \quad \longleftarrow \quad y_{p1}$$

$$\frac{d^2 y}{d x^2} + b\frac{d y}{d x} + c\,y = \cos 8x \quad \longleftarrow \quad y_{p2}$$

$$\frac{d^2 y_c}{d x^2} + b\frac{d y_c}{d x} + c\,y_c = 0$$

$$\frac{d^2 y_{p1}}{d x^2} + b\frac{d y_{p1}}{d x} + c\,y_{p1} = (2x^2 + 3)$$

$$\frac{d^2 y_{P2}}{d x^2} + b\frac{d y_{p2}}{d x} + c\,y_{P2} = \cos 8x$$

$$\frac{d^2}{d x^2}[y_c + y_{p1} + y_{p2}] + b\frac{d}{d x}[y_c + y_{p1} + y_{p2}] + c[y_c + y_{p1} + y_{p2}] = 2x^2 + 3 + \cos 8x$$

Young Won Lim
4/30/15

Superposition (2)

DEQ

$$a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = (2x^2 + 3)\cdot\cos 8x$$

$$y_p = (Ax^2 + Bx + C)\cdot(\cos 8x + \sin 8x)$$

$$\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = 0 \quad \leftarrow \quad y_c$$

$$\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = (2x^2 + 3) \quad \leftarrow \quad y_{p1}$$

$$\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = \cos 8x \quad \leftarrow \quad y_{p2}$$

$$\frac{d^2}{dx^2}[y_c + y_{p1} + y_{p2}] + b\frac{d}{dx}[y_c + y_{p1} + y_{p2}] + c[y_c + y_{p1} + y_{p2}] = (2x^2 + 3)\cdot\cos 8x$$

Finite Number of Derivative Functions

$$y = x e^{mx}$$

$$\dot{y} = e^{mx} + m x e^{mx}$$

$$\ddot{y} = m e^{mx} + m\left(e^{mx} + m x e^{mx}\right) = 2 m e^{mx} + m^2 x e^{mx}$$

$$\dddot{y} = 2 m e^{mx} + m^2\left(e^{mx} + m x e^{mx}\right) = \left(m^2 + 2m\right) e^{mx} + m^3 x e^{mx}$$

-
-
-

$$\left\{e^{mx},\ \ x e^{mx}\right\}$$

$$y = 2x^2 + 3x + 4$$

$$\dot{y} = 4x + 3$$

$$\ddot{y} = 4$$

$$\dddot{y} = 0$$

$$\left\{2x^2 + 3x + 4,\ \ 4x + 3,\ \ 4\right\}$$

Infinite Number of Derivative Functions

$y = +x^{-1}$

$\dot{y} = -x^{-2}$

$\ddot{y} = +2x^{-3}$

$\dddot{y} = -6x^{-4}$

•
•
•

$y = \ln x$

$y = +x^{-1}$

$\dot{y} = -x^{-2}$

$\ddot{y} = +2x^{-3}$

$\dddot{y} = -6x^{-4}$

•
•
•

Finding a Particular Solution
- Variation of Parameters

Variation of Parameter [c → u(x)]

$$y' + P(x)y = 0$$

$$y = c\,e^{-\int P(x)dx}$$

$$y_h = c\,y_1$$

$$y' + P(x)y = Q(x)$$

$$y_p = u(x)\,y_1$$

Integrating factor

$$\frac{1}{y_1} = e^{+\int P(x)dx}$$

$$y_1 = e^{-\int P(x)dx}$$

$$y'' + P(x)y' + Q(x)y = 0$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y_p = u_1(x)\,y_1 + u_2(x)\,y_2$$

Variation of Parameter [c → u(x)]

$$y'' + P(x)\,y' + Q(x)\,y = 0$$

$$y'' + P(x)\,y' + Q(x)\,y = f(x)$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1(x)\,y_1 + u_2(x)\,y_2$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

$$y_p'' = u_1'' y_1 + u_1' y_1' + u_1' y_1' + u_1 y_1'' + u_2'' y_2 + u_2' y_2' + u_2' y_2' + u_2 y_2''$$

$$y_p'' + P(x)\,y_p' + Q(x)\,y_p =$$

not a matrix notation

$$\begin{pmatrix} + u_1'' y_1 + u_1' y_1' + u_1' y_1' + u_1 y_1'' \\ + u_2'' y_2 + u_2' y_2' + u_2' y_2' + u_2 y_2'' \end{pmatrix} + P \begin{pmatrix} + u_1' y_1 + u_1 y_1' \\ + u_2' y_2 + u_2 y_2' \end{pmatrix} + Q \begin{pmatrix} + u_1 y_1 \\ + u_2 y_2 \end{pmatrix}$$

$$u_1\left(y_1'' + P\,y_1' + Q\,y_1\right) = 0$$
$$u_2\left(y_2'' + P\,y_2' + Q\,y_2\right) = 0$$

