Hybrid CORDIC 2.A Sine/Cosine Generator

20170706

Wilson ROM based Sine/Cosine Generation

[24] Fu & Willson Sine / Cosine Generation

ROM-based

For high resolution, ROM size grows exponentially

quater-wave symmetry

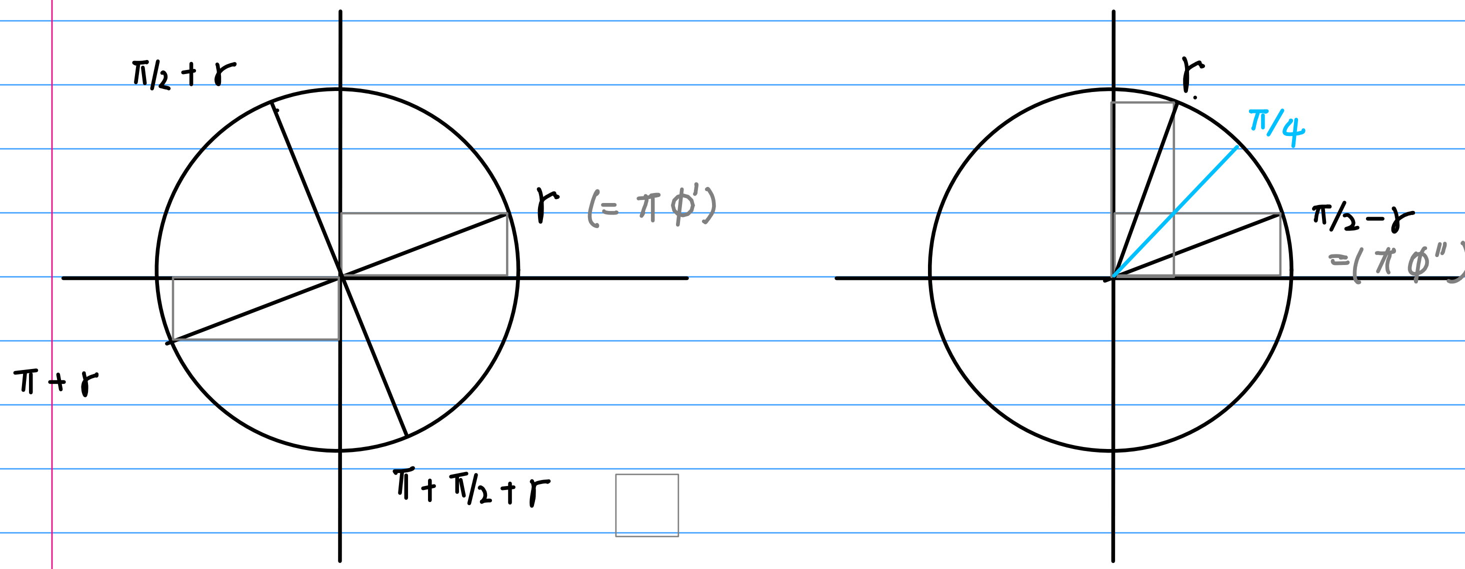
$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\phi \quad [0, 2\pi] \longrightarrow [0, \frac{\pi}{4}]$$

conditionally interchanging inputs X_0 & Y_0
Conditionally interchanging and negating outputs X & Y

$$X = X_0 \cos\phi - Y_0 \sin\phi$$
$$Y = Y_0 \cos\phi + X_0 \sin\phi$$

Madisetti VLSI arch



for frequency synthesis

argument: signed normalized by π angle [-1, 1]

binary representation of a radian angle required

$$\underset{\phi}{[-1, 1]} \longrightarrow \underset{\theta}{[0, \pi/4]} \longrightarrow \text{sine/cosine generator}$$

$$\pi\phi \longleftarrow$$

① a phase accumulator ϕ [-1, 1]
② a radian converter $\phi \to \theta$
③ a sine/cosine generator $\sin\theta$, $\cos\theta$
④ an output stage $\sin\theta$, $\cos\theta$

$\sin\theta \to \sin\pi\phi$, $\cos\theta \to \cos\pi\phi$

.

output stage $\quad \sin\theta \rightarrow \sin\pi\phi \quad [-\pi, +\pi]$

$\cos\theta \rightarrow \cos\pi\phi$

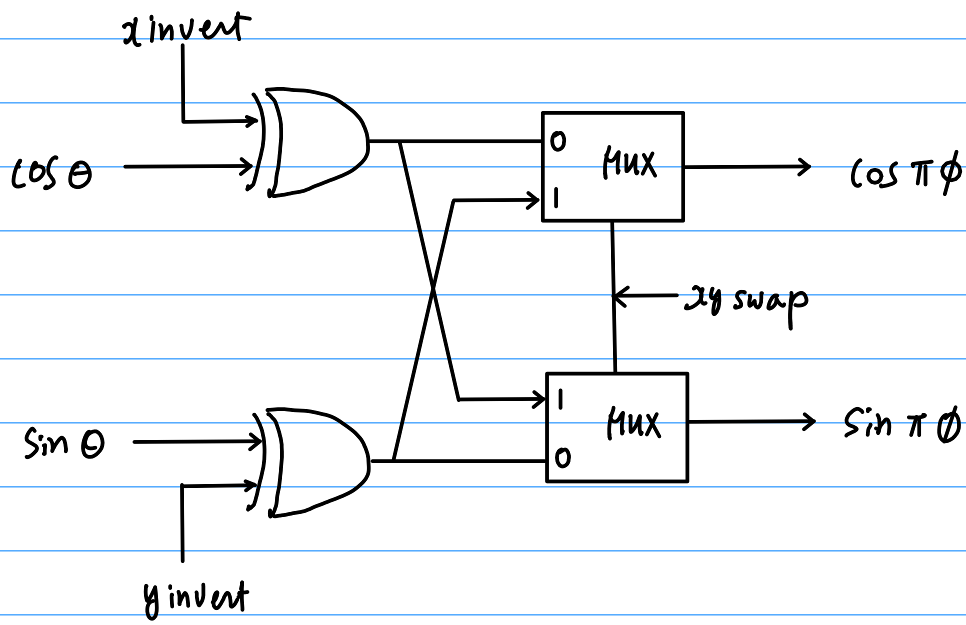
negation / interchange

MSB₁ → x invert $\quad$ the negation of $\cos\theta = X_{N+1}$

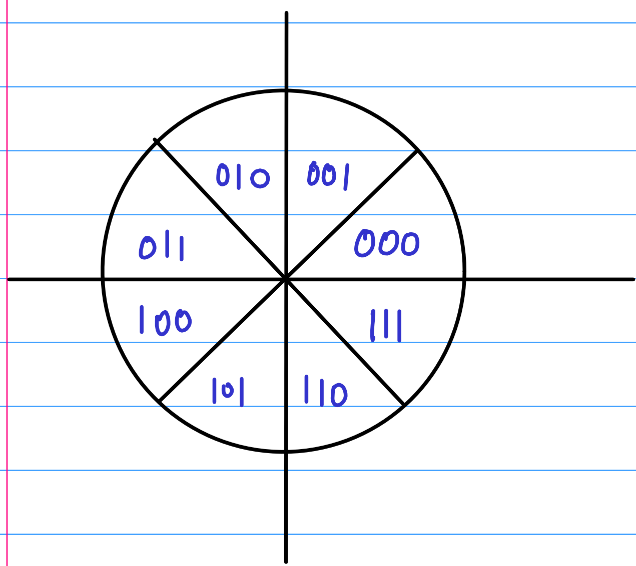
MSB₂ → y invert $\quad \sin\theta = Y_{N+1}$

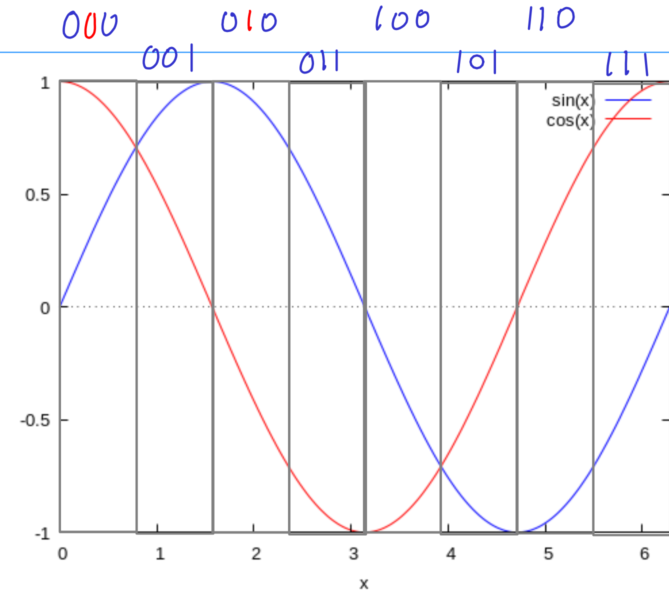
MSB₃ → xy swap $\quad$ Interchange

negate before swap



Madisetti & Willson, DDS Freq synthesizer





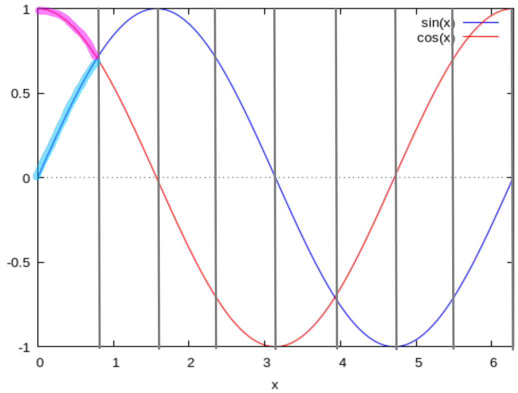
	cos x inv	sin. y inv	swap	$\cos \pi\theta$	$\sin \pi\phi$
000	0	0	0	$\cos\theta$	$\sin\theta$
001	0	0	1	$\sin\theta$	$\cos\theta$
010	0	1	1	$-\sin\theta$	$\cos\theta$
011	1	0	0	$-\cos\theta$	$\sin\theta$
100	1	1	0	$-\cos\theta$	$-\sin\theta$
101	1	1	1	$-\sin\theta$	$-\cos\theta$
110	1	0	1	$\sin\theta$	$-\cos\theta$
111	0	1	0	$\cos\theta$	$-\sin\theta$

Madisetti & Willson, DDS Freq synthesizer

0

$\cos\theta$

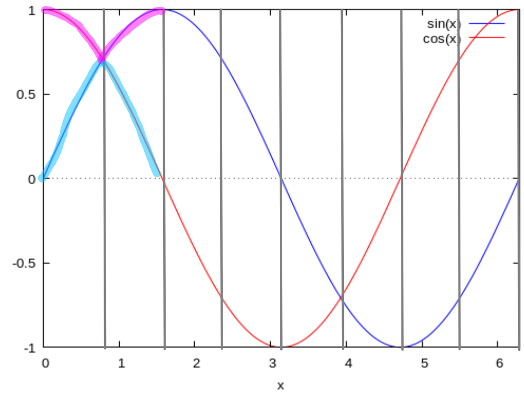
$\sin\theta$



1

$\sin\theta$

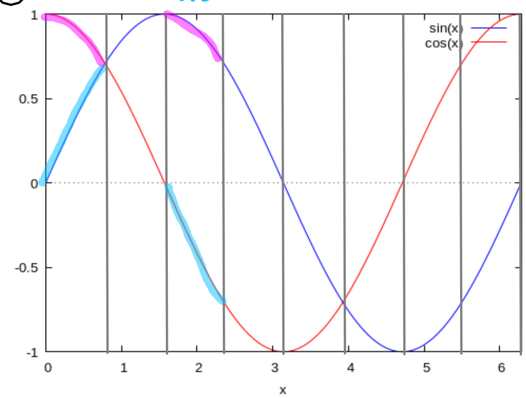
$\cos\theta$



2

$-\sin\theta$

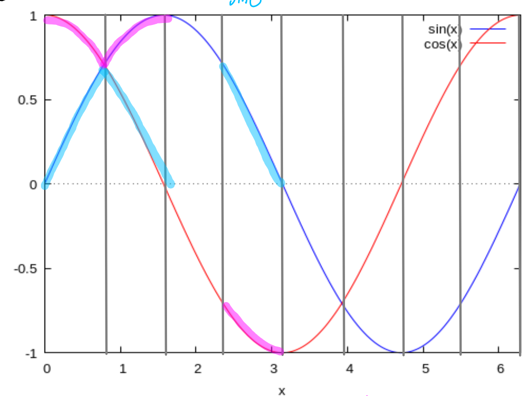
$\cos\theta$



3

$-\cos\theta$

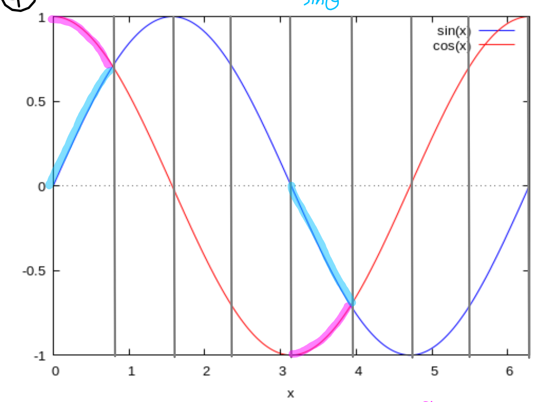
$\sin\theta$



4

$-\cos\theta$

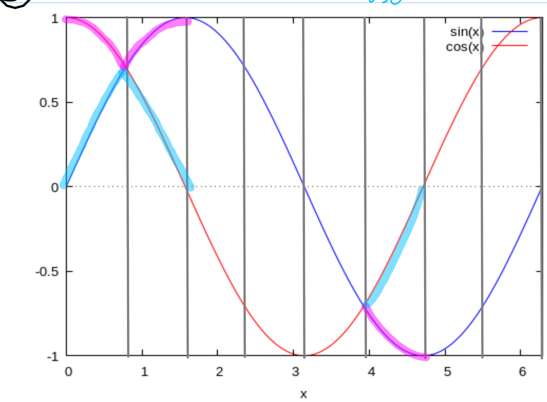
$-\sin\theta$



5

$-\sin\theta$

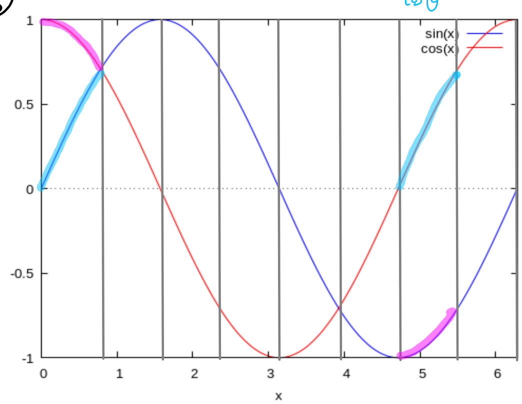
$-\cos\theta$



6

$\sin\theta$

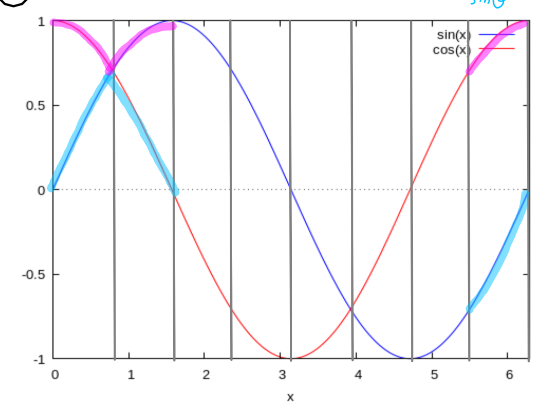
$-\cos\theta$



7

$\cos\theta$

$-\sin\theta$

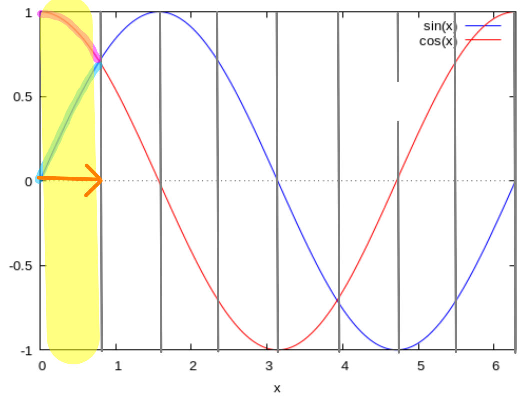


Madisetti & Willson, DDS Freq synthesizer

$$\begin{cases} \cos\phi \\ \sin\phi \end{cases}$$

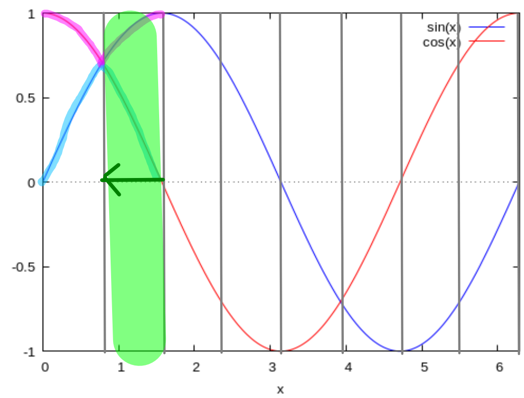
0

$\cos\theta$
$\sin\theta$



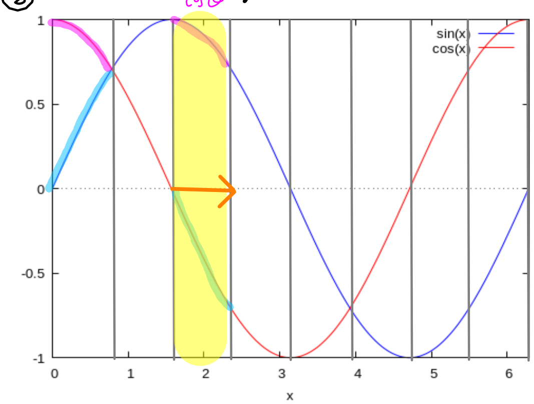
1

$\sin\theta$
$\cos\theta$



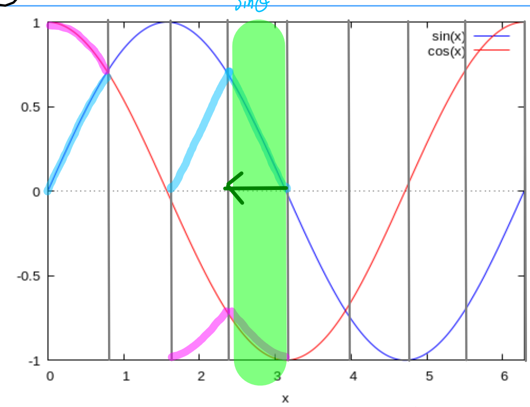
2

$-\sin\theta$
$\cos\theta$



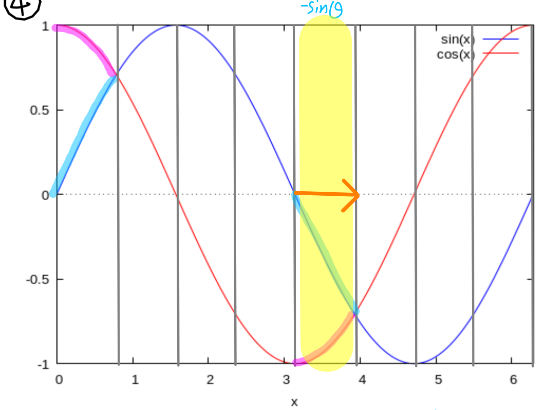
3

$-\cos\theta$
$\sin\theta$



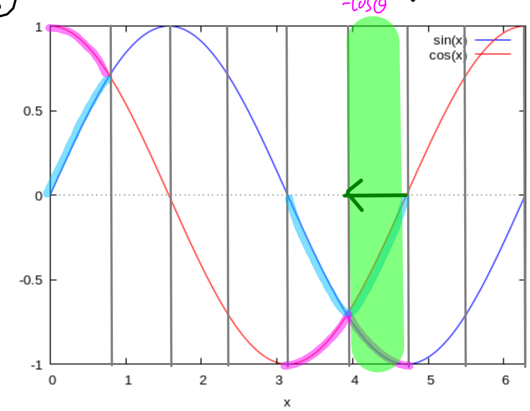
4

$-\cos\theta$
$-\sin\theta$



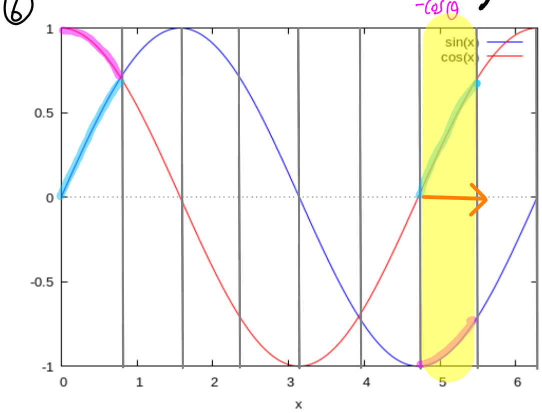
5

$-\sin\theta$
$-\cos\theta$



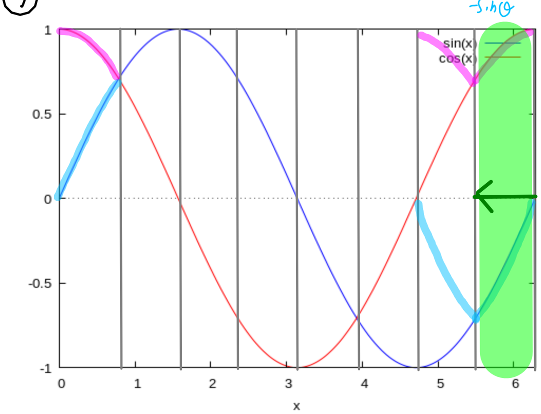
6

$\sin\theta$
$-\cos\theta$

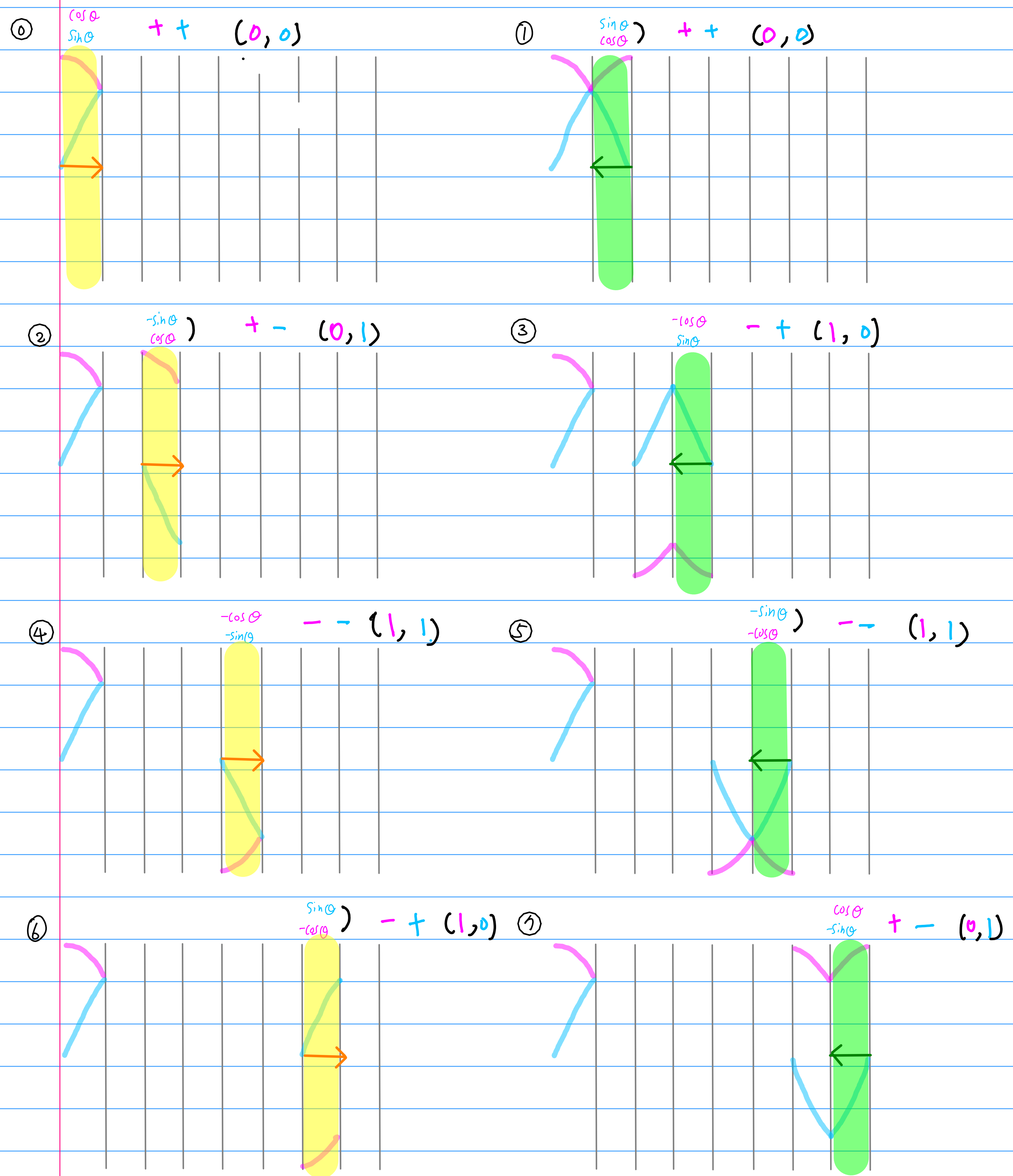


7

$\cos\theta$
$-\sin\theta$



sin φ



0
cos θ
sin θ
+ +
(0, 0)
1
sin θ
cos θ
+ +
(0, 0)
2
-sin θ
cos θ
+ -
(0, 1)
3
-cos θ
sin θ
- +
(1, 0)
4
-cos θ
-sin θ
- -
(1, 1)
5
-sin θ
-cos θ
- -
(1, 1)
6
sin θ
-cos θ
- +
(1, 0)
7
cos θ
-sin θ
+ -
(0, 1)

Madisetti & Willson, DDS Freq synthesizer

	xinv	yinv	swap	$\cos \pi \phi$	$\sin \pi \phi$
0 0 0	0	0	0	$\cos\theta$	$\sin\theta$
0 0 1	0	0	1	$\sin\theta$	$\cos\theta$
0 1 0	0	1	1	$-\sin\theta$	$\cos\theta$
0 1 1	1	0	0	$-\cos\theta$	$\sin\theta$
1 0 0	1	1	0	$-\cos\theta$	$-\sin\theta$
1 0 1	1	1	1	$-\sin\theta$	$-\cos\theta$
1 1 0	1	0	1	$\sin\theta$	$-\cos\theta$
1 1 1	0	1	0	$\cos\theta$	$-\sin\theta$

0 0	0 0
0 1	1 0
1 1	1 1
1 0	0 1

0 0 0 0

0 1 1 0

1 1 1 1

1 0 0 1

Madisetti & Willson, DDS Freq synthesizer

$$\theta = \sum_{k=1}^{N} b_k \theta_k$$

b_k sign + N bit — (N+1) bit fractional b

$b_k \in \{0, 1\}$

$\theta_k = 2^{-k}$

θ is constrained to be positive $\boxed{b_0 = 0}$

$$\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

$r_k \in \{-1, +1\}$ signed digits

ϕ_0 constant

⊕ subrotation by 2^{-k}

2 equal ⊕ half rotations by 2^{-k-1}

⊖ subrotation

2 equal opposite half rotations by $\pm 2^{-k-1}$

Binary Representation

$b_k = 1$: rotation by 2^{-k}

$b_k = 0$: zero rotation

k-th rotation

- fixed rotation by 2^{-k-1}
- pos rotation ← $b_k = 1$
  neg rotation ← $b_k = 0$

Combining all the fixed rotations

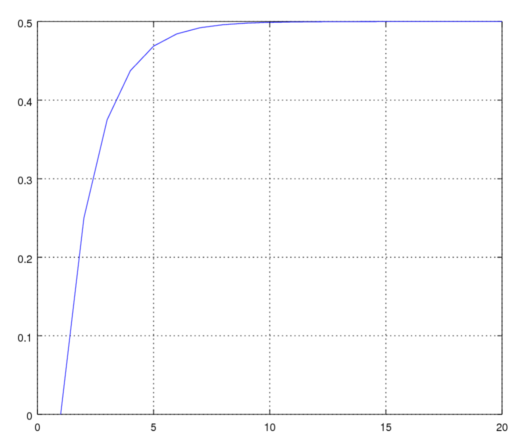
→ initial fixed rotation

	b_1 2^{-1}	b_2 2^{-2}	b_3 2^{-3}		b_N 2^{-N}
fixed $\Rightarrow$	$+2^{-2}$	$+2^{-3}$	$+2^{-4}$		$+2^{-N-1}$
	$(b_1=1)$ $+2^{-2}$	$(b_2=1)$ $+2^{-3}$	$(b_3=1)$ $+2^{-4}$		$(b_N=1)$ $+2^{-N-1}$
	$(b_1=0)$ -2^{-2}	$(b_2=0)$ -2^{-3}	$(b_3=0)$ -2^{-4}		$(b_N=0)$ -2^{-N-1}

initial fixed rotation

$$\phi_0 = \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{N+1}}$$

$$= \frac{\frac{1}{2^2}\left(1-\frac{1}{2^N}\right)}{\left(1-\frac{1}{2}\right)} = \frac{1}{2}\left(1-\frac{1}{2^N}\right) = \frac{1}{2} - \frac{1}{2^{N+1}}$$



Signed Digit Recoding

the rotation after recoding

— a fixed initial rotation ϕ_0

a sequence of ⊕/⊖ rotations

$b_k = 1 \quad + 2^{-k-1}$ rotation

$b_k = 0 \quad - 2^{-k-1}$ rotation

$$r_k = (2b_{k-1} - 1)$$

$2 \cdot 1 - 1 = +1 \qquad b_{k-1} = 1 \longrightarrow r_k = +1$

$2 \cdot 0 - 1 = -1 \qquad b_{k-1} = 0 \longrightarrow r_k = -1$

The recoding need not be explicitly performed

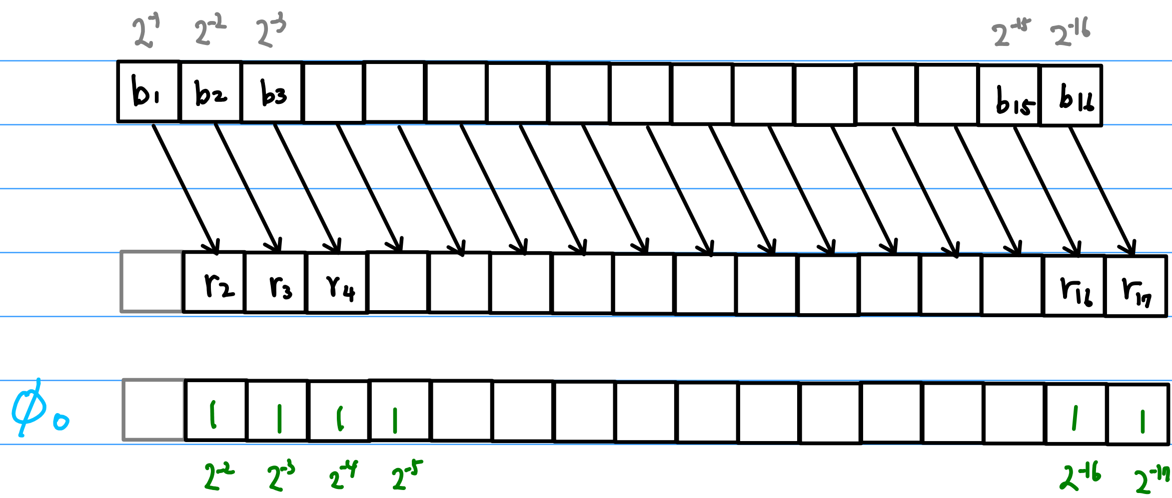
Simply replacing $b_k = 0$ with -1

This recoding maintains

a constant scaling factor K

$$\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

Binary Representation $\{b_k\}$



Signed Digit Recoding $\{r_k\}$

The Scaling K.
The initial rotation ϕ_0

rotation starting point
$$(X_0, Y_0) = (K\cos\phi_0, K\sin\phi)$$

— fixed
— no error buildup

— rotation direction
immediately obtained from the binary representation
→ no need for comparison

the subangles $\boxed{\theta_k = 2^{-k}}$ used in recoding
the subangles $\boxed{\theta_k = \tan^{-1}(2^{-k})}$ used in CORDIC

$\tan\theta_k$ multipliers used
in the first few subrotation stages
cannot be implemented
as a simple shift-and-add operations

→ ROM implementation

reduced chip area
higher operating speed.

Architecture

1. phase accumulator $\phi \in [-1, +1]$
2. radian converter $\phi \rightarrow \theta \in [0, \frac{\pi}{4}]$
3. sine/cosine generator $\sin(\theta)$ $\cos(\theta)$
4. output stage $\sin(\pi\phi)$ $\cos(\pi\phi)$

Overflowing 2's complement accumulator

normalized by π angle ϕ

need radian angle $\theta \in [0, \frac{\pi}{4}]$

$0 < \theta < 1$ rad

N-bit binary representation of θ

controls the direction of subrotation

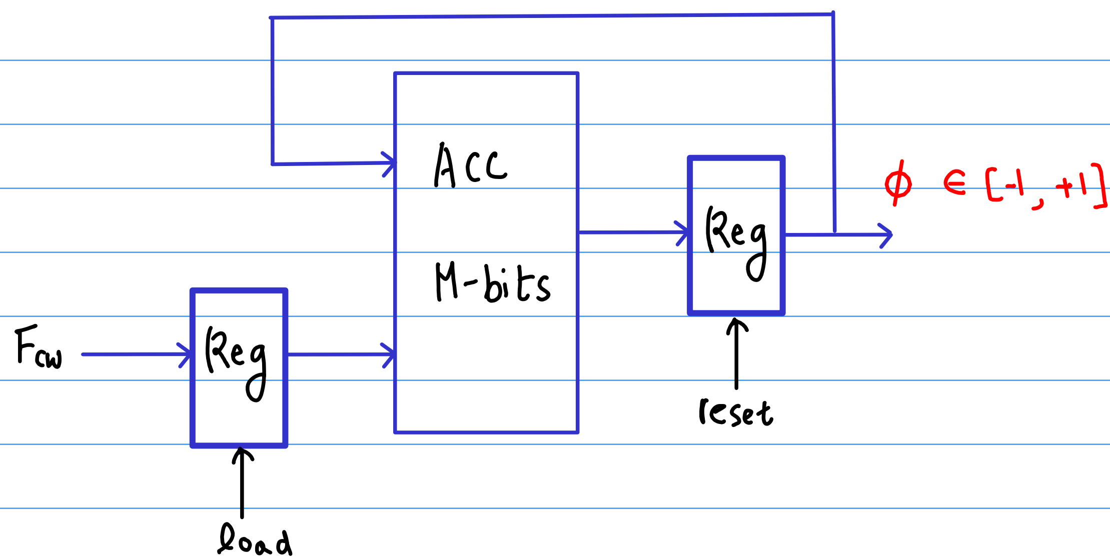
N-bit precision of $\cos\theta$ & $\sin\theta$

Output stage $\quad \theta \rightarrow \pi\phi$

$\sin\theta \rightarrow \sin\pi\phi$

$\cos\theta \rightarrow \cos\pi\phi$

phase accumulator



M-bit adder

repeatedly increments the phase angle by F_{cw} at each clock cycle

frequency control word

at time n, $\phi = n F_{cw} / 2^M$

$$\cos\phi = \cos(n F_{cw} / 2^M)$$
$$\sin\phi = \sin(n F_{cw} / 2^M)$$

Radian Converter

normalized angle ϕ

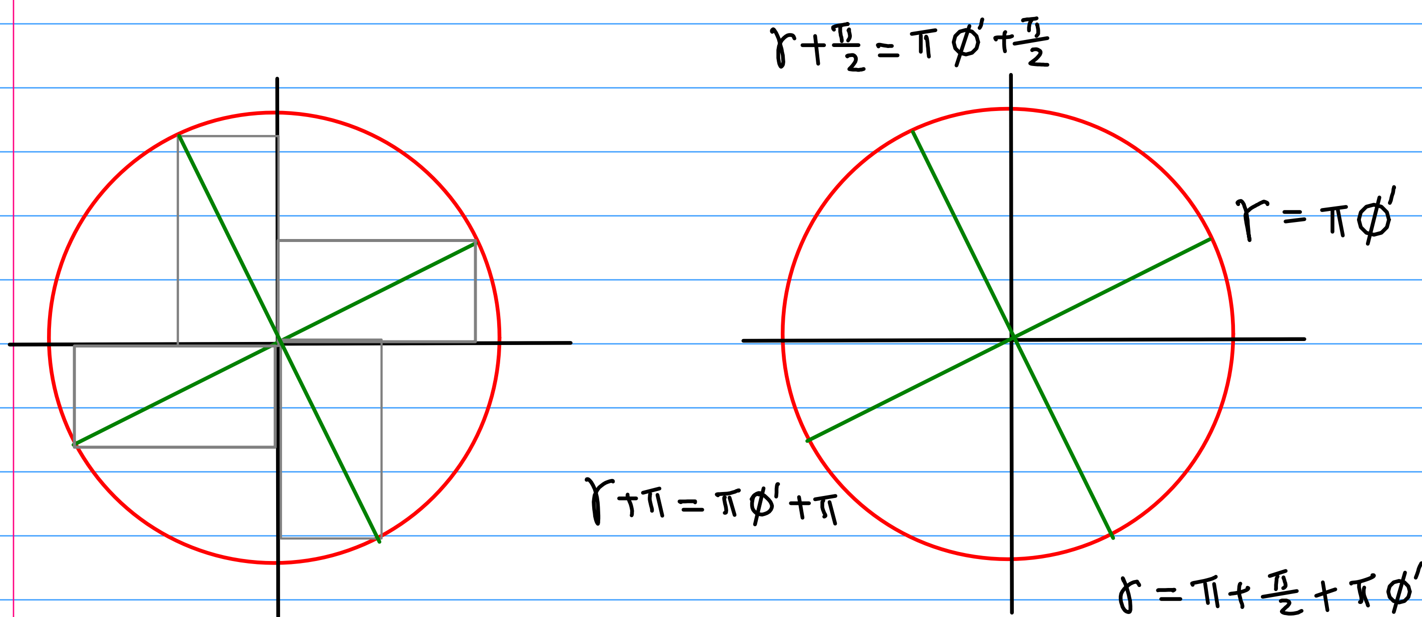
ϕ	MSB1	MSB2	MSB3					...	

MSB1–MSB2: Quadrant of $\pi\phi$

MSB3: U. half / L. half

MSB1–MSB3: Stored to interchange / negate

ϕ'	0	0	MSB3					...	



$\phi \rightarrow \phi' \rightarrow$

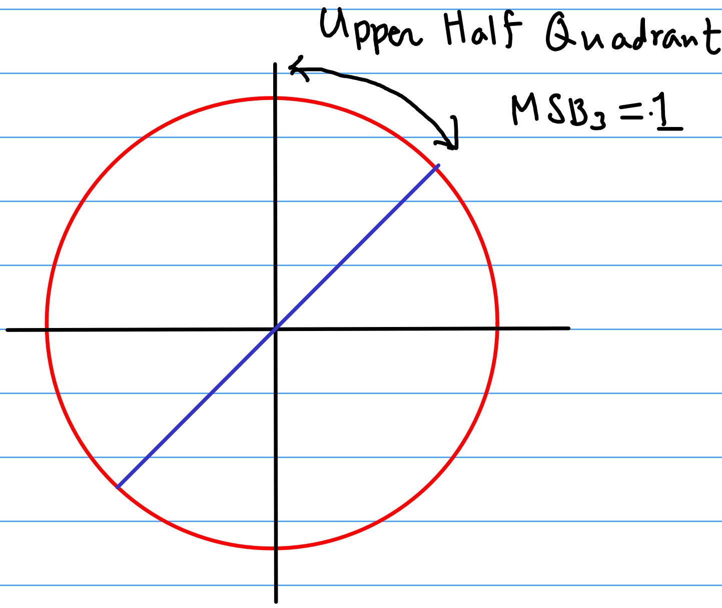
$\pi\phi'$	$+\ 0 \cdot \frac{\pi}{2}$	00
$\pi\phi'$	$+\ 1 \cdot \frac{\pi}{2}$	01
$\pi\phi'$	$+\ 2 \cdot \frac{\pi}{2}$	10
$\pi\phi'$	$+\ 3 \cdot \frac{\pi}{2}$	11

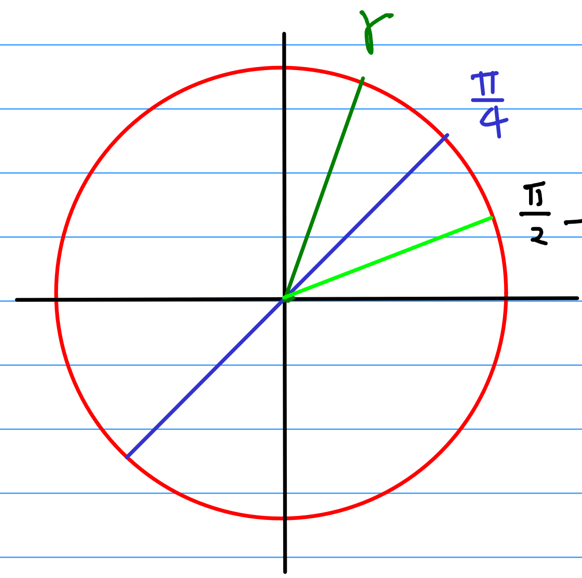
$\uparrow$ 1st Quad

1st Quadrant

ϕ'

0	0	MSB3					...	



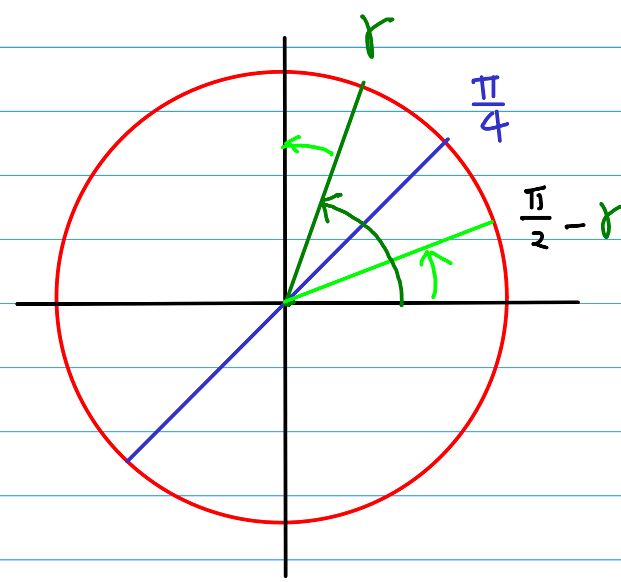


$\gamma > \frac{\pi}{4}$: Upper Half ($MSB_3 = 1$)

$\gamma < \frac{\pi}{4}$: Lower Half ($MSB_3 = 0$)

$$\cos\gamma = \sin\left(\frac{\pi}{2} - \gamma\right)$$

$$\sin\gamma = \cos\left(\frac{\pi}{2} - \gamma\right)$$



$\gamma > \frac{\pi}{4}$ $\quad \frac{\pi}{2} - \gamma < \frac{\pi}{4}.$

ϕ'									
	0	0	MSB3					. . .	

$MSB3 = 1 \qquad \phi' > \frac{\pi}{4}$

$$\phi'' = \frac{\pi}{2} - \phi'$$

0.5	1	0	0					. . .	
$\ominus$ ϕ'	0	0	MSB3					. . .	

$$\begin{cases} MSB3 = 0 & \phi'' = \phi' \\ MSB_3 = 1 & \phi'' = 0.5 - \phi' \end{cases}$$

$\theta = \pi \phi''$ (Handwired Multiplier)

$0 < \theta < \frac{\pi}{4}$

$$\phi \longrightarrow \phi' \longrightarrow \phi''$$

1st Quad Lower Half

Sine / Cosine Generator

Subrotation

$$X_{k+1} = X_k - (r_k \tan\theta_k) Y_k$$

$$Y_{k+1} = Y_k + (r_k \tan\theta_k) X_k$$

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

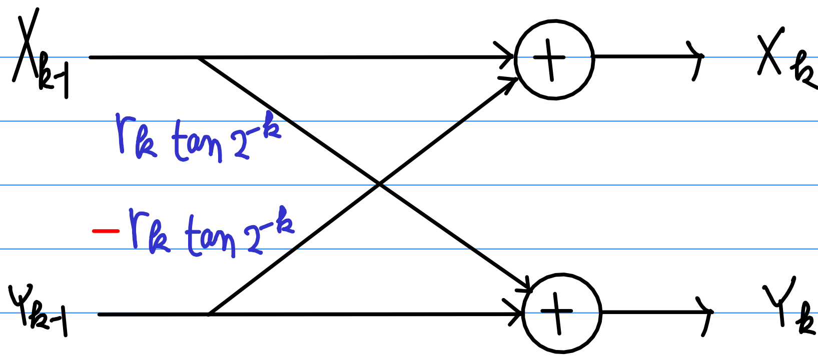
$$= \cos\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

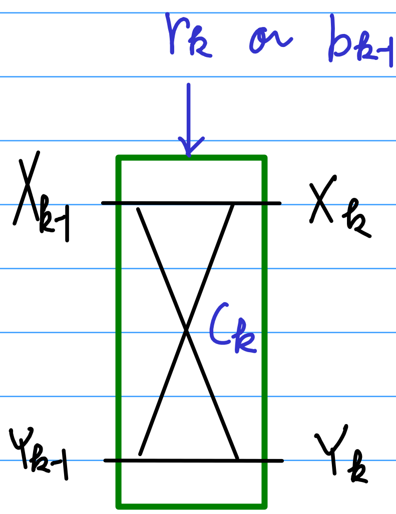
$$\theta = \sigma_0\theta_0 + \sigma_1\theta_1 + \cdots + \sigma_N\theta_N$$

$$\sigma_k = \{-1, 0, +1\}$$

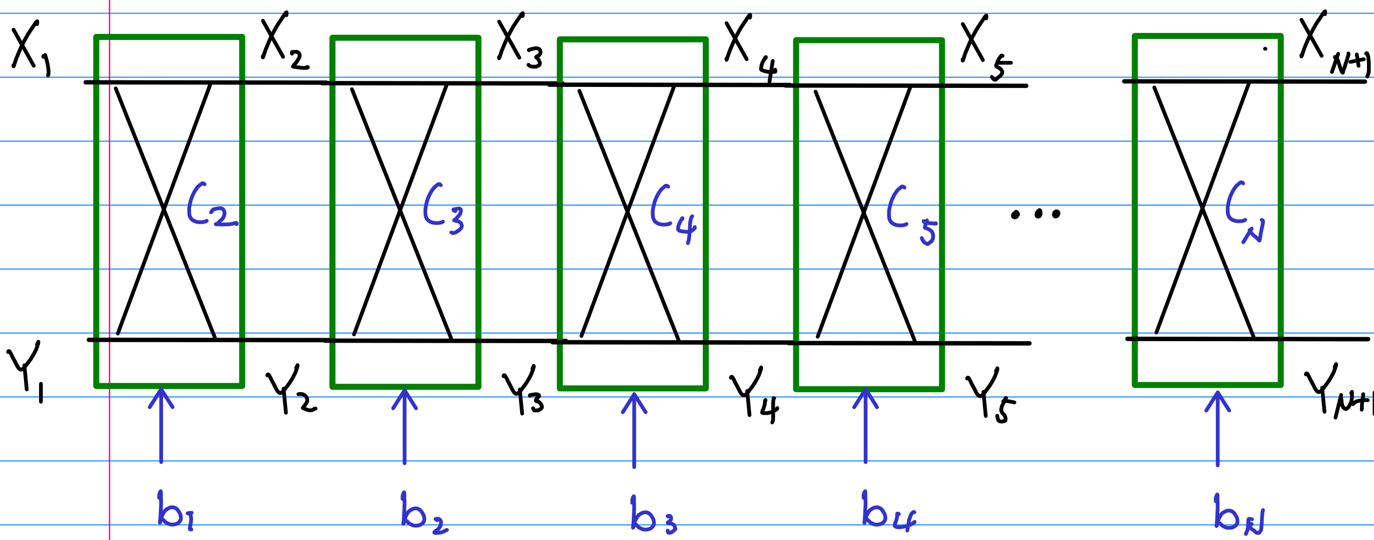
$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = K \begin{bmatrix} 1 & -\tan\sigma_N\theta_N \\ \tan\sigma_N\theta_N & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan\sigma_0\theta_0 \\ \tan\sigma_0\theta_0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$K = \cos\sigma_0\theta_0 \cdot \cos\sigma_1\theta_1 \cdots \cos\sigma_N\theta_N$$





Butterfly



$c_2 = \tan\left(\frac{1}{2^2}\right)$ $c_3 = \tan\left(\frac{1}{2^3}\right)$ $c_4 = \tan\left(\frac{1}{2^4}\right)$ $c_5 = \tan\left(\frac{1}{2^5}\right)$

$K\cos\phi_0 \to X_1$ $X_{N+1} \to \cos\theta$

$K\sin\phi_0 \to Y_1$ $Y_{N+1} \to \sin\theta$

$\theta \to \{b_1, b_2, \dots, b_N\}$

the initial (X_0, Y_0) always the same

merge the first B/3 butterflies

→ $2^{B/3}$ words ROM implementation

→ no need $\tan \theta_k$ multipliers

→ $\{b_1, b_2, \cdots, b_{B/3}\} \Rightarrow$ address

accesses

$$\cos\left(\phi_0 + \sