Computational Aspects (1B)

of Fourier Analysis Types

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Fourier Transform Types

Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

Discrete Time Fourier **Series**

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n} \quad \Longleftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \gamma[k]\, e^{+jk\hat{\omega}_0 n}$$

Continuous Time Fourier **Transform**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Discrete Time Fourier **Transform**

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(j\hat{\omega})\, e^{+j\hat{\omega} n}\, d\hat{\omega}$$

DTFS and DFT coefficients relationship

Discrete Time Fourier Series DTFS

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \gamma[k]\, e^{+j(2\pi/N)kn}$$

$$X[k] = N \cdot \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \frac{1}{N} X[k]\, e^{+j(2\pi/N)kn}$$

$$X[k] = N\,\gamma[k] \qquad \gamma[k] = \frac{1}{N} X[k]$$

Discrete Fourier Transform DFT

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j\frac{2\pi}{N}kn}$$

Computations using DFT

CTFS — ***Periodic x(t)***

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k \approx \frac{1}{N}\, \mathbf{DFT}\{x(nT_s)\}$$

$k\omega_0$

$$@ \quad k\omega_0 = k\left(\frac{2\pi}{T}\right) \quad rad/sec$$

CTFT — ***Aperiodic x(t)***

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(jk\omega_0) \approx T_s\, \mathbf{DFT}\{x(nT_s)\}$$

$\omega \leftarrow k\omega_0$

$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{NT_s}\right) \quad rad/sec$$

DTFS — ***Periodic x[n]***

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n}$$

$$\gamma[k] = \frac{1}{N}\, \mathbf{DFT}\{x[n]\}$$

$k\hat{\omega}_0$

$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{NT_s}\right) \quad rad/sec$$

DTFT — ***Aperiodic x[n]***

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

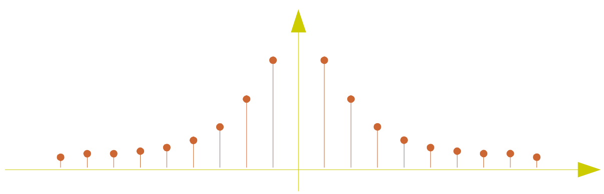
$$X(jk\hat{\omega}_0) \approx \mathbf{DFT}\{x[n]\}$$

$\hat{\omega} \leftarrow k\hat{\omega}_0$

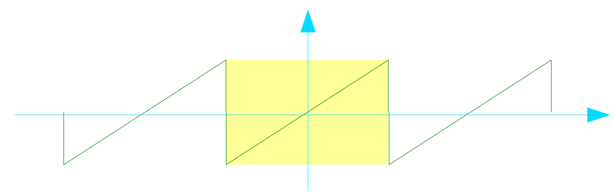
$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{NT_s}\right) \quad rad/sec$$

Continuous Time Fourier Series

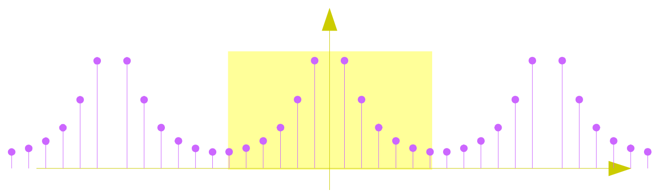
$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

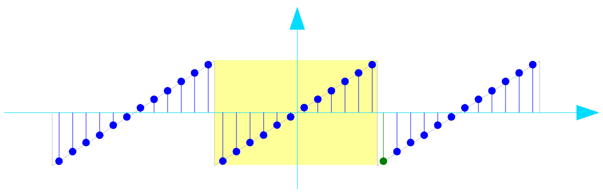


$$\omega_0 = \frac{2\pi}{T}$$



$$C_k \approx \frac{1}{N}\, \mathbf{DFT}\{x(nT_s)\} \qquad x(nT_s) \approx N\, \mathbf{IDFT}\{C_k\}$$

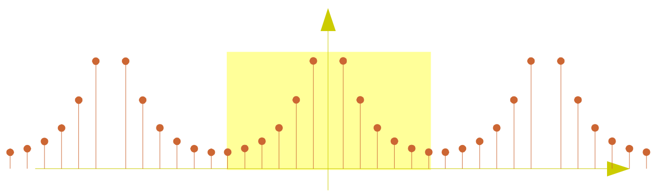




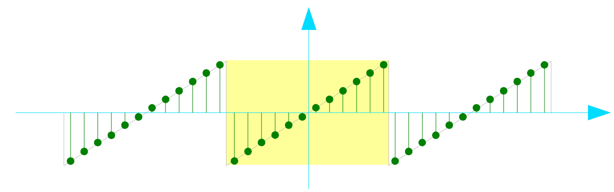
$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \quad \Longleftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j\frac{2\pi}{N}kn}$$

Discrete Time Fourier **Series**

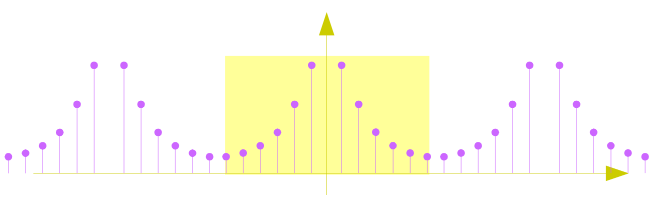
$$y[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n} \quad \Longleftrightarrow \quad x[n] = \sum_{k=0}^{N-1} y[k]\, e^{+jk\hat{\omega}_0 n}$$

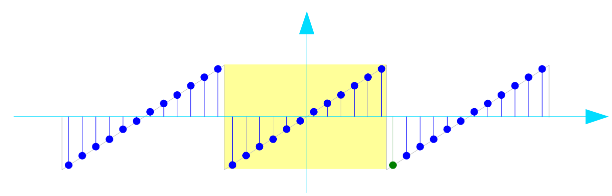


$$\hat{\omega}_0 = \frac{2\pi}{N}$$



$$y[k] = \frac{1}{N}\, \boldsymbol{DFT}\{x[n]\} \qquad x[n] = N\, \boldsymbol{IDFT}\{y_k\}$$



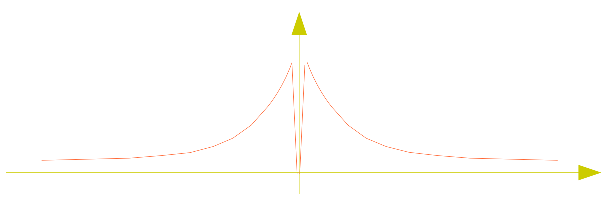


$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \quad \Longleftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j\frac{2\pi}{N}kn}$$

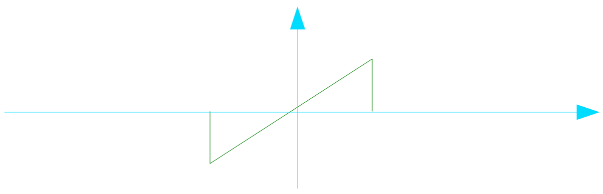
Continuous Time – **CTFT** computation

Continuous Time Fourier Transform

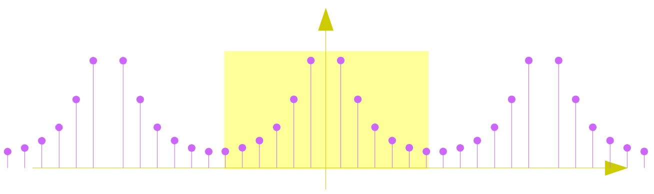
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

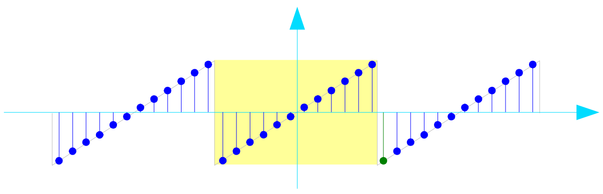


$$\omega_0 = \frac{2\pi}{T}$$



$$X(jk\omega_0) \approx T_s\, \mathbf{DFT}\{x(nT_s)\} \qquad x(nT_s) \approx \frac{1}{T_s}\, \mathbf{IDFT}\{X(jk\omega_0)\}$$



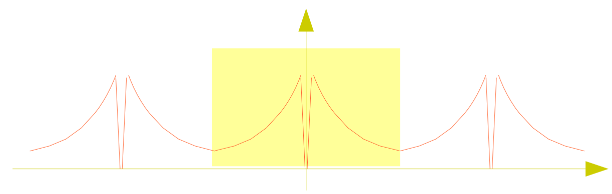


$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j\frac{2\pi}{N}kn}$$

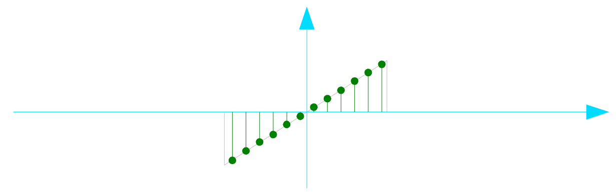
Discrete Time – **DTFT** computation

Discrete Time Fourier Transform

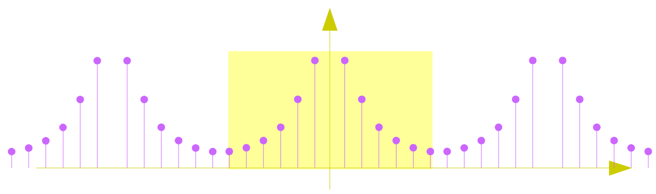
$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(j\hat{\omega})\, e^{+j\hat{\omega} n}\, d\hat{\omega}$$

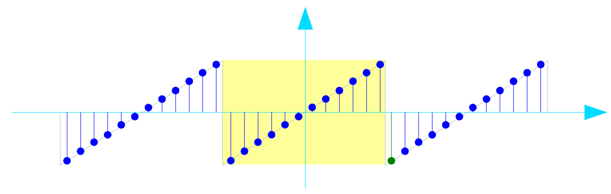


$$\hat{\omega}_0 = \frac{2\pi}{N}$$



$$X(jk\hat{\omega}_0) \approx \boldsymbol{DFT}\{x[n]\} \qquad x[n] \approx \boldsymbol{IDFT}\{X(jk\hat{\omega}_0)\}$$



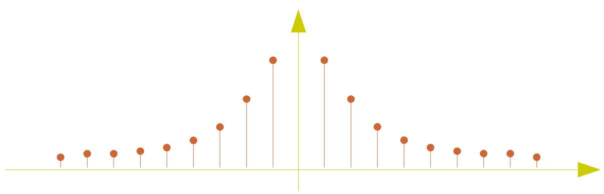


$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \quad \Longleftrightarrow \quad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+j\frac{2\pi}{N}kn}$$

Continuous Time – **CTFS** Computation

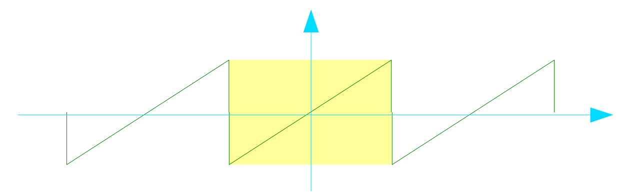
Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$



$$\omega_0 = \frac{2\pi}{T}$$

$$\hat{\omega}_0 = \frac{2\pi}{N}$$



$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k \approx \frac{1}{T}\sum_{n=0}^{N-1}\left[\int_{nT_s}^{(n+1)T_s} x(t)\, e^{-jk\omega_0 t}\, dt\right]$$

$$\frac{T_s}{T} = \frac{1}{N}$$

$$C_k \approx e^{-jk\hat{\omega}_0/2}\,\frac{\sin(k\hat{\omega}_0/2)}{k\hat{\omega}_0/2}\left[\frac{1}{N}\sum_{n=0}^{N-1} x(nT_s)\, e^{-jk\hat{\omega}_0 n}\right]$$

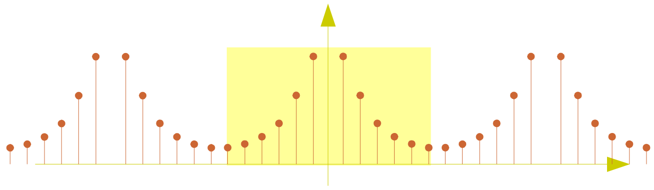
$$C_k \approx \frac{1}{N}\, \boldsymbol{DFT}\{x(nT_s)\}$$

$$x(nT_s) \approx N\, \boldsymbol{IDFT}\{C_k\}$$

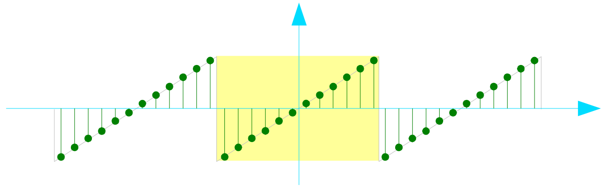
Discrete Time – **DTFS** computation

Discrete Time Fourier Series

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n} \quad \Longleftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \gamma[k]\, e^{+jk\hat{\omega}_0 n}$$



$$\omega_0 = \frac{2\pi}{T} \qquad \hat{\omega}_0 = \frac{2\pi}{N}$$



$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

$$X[k] = N\cdot\frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

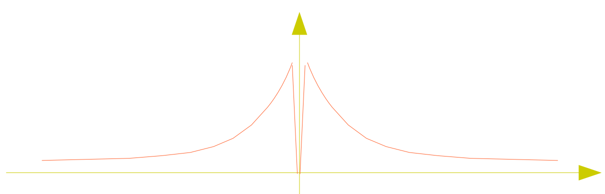
$$\gamma[k] = \frac{1}{N}\,\boldsymbol{DFT}\{x[n]\}$$

$$x[n] = N\,\boldsymbol{IDFT}\{\gamma_k\}$$

Continuous Time – **CTFT** computation

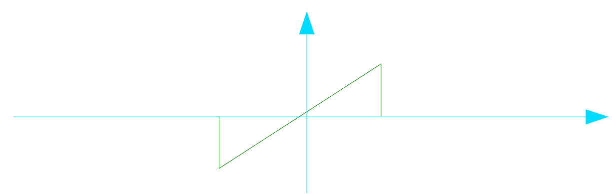
Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$



$$\omega_0 = \frac{2\pi}{T}$$

$$\hat{\omega}_0 = \frac{2\pi}{N}$$



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(j\omega) \approx \sum_{n=0}^{N-1} \left[\int_{nT_s}^{(n+1)T_s} x(t)\, e^{-j\omega t}\, dt \right] \qquad \omega \leftarrow k\hat{\omega}_0 \qquad T_s$$

$$X(jk\hat{\omega}_0) \approx e^{-jk\hat{\omega}_0/2} \frac{\sin(k\hat{\omega}_0/2)}{k\hat{\omega}_0/2} \left[T_s \sum_{n=0}^{N-1} x(nT_s) e^{-jk\hat{\omega}_0 n} \right]$$

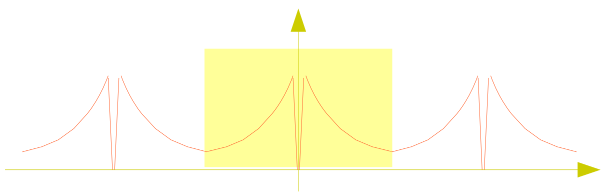
$$X(jk\omega_0) \approx T_s\, \boldsymbol{DFT}\{x(nT_s)\}$$

$$x(nT_s) \approx \frac{1}{T_s}\, \boldsymbol{IDFT}\{X(jk\omega_0)\}$$

Discrete Time – **DTFT** computation

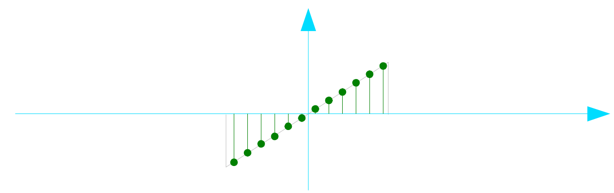
Discrete Time Fourier Transform

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(j\hat{\omega})\, e^{+j\hat{\omega} n}\, d\hat{\omega}$$



$$\omega_0 = \frac{2\pi}{T}$$

$$\hat{\omega}_0 = \frac{2\pi}{N}$$



$$X(j\hat{\omega}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\hat{\omega} n} \qquad \hat{\omega} \leftarrow k\hat{\omega}_0$$

$$X(jk\hat{\omega}_0) = \sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

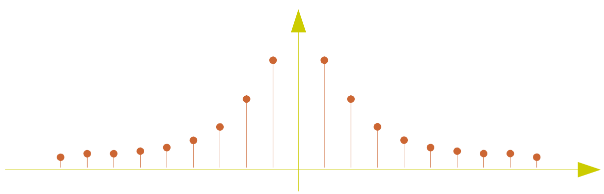
$$X(jk\hat{\omega}_0) \approx \boldsymbol{DFT}\{x[n]\}$$

$$x[n] \approx \boldsymbol{IDFT}\{X(jk\hat{\omega}_0)\}$$

Continuous Time – **CTFS** Computation

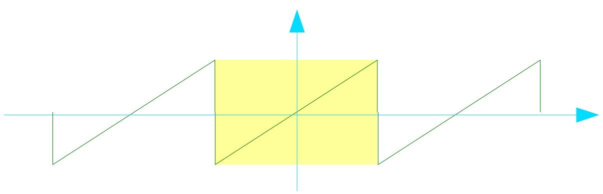
Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$



$$\omega_0 = \frac{2\pi}{T}$$

$$\hat{\omega}_0 = \frac{2\pi}{N}$$



$$C_k \approx \frac{1}{N}\, \boldsymbol{DFT}\{x(nT_s)\} \qquad x(nT_s) \approx N\, \boldsymbol{IDFT}\{C_k\}$$

Forward **CTFS** Approximation (1)

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

CTFS

$$C_k \approx \frac{1}{T_0}\sum_{n=0}^{N-1}\int_{nT_s}^{(n+1)T_s} x(t)\, e^{-jk\omega_0 t}\, dt$$

CTFS approximated at $k\omega_0$

$$\approx \frac{1}{T_0}\sum_{n=0}^{N-1} x(nT_s)\int_{nT_s}^{(n+1)T_s} e^{-jk\omega_0 t}\, dt$$

$$\approx \left(\frac{T_s}{T_0}\right)\frac{1}{jk2\pi/N}\left[1 - e^{-j\frac{2\pi}{N}k}\right]\sum_{n=0}^{N-1} x(nT_s)\, e^{-j\frac{2\pi}{N}kn}$$

$$k\omega_0 = k\left(\frac{2\pi}{T_0}\right)$$

$$0 < t < T_0$$
$$0 < k < N$$

$$\frac{T_s}{T_0} = \frac{T_s}{NT_s} = \frac{1}{N}$$

$$\approx e^{-j\pi k/N}\, sinc(k/N)\, \frac{1}{N}\sum_{n=0}^{N-1} x(nT_s)\, e^{-j\frac{2\pi}{N}kn}$$

$$C_k \approx e^{-j\pi\frac{k}{N}}\, sinc\left(\frac{k}{N}\right)\frac{1}{N}\sum_{n=0}^{N-1} x(nT_s)\, e^{-j2\pi\left(\frac{k}{N}\right)n}$$

$$\frac{1}{N}X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

DFT scaled by $1/N$

Forward **CTFS** Approximation (2)

$$C_k \approx \frac{1}{T_0} \sum_{n=0}^{N-1} x(nT_s) \int_{nT_s}^{(n+1)T_s} e^{-jk\omega_0 t}\, dt$$

$$\int_{nT_s}^{(n+1)T_s} e^{-jk\omega_0 t}\, dt = -\left[\frac{1}{jk\omega_0} e^{-jk\omega_0 t}\right]_{nT_s}^{(n+1)T_s} = \frac{1}{jk\omega_0}\left[-e^{-jk\omega_0 (n+1)T_s} + e^{-jk\omega_0 nT_s}\right]$$

$$= \frac{1}{jk\omega_0}\left[1 - e^{-jk\omega_0 T_s}\right] e^{-jk\omega_0 nT_s} = \frac{1}{jk2\pi/T_0}\left[1 - e^{-j2\pi k T_s/T_0}\right] e^{-j2\pi k n T_s/T_0}$$

$$\approx \left(\frac{T_s}{T_0}\right) \frac{1}{jk2\pi/N}\left[1 - e^{-j\frac{2\pi}{N}k}\right] \sum_{n=0}^{N-1} x(nT_s)\, e^{-j\frac{2\pi}{N}kn}$$

$$\frac{1}{jk2\pi/N}\left[1 - e^{-j\frac{2\pi}{N}k}\right] = \frac{\left[1 - e^{-j2\pi k/N}\right]}{j2\pi k/N} = e^{-j\pi k/N} \frac{\left[e^{+j\pi k/N} - e^{-j\pi k/N}\right]}{j2\pi k/N}$$

$$\approx e^{-j\pi k/N}\, \mathrm{sinc}(k/N)\, \frac{1}{N} \sum_{n=0}^{N-1} x(nT_s)\, e^{-j\frac{2\pi}{N}kn}$$

Inverse **CTFS** Approximation (1)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

ICTFS

$$x(t) \approx \sum_{k=-k_m}^{+k_m} C_k\, e^{+jk\omega_0 t}$$

ICTFS *approximated at* $k\,\omega_0$

$$x(nT_s) \approx \sum_{k=-k_m}^{+k_m} C_k\, e^{+jk\frac{2\pi}{T_0}nT_s}$$

$t \leftarrow nT_s$

$$\omega_o t \leftarrow \left(\frac{2\pi}{NT_s}\right) nT_s = \left(\frac{2\pi}{N}\right) n$$

$$\frac{t}{T_0} \leftarrow \frac{nT_s}{NT_s} = \left(\frac{n}{N}\right)$$

$$\approx \sum_{k=-k_m}^{+k_m} C_k\, e^{+j2\pi\left(\frac{T_s}{T_0}\right)kn}$$

$$\approx \sum_{k=0}^{N-1} C_k\, e^{+j\frac{2\pi}{N}kn}$$

$0 \le t < T_0$
$0 \le k < N$

$$x(nT_s) \approx \sum_{k=0}^{N-1} C_k\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} N C_k\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

IDFT *scaled by* N

Inverse **CTFS** Approximation (2)

$$x(t) \approx \frac{1}{2\pi} \sum_{k=-k_m}^{+k_m} X(jk\omega_0) \int_{k\omega_0}^{(k+1)\omega_0} e^{+j\omega t} d\omega$$

$$\int_{k\omega_0}^{(k+1)\omega_0} e^{+j\omega t} d\omega = \left[\frac{1}{jt} e^{+j\omega t}\right]_{k\omega_0}^{(k+1)\omega_0} = \frac{1}{jt}\left[e^{+j(k+1)\omega_0 t} - e^{+jk\omega_0 t}\right]$$

$$= \frac{1}{jt} e^{+jk\omega_0 t}\left[e^{+j\omega_0 t} - 1\right] = \frac{1}{jt}\left[e^{+j\omega_0 t} - 1\right] e^{+jk\omega_0 t}$$

$$\approx \frac{1}{j2\pi t}\left[e^{+j\omega_0 t} - 1\right] \sum_{k=-k_m}^{+k_m} X(jk\omega_0) e^{+jk\omega_0 t}$$

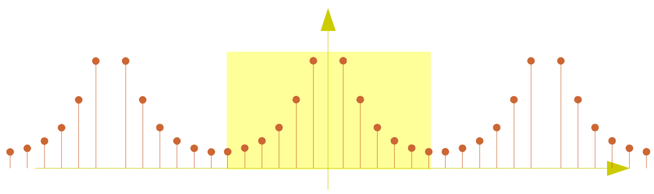
$$\frac{1}{j2\pi t}\left[e^{+j\omega_0 t} - 1\right] = \frac{\left[e^{+j2\pi t/T_0} - 1\right]}{j2\pi t} = \frac{e^{+j\pi t/T_0}}{T_0} \frac{\left[e^{+j\pi t/T_0} - e^{-j\pi t/T_0}\right]}{j2\pi t/T_0}$$

$$\approx e^{+j\pi\frac{t}{T_0}} \, sinc\left(\frac{t}{T_0}\right) \frac{1}{T_0} \sum_{k=-N/2}^{+N/2} X(jk\omega_0) e^{+jk\omega_0 t}$$

Discrete Time – **DTFS** computation

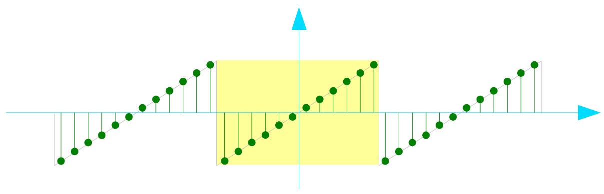
Discrete Time Fourier Series

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n} \quad \Longleftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \gamma[k]\, e^{+jk\hat{\omega}_0 n}$$



$$\omega_0 = \frac{2\pi}{T}$$

$$\hat{\omega}_0 = \frac{2\pi}{N}$$



$$\gamma[k] = \frac{1}{N}\, \mathbf{DFT}\{x[n]\} \qquad\qquad x[n] = N\, \mathbf{IDFT}\{\gamma_k\}$$

Forward **DTFS** Approximation

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n}$$

DTFS

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

DTFS at $k\hat{\omega}_0$

$$k\hat{\omega}_0 = k\left(\frac{2\pi}{T_0}\right)T_s = k\left(\frac{2\pi}{N}\right)$$

$0 \le t < T_0$
$0 < k \le N$

$$\frac{1}{N}X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

DFT scaled by $1/N$

Inverse DTFS Approximation

$$x[n] = \sum_{k=0}^{N-1} \gamma[k]\, e^{+jk\hat{\omega}_0 n}$$

$$= \sum_{k=0}^{N-1} \gamma[k]\, e^{+j\frac{2\pi}{N}kn}$$

ICTFS

IDTFS *at* $k\hat{\omega}_0$

$$k\hat{\omega}_0 = k\left(\frac{2\pi}{T_0}\right)T_s = k\left(\frac{2\pi}{N}\right)$$

$0 \leq t < T_0$

$0 < k \leq N$

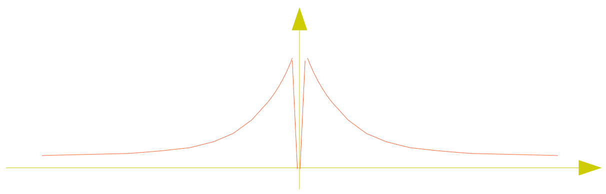
$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} N\gamma[k]\, e^{+j\frac{2\pi}{N}kn}$$

IDFT *scaled by* N

Continuous Time – **CTFT** computation

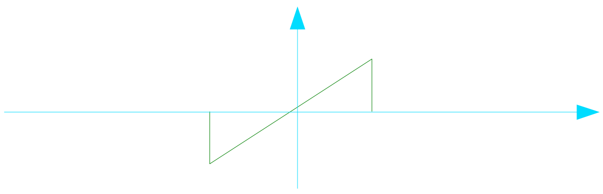
Continuous Time Fourier **Transform**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$



$$\omega_0 = \frac{2\pi}{T}$$

$$\hat{\omega}_0 = \frac{2\pi}{N}$$



$$X(jk\omega_0) \approx T_s\, \boldsymbol{DFT}\{x(nT_s)\} \qquad x(nT_s) \approx \frac{1}{T_s}\, \boldsymbol{IDFT}\{X(jk\omega_0)\}$$

Forward **CTFT** Approximation (1)

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

CTFT

$$X(j\omega) \approx \sum_{n=0}^{\infty} \int_{nT_s}^{(n+1)T_s} x(t)\, e^{-j\omega t}\, dt$$

CTFT *approximated at* $k\omega_0$

$$\approx \sum_{n=0}^{\infty} x(nT_s) \int_{nT_s}^{(n+1)T_s} e^{-j\omega t}\, dt$$

$$\approx \frac{1}{j2\pi/t}\left[1 - e^{-j2\pi T_s/t}\right] \sum_{n=0}^{\infty} x(nT_s)\, e^{-j2\pi n T_s/t}$$

$$\omega = \frac{2\pi}{t}$$

$$\approx e^{-j\pi T_s/t}\, sinc(T_s/t)\, T_s \sum_{n=0}^{\infty} x(nT_s)\, e^{-j2\pi n T_s/t}$$

$$\omega \leftarrow k\omega_0 = k\left(\frac{2\pi}{T_0}\right)$$

$0 < t < T_0$

$0 < k < N$

$$X(jk\omega_0) \approx e^{-j\pi\frac{k}{N}}\, sinc\left(\frac{k}{N}\right) T_s \sum_{n=0}^{\infty} x(nT_s)\, e^{-j2\pi\left(\frac{k}{N}\right)n}$$

$$\frac{2\pi}{t} \leftarrow k\frac{2\pi}{T_0} = k\frac{2\pi}{NT_s}$$

$$\frac{T_s}{t} \leftarrow \frac{k}{N}$$

$$T_s X[k] = T_s \sum_{n=0}^{N-1} x(nT_s)\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

DFT *scaled by* T_s

Forward **CTFT** Approximation (2)

$$X(j\omega) \approx \sum_{n=0}^{\infty} x(nT_s) \int_{nT_s}^{(n+1)T_s} e^{-j\omega t}\, dt$$

$$\int_{nT_s}^{(n+1)T_s} e^{-j\omega t}\, dt = -\left[\frac{1}{j\omega} e^{-j\omega t}\right]_{nT_s}^{(n+1)T_s} = \frac{1}{j\omega}\left[-e^{-j\omega(n+1)T_s} + e^{-j\omega n T_s}\right]$$

$$= \frac{1}{j\omega}\left[1 - e^{-j\omega T_s}\right] e^{-j\omega n T_s} = \frac{1}{j2\pi/t}\left[1 - e^{-j2\pi T_s/t}\right] e^{-j2\pi n T_s/t}$$

$$\approx \frac{1}{j2\pi/t}\left[1 - e^{-j2\pi T_s/t}\right] \sum_{n=0}^{\infty} x(nT_s)\, e^{-j2\pi n T_s/t}$$

$$\frac{\left[1 - e^{-j2\pi T_s/t}\right]}{j2\pi/t} = T_s \frac{\left[1 - e^{-j2\pi T_s/t}\right]}{j2\pi T_s/t} = T_s e^{-j\pi T_s/t} \frac{\left[e^{+j\pi T_s/t} - e^{-j\pi T_s/t}\right]}{j2\pi T_s/t}$$

$$\approx e^{-j\pi T_s/t}\, sinc(T_s/t)\, T_s \sum_{n=0}^{\infty} x(nT_s)\, e^{-j2\pi n T_s/t}$$

Inverse **CTFT** Approximation (1)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

ICTFT

$$x(t) \approx \frac{1}{2\pi} \sum_{k=-k_m}^{+k_m} \int_{k\omega_0}^{(k+1)\omega_0} X(jk\omega_0)\, e^{+j\omega t}\, d\omega$$

***ICTFT** approximated at* $k\omega_0$

$$x(t) \approx \frac{1}{2\pi} \sum_{k=-k_m}^{+k_m} X(jk\omega_0) \int_{k\omega_0}^{(k+1)\omega_0} e^{+j\omega t}\, d\omega$$

$$\approx \frac{1}{j2\pi t}\left[e^{+j\omega_0 t} - 1\right] \sum_{k=-k_m}^{+k_m} X(jk\omega_0)\, e^{+jk\omega_0 t}$$

$t \leftarrow nT_s$ $\qquad$ $\omega_o t \leftarrow \left(\frac{2\pi}{NT_s}\right) nT_s = \left(\frac{2\pi}{N}\right) n$

$$\approx e^{+j\pi\frac{t}{T_0}}\, sinc\left(\frac{t}{T_0}\right) \frac{1}{T_0} \sum_{k=-N/2}^{+N/2} X(jk\omega_0)\, e^{+jk\omega_0 t}$$

$\frac{t}{T_0} \leftarrow \frac{nT_s}{NT_s} = \left(\frac{n}{N}\right)$ $\qquad$ $0 < t < T_0$, $0 < k < N$

$$x(nT_s) \approx e^{+j\pi\frac{n}{N}}\, sinc\left(\frac{n}{N}\right) \frac{1}{NT_s} \sum_{k=0}^{N-1} X(jk\omega_0)\, e^{+j\frac{2\pi}{N}kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{T_s} X(jk\omega_0)\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

***IDFT** scaled by* $1/T_s$

Inverse **CTFT** Approximation (2)

$$x(t) \approx \frac{1}{2\pi} \sum_{k=-k_m}^{+k_m} X(jk\omega_0) \int_{k\omega_0}^{(k+1)\omega_0} e^{+j\omega t} d\omega$$

$$\int_{k\omega_0}^{(k+1)\omega_0} e^{+j\omega t} d\omega = \left[\frac{1}{jt} e^{+j\omega t}\right]_{k\omega_0}^{(k+1)\omega_0} = \frac{1}{jt}\left[e^{+j(k+1)\omega_0 t} - e^{+jk\omega_0 t}\right]$$

$$= \frac{1}{jt} e^{+jk\omega_0 t}\left[e^{+j\omega_0 t} - 1\right] = \frac{1}{jt}\left[e^{+j\omega_0 t} - 1\right] e^{+jk\omega_0 t}$$

$$\approx \frac{1}{j2\pi t}\left[e^{+j\omega_0 t} - 1\right] \sum_{k=-k_m}^{+k_m} X(jk\omega_0)\, e^{+jk\omega_0 t}$$

$$\frac{1}{j2\pi t}\left[e^{+j\omega_0 t} - 1\right] = \frac{\left[e^{+j2\pi t/T_0} - 1\right]}{j2\pi t} = \frac{e^{+j\pi t/T_0}}{T_0} \frac{\left[e^{+j\pi t/T_0} - e^{-j\pi t/T_0}\right]}{j2\pi t/T_0}$$

$$\approx e^{+j\pi\frac{t}{T_0}} \mathrm{sinc}\left(\frac{t}{T_0}\right) \frac{1}{T_0} \sum_{k=-N/2}^{+N/2} X(jk\omega_0)\, e^{+jk\omega_0 t}$$

CTFT Approximation Summary

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(j\omega) \approx \sum_{n=0}^{\infty} \int_{nT_s}^{(n+1)T_s} x(t)\, e^{-j\omega t}\, dt$$

$$X(j\omega) \approx \sum_{n=0}^{\infty} x(nT_s) \int_{nT_s}^{(n+1)T_s} e^{-j\omega t}\, dt$$

$$X(jk\omega_0) \approx e^{-j\pi\frac{k}{N}} sinc\left(\frac{k}{N}\right) \cdot T_s \sum_{n=0}^{\infty} x(nT_s)\, e^{-j2\pi\left(\frac{k}{N}\right)n}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$x(t) \approx \frac{1}{2\pi} \sum_{k=-k_m}^{+k_m} \int_{k\omega_0}^{(k+1)\omega_0} X(jk\omega_0)\, e^{+j\omega t}\, d\omega$$

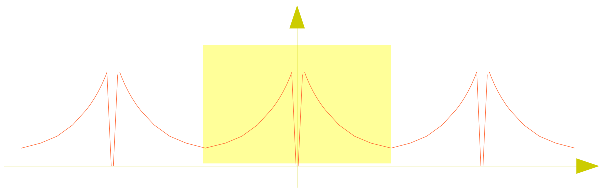
$$x(t) \approx \frac{1}{2\pi} \sum_{k=-k_m}^{+k_m} X(jk\omega_0) \int_{k\omega_0}^{(k+1)\omega_0} e^{+j\omega t}\, d\omega$$

$$x(nT_s) \approx e^{+j\pi\frac{n}{N}} sinc\left(\frac{n}{N}\right) \cdot \frac{1}{NT_s} \sum_{k=0}^{N-1} X(jk\omega_0)\, e^{+j\frac{2\pi}{N}kn}$$

Discrete Time – **DTFT** computation

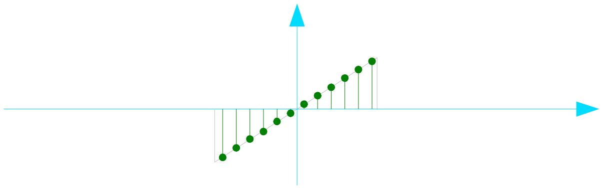
Discrete Time Fourier **Transform**

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega}n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(j\hat{\omega})\, e^{+j\hat{\omega}n}\, d\hat{\omega}$$



$$\omega_0 = \frac{2\pi}{T}$$

$$\hat{\omega}_0 = \frac{2\pi}{N}$$



$$X(jk\hat{\omega}_0) \approx \boldsymbol{DFT}\{x[n]\} \qquad x[n] \approx \boldsymbol{IDFT}\{X(jk\hat{\omega}_0)\}$$

Forward **DTFT** Approximation

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$ ***DTFT***

$$X(j\hat{\omega}) \approx \sum_{n=0}^{N-1} x[n]\, e^{-j\hat{\omega} n}$$

$$\hat{\omega} \leftarrow k\hat{\omega}_0 = k\left(\frac{2\pi}{N}\right) = 2\pi\left(\frac{k}{N}\right)$$

$$X(jk\hat{\omega}_0) \approx \sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n}$$ ***DTFT*** *approximated at* $k\hat{\omega}_0$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$ ***DFT***

$$X(jk\hat{\omega}_0) \approx DFT(x[n])$$

Inverse **DTFT** Approximation (1)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(j\hat{\omega})\, e^{+j\hat{\omega}n} d\hat{\omega}$$

IDTFT

$$x[n] \approx \frac{1}{2\pi}\sum_{k=0}^{N-1}\int_{k\hat{\omega}_0}^{(k+1)\hat{\omega}_0} X(j\hat{\omega})\, e^{+j\hat{\omega}n} d\hat{\omega}$$

$$\hat{\omega} \leftarrow k\hat{\omega}_0 = k\left(\frac{2\pi}{N}\right) = 2\pi\left(\frac{k}{N}\right)$$

$$x[n] \approx \frac{1}{2\pi}\sum_{k=0}^{N-1} X(jk\hat{\omega}_0)\int_{k\hat{\omega}_0}^{(k+1)\hat{\omega}_0} e^{+j\hat{\omega}n} d\hat{\omega}$$

IDTFT *approximated at* $k\hat{\omega}_0$

$$\approx \frac{\left[e^{+j\hat{\omega}_0 n} - 1\right]}{j2\pi n}\sum_{k=0}^{N-1} X(jk\hat{\omega}_0)\, e^{+jk\hat{\omega}_0 n}$$

$$\approx e^{+j\pi\frac{n}{N}} sinc\left(\frac{n}{N}\right) \frac{1}{N}\sum_{k=0}^{N-1} X(jk\hat{\omega}_0)\, e^{+jk\hat{\omega}_0 n}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(jk\hat{\omega}_0)\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

IDFT

Inverse **DTFT** Approximation (2)

$$x[n] \approx \frac{1}{2\pi} \sum_{k=0}^{N-1} X(jk\hat{\omega}_0) \int_{k\hat{\omega}_0}^{(k+1)\hat{\omega}_0} e^{+j\hat{\omega}n}\, d\hat{\omega}$$

$$\int_{k\hat{\omega}_0}^{(k+1)\hat{\omega}_0} e^{+j\hat{\omega}n}\, d\hat{\omega} = \left[\frac{1}{jn} e^{+j\hat{\omega}n}\right]_{k\hat{\omega}_0}^{(k+1)\hat{\omega}_0} = \frac{1}{jn}\left[e^{+j(k+1)\hat{\omega}_0 n} - e^{+jk\hat{\omega}_0 n}\right]$$

$$= \frac{1}{jn} e^{+jk\hat{\omega}_0 n}\left[e^{+j\hat{\omega}_0 n} - 1\right] = \frac{1}{jn}\left[e^{+j\hat{\omega}_0 n} - 1\right] e^{+jk\hat{\omega}_0 n}$$

$$\approx \frac{\left[e^{+j\hat{\omega}_0 n} - 1\right]}{j2\pi n} \sum_{k=0}^{N-1} X(jk\hat{\omega}_0)\, e^{+jk\hat{\omega}_0 n}$$

$$\frac{\left[e^{+j\hat{\omega}_0 n} - 1\right]}{j2\pi n} = \frac{\left[e^{+j2\pi n/N} - 1\right]}{j2\pi n} = \frac{e^{+j\pi n/N}}{N} \frac{\left[e^{+j\pi n/N} - e^{-j\pi n/N}\right]}{j2\pi n/N}$$

$$\approx e^{+j\pi\frac{n}{N}} sinc\left(\frac{n}{N}\right) \frac{1}{N} \sum_{k=0}^{N-1} X(jk\hat{\omega}_0)\, e^{+jk\hat{\omega}_0 n}$$

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