Variation of Parameter [c → u(x)]

not a matrix notation

$$y_p'' + P(x)\,y_p' + Q(x)\,y_p = \begin{pmatrix} +u_1''y_1 + u_1'y_1' + u_1'y_1' \\ +u_2''y_2 + u_2'y_2' + u_2'y_2' \end{pmatrix} + P\begin{pmatrix} +u_1'y_1 \\ +u_2'y_2 \end{pmatrix}$$

$$= \begin{pmatrix} +u_1''y_1 + u_1'y_1' \\ +u_2''y_2 + u_2'y_2' \end{pmatrix} + P\begin{pmatrix} +u_1'y_1 \\ +u_2'y_2 \end{pmatrix} + \begin{pmatrix} +u_1'y_1' \\ +u_2'y_2' \end{pmatrix}$$

$$= \frac{d}{dx}\begin{pmatrix} +u_1'y_1 \\ +u_2'y_2 \end{pmatrix} + P\begin{pmatrix} +u_1'y_1 \\ +u_2'y_2 \end{pmatrix} + \begin{pmatrix} +u_1'y_1' \\ +u_2'y_2' \end{pmatrix} = f(x)$$

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = f(x) \end{cases} \qquad \begin{cases} y_1u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = f(x) \end{cases} \qquad \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \qquad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

Variation of Parameter [c → u(x)]

$$y'' + P(x)y' + Q(x)y = 0$$

$$y_h = c_1 y_1 + c_2 y_2$$

If the associated homogeneous solution can be solved

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

then, always a particular solution can be found

No restriction

~~constant coefficients~~

~~A constant or~~
~~A polynomial or~~
~~An exponential function or~~
~~A sine and cosine functions or~~

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{W_1}{W}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{W_2}{W}$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$$

$$u_1'(x) = -\frac{y_2(x)f(x)}{W}$$

$$u_2'(x) = \frac{y_1(x)f(x)}{W}$$

Homogeneous Linear Equations
with variable coefficients

Cauchy-Euler Equation

Second Order Linear Equations with Constant Coefficients

$$a_2 \frac{d^2 y}{d x^2} + a_1 \frac{d y}{d x} + a_0 y = g(x)$$

$$a_2 y'' + a_1 y' + a_0 y = g(x)$$

Second Order Linear Equations with <u>Variable</u> Coefficients

$$a_2(x) \frac{d^2 y}{d x^2} + a_1(x) \frac{d y}{d x} + a_0(x) y = g(x)$$

$$a_2(x) y'' + a_1(x) y' + a_0(x) y = g(x)$$

Cauchy-Euler Equation

$$a_2 x^2 \frac{d^2 y}{d x^2} + a_1 x \frac{d y}{d x} + a_0 y = g(x)$$

$$a_2 x^2 y'' + a_1 x y' + a_0 y = g(x)$$

Auxiliary Equation of Cauchy-Euler Equation

Homogeneous Second Order Cauchy-Euler Equation

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

try a solution $y = x^m$

$$a x^2 \frac{d^2}{d x^2}\{x^m\} + b x \frac{d}{d x}\{x^m\} + c\{x^m\} = 0$$

$$a\{m(m-1)x^m\} + b\{m x^m\} + c\{x^m\} = 0$$

$$(a m^2 + (b-a)m + c) \cdot x^m = 0$$

$$a x^2 \{x^m\}'' + b x \{x^m\}' + c\{x^m\} = 0$$

$$a\{m(m-1)x^m\} + b\{m x^m\} + c\{x^m\} = 0$$

$$(a m^2 + (b-a)m + c) \cdot x^m = 0$$

auxiliary equation

$$(a m^2 + (b-a)m + c) = 0$$

$$(a m^2 + (b-a)m + c) = 0$$

Homogeneous Second Order Cauchy-Euler Equation

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

try a solution $y = x^m$

auxiliary equation

$$(a m^2 + (b-a) m + c) = 0$$

$$m_1 = \{-(b-a) + \sqrt{(b-a)^2 - 4ac}\}/2a$$
$$m_2 = \{-(b-a) - \sqrt{(b-a)^2 - 4ac}\}/2a$$

(A) $(b-a)^2 - 4ac > 0$ Real, distinct m_1, m_2

(B) $(b-a)^2 - 4ac = 0$ Real, equal m_1, m_2

(C) $(b-a)^2 - 4ac < 0$ Conjugate complex m_1, m_2

$$y = C_1 x^{m_1} + C_2 x^{m_2}$$
$$y = C_1 x^{m_1} + C_2 x^{m_1} \ln x$$
$$y = C_1 x^{m_1} + C_2 x^{m_2} = C_1 x^{(\alpha + i\beta)} + C_2 x^{(\alpha - i\beta)}$$

Homogeneous Second Order Cauchy-Euler Equation

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

$$m_1 = \{-(b-a) + \sqrt{4ac-(b-a)^2}\, i\}/2a \quad \Rightarrow \quad m_1 = \alpha + i\beta$$

$$m_2 = \{-(b-a) - \sqrt{4ac-(b-a)^2}\, i\}/2a \quad \Rightarrow \quad m_2 = \alpha - i\beta$$

$$y_1 = x^{m_1} = x^{\alpha + i\beta}$$

$$y_2 = x^{m_2} = x^{\alpha - i\beta}$$

$$\begin{cases} x^{m_1} = x^{\alpha} \cdot x^{+i\beta} = x^{\alpha} \cdot e^{+i\beta \ln x} \\ x^{m_2} = x^{\alpha} \cdot x^{-i\beta} = x^{\alpha} \cdot e^{-i\beta \ln x} \end{cases}$$

$$x = e^{\ln x}$$
$$x^{i\beta} = e^{+i\beta \ln x}$$

$$y = C_1 x^{m_1} + C_2 x^{m_2} = C_1 x^{\alpha} \cdot e^{+i\beta \ln x} + C_2 x^{\alpha} \cdot e^{+i\beta \ln x}$$

Pick **two** homogeneous solution

$$y_1 = x^{\alpha}\{e^{+i\beta \ln x} + e^{-i\beta \ln x}\}/2 = x^{\alpha} \cos(\beta \ln x) \qquad (C_1 = +1/2, \quad C_2 = +1/2)$$

$$y_2 = x^{\alpha}\{e^{+i\beta \ln x} - e^{-i\beta \ln x}\}/2i = x^{\alpha} \sin(\beta \ln x) \qquad (C_1 = +1/2i, \quad C_2 = -1/2i)$$

$$y = C_3 x^{\alpha} \cos(\beta \ln x) + C_4 x^{\alpha} \sin(\beta \ln x) = x^{\alpha}(C_3 \cos(\beta \ln x) + C_4 \sin(\beta \ln x))$$

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

$$a x^2 y'' + b x y' + c y = 0$$

$$\begin{cases} a x^2 y_2'' + b x y_2' + c y_2 = 0 \\ a x^2 y_1'' + b x y_1' + c y_1 = 0 \end{cases}$$

$$\begin{cases} y_2 = u y_1 \\ y_2' = u' y_1 + u y_1' \\ y_2'' = u'' y_1 + 2u' y_1' + u y_1'' \end{cases}$$

$$\Rightarrow \begin{cases} a x^2 [u'' y_1 + 2u' y_1' + \underline{u y_1''}] + b x [u' y_1 + \underline{u y_1'}] + \underline{c u y_1} = 0 \\ u [\underline{a x^2 y_1'' + b x y_1' + c y_1}] + a x^2 [u'' y_1 + 2u' y_1'] + b x [u' y_1] = 0 \end{cases}$$

Condition for $y_2(t)$ to be a solution

$$y_2(x) = u(x) y_1(x) \quad \Rightarrow \quad a x u'' y_1 + u' [2 a x y_1' + b y_1] = 0$$

We know one solution $y_1(x)$

$$a x^2 \frac{d^2 y}{d x^2} + b x \frac{d y}{d x} + c y = 0$$

Suppose the other solution $y_2(x) = u(x) y_1(x)$

$$a x^2 y'' + b x y' + c y = 0$$

$$a x^2 y_2'' + b x y_2' + c y_2 = 0 \Rightarrow a x u'' y_1 + u'[2 a x y_1' + b y_1] = 0$$

$$w(x) = u'(x) \Rightarrow a x w' y_1 + w[2 a x y_1' + b y_1] = 0$$

$$u = c_1 \int \frac{x^{-(b/a)}}{y_1^2} dx + c_2 \Rightarrow y_2 = y_1 \int \frac{x^{-(b/a)}}{y_1^2} dx$$

$$y_1^2 = x^{-\frac{(b-a)}{a}}$$

$$u = c_1 \ln x + c_2$$

$$y_2 = c_1 y_1 \int \frac{x^{-(b/a)}}{y_1^2} dx + c_2 y_1 \qquad (c_1 = 1, \; c_2 = 0)$$

$$y_2 = y_1 \ln x$$

Constant v.s. Non-constant Coefficients

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

(A) $b^2 - 4ac > 0$ ⇒ $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

(B) $b^2 - 4ac = 0$ ⇒ $y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$

(C) $b^2 - 4ac < 0$ ⇒ $y = C_1 e^{\alpha x} e^{+i\beta x} + C_2 e^{\alpha x} e^{-i\beta x} = e^{\alpha x}(C_3 \cos(\beta x) + C_4 \sin(\beta x))$

Homogeneous Second Order Cauchy-Euler Equation

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

$$ax^2 y'' + bxy' + cy = 0$$

$$x = e^{\ln x}$$
$$x^{i\beta} = e^{+i\beta \ln x}$$

(A) $(b-a)^2 - 4ac > 0$ ⇒ $y = C_1 x^{m_1} + C_2 x^{m_2}$

(B) $(b-a)^2 - 4ac = 0$ ⇒ $y = C_1 x^{m_1} + C_2 x^{m_1} \ln x$

(C) $(b-a)^2 - 4ac < 0$ ⇒ $y = C_1 x^{\alpha} \cdot e^{+i\beta \ln x} + C_2 x^{\alpha} \cdot e^{+i\beta \ln x} = x^{\alpha}(C_3 \cos(\beta \ln x) + C_4 \sin(\beta \ln x))$

A Unifying View

Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

(A) $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

(B) $y = C_1 e^{m_1 x} + C_2\, x\, e^{m_1 x}$

(C) $y = C_1 e^{\alpha x} e^{+i\beta x} + C_2 e^{\alpha x} e^{-i\beta x}$

$= e^{\alpha x}\left(C_3\cos(\beta x) + C_4\sin(\beta x)\right)$

x

Non-constant Coefficients

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

$x = e^{\ln x}$

$x^{i\beta} = e^{+i\beta\ln x}$

(A) $y = C_1 x^{m_1} + C_2 x^{m_2}$

(B) $y = C_1 x^{m_1} + C_2 x^{m_1}\ln x$

(C) $y = C_1 x^{\alpha}\cdot e^{+i\beta\ln x} + C_2 x^{\alpha}\cdot e^{+i\beta\ln x}$

$= x^{\alpha}\left(C_3\cos(\beta\ln x) + C_4\sin(\beta\ln x)\right)$

(A) $y = C_1 e^{m_1\ln x} + C_2 e^{m_2\ln x}$

(B) $y = C_1 e^{m_1\ln x} + C_2 e^{m_1\ln x}\ln x$

(C) $y = C_1 e^{\alpha\ln x}\cdot e^{+i\beta\ln x} + C_2 e^{\alpha\ln x}\cdot e^{+i\beta\ln x}$

$= e^{\alpha\ln x}\left(C_3\cos(\beta\ln x) + C_4\sin(\beta\ln x)\right)$

$\ln x$

Green's Function

Initial Value Problems

$$y'' + P(x)y' + Q(x)y = 0$$
$$y'(x_0) = y_1$$
$$y(x_0) = y_0$$

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y'(x_0) = 0$$
$$y(x_0) = 0$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

$$y_p(x_0) = 0$$

$$u_1'(x) = \frac{W_1}{W} = -\frac{y_2(x) f(x)}{W}$$

$$u_2'(x) = \frac{W_2}{W} = \frac{y_1(x) f(x)}{W}$$

$$u_1(x) = \int u_1'(x)\, dx$$

anti-derivative $= \int -\frac{y_2(t) f(t)}{W(t)}\, dt + c_1$

$$= \int_{x_0}^{x} -\frac{y_2(t) f(t)}{W(t)}\, dt$$

$$u_1(x_0) = 0 \quad \Rightarrow \quad u_1(x_0) y_1(x_0) = 0$$

$$u_2(x) = \int u_2'(x)\, dx$$

anti-derivative $= \int \frac{y_1(t) f(t)}{W(t)}\, dt + c_2$

$$= \int_{x_0}^{x} \frac{y_1(t) f(t)}{W(t)}\, dt$$

$$u_2(x_0) = 0 \quad \Rightarrow \quad u_2(x_0) y_2(x_0) = 0$$

Green's Function and IVP's (1)

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$[x_0, \ x] \subset I$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(x)$$

$$u_1'(x) = -\frac{y_2(x)f(x)}{W(x)} \qquad u_1(x) = \int_{x_0}^{x} -\frac{y_2(t)f(t)}{W(t)}\,dt$$

$$u_2'(x) = \frac{y_1(x)f(x)}{W(x)} \qquad u_2(x) = \int_{x_0}^{x} \frac{y_1(t)f(t)}{W(t)}\,dt$$

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

$$= \left[\int_{x_0}^{x} -\frac{y_2(t)f(t)}{W(t)}\,dt\right] y_1(x) + \left[\int_{x_0}^{x} \frac{y_1(t)f(t)}{W(t)}\,dt\right] y_2(x)$$

$$= \left[\int_{x_0}^{x} -\frac{y_1(x)y_2(t)}{W(t)} f(t)\,dt\right] + \left[\int_{x_0}^{x} \frac{y_1(t)y_2(x)}{W(t)} f(t)\,dt\right]$$

$$= \int_{x_0}^{x} \left[\frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}\right] f(t)\,dt$$

$$= \int_{x_0}^{x} G(x, \ t) f(t)\,dt$$

Green's Function and IVP's (2)

$$y'' + P(x)y' + Q(x)y = 0$$
$$y'(x_0) = y_1$$
$$y(x_0) = y_0$$

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y'(x_0) = 0$$
$$y(x_0) = 0$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

$$y_p = u_1(x) y_1 + u_2(x) y_2 = \int_{x_0}^{x} \left[\frac{y_1(t) y_2(x) - y_1(x) y_2(t)}{W(t)} \right] f(t)\, dt = \int_{x_0}^{x} G(x, t) f(t)\, dt$$

at the end, this x will replace the literal t

$$= \int_{x_0}^{x} G(x, t) f(t)\, dt$$

this x and t appear in the indefinite integral

Green's Function

$$G(x,\ t) = \left[\frac{y_1(t)\,y_2(x) - y_1(x)\,y_2(t)}{W(t)}\right] \qquad W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

$$y'' + P(x)y' + Q(x)y = 0 \quad \rightarrow \quad y_1,\ y_2$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y'' + P(x)y' + Q(x)y = g(x)$$

$$y'' + P(x)y' + Q(x)y = h(x)$$

the same Green's function

$$G(x,\ t) = \left[\frac{y_1(t)\,y_2(x) - y_1(x)\,y_2(t)}{W(t)}\right]$$

Three Initial Value Problem

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$

$$y'' + P(x)y' + Q(x)y = 0$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$

Homogeneous DEQ

Nonhomogeneous Initial Conditions

Nonzero Initial Conditions

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(x_0) = 0$$
$$y'(x_0) = 0$$

Nonhomogeneous DEQ

Zero Initial Conditions

Initially at rest

Rest Solution

General Solutions of the Initial Value Problem

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$

$$y = y_h + y_p$$

$$y(x_0) = y_h(x_0) + y_p(x_0) = y_0 + 0 = y_0$$
$$y'(x_0) = y_h'(x_0) + y_p'(x_0) = y_1 + 0 = y_1$$

$$y'' + P(x)y' + Q(x)y = 0$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$

$$y_h$$

Nonhomogeneous Initial Conditions

Nonzero Initial Conditions

Response due to the initial conditions

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(x_0) = 0$$
$$y'(x_0) = 0$$

$$y_p = \int_{x_0}^{x} G(x, t) f(t)\, dt$$

Response due to the forcing function f

Zero Initial Conditions

Initially at rest

Rest Solution

Rest Solution

$$y'' + P(x)y' + Q(x)y = f(x)$$
$$y(x_0) = 0$$
$$y'(x_0) = 0$$

Nonhomogeneous DEQ

Zero Initial Conditions

Initially at rest

Rest Solution

$$y_p = u_1(x)y_1 + u_2(x)y_2 = \int_{x_0}^{x} \left[\frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}\right] f(t)\,dt = \int_{x_0}^{x} G(x, t) f(t)\,dt$$

$$y_p(x) = \int_{x_0}^{x} G(x, t) f(t)\,dt$$

$$y_p'(x) = G(x, x) f(x) + \int_{x_0}^{x} \frac{\partial}{\partial x}[G(x, t) f(t)]\,dt = \int_{x_0}^{x} \left[\frac{y_1(t)y_2'(x) - y_1'(x)y_2(t)}{W(t)}\right] f(t)\,dt$$

$$y_p(x_0) = \int_{x_0}^{x_0} G(x, t) f(t)\,dt = 0$$

$$y_p'(x_0) = \int_{x_0}^{x_0} \left[\frac{y_1(t)y_2'(x_0) - y_1'(x_0)y_2(t)}{W(t)}\right] f(t)\,dt = 0$$

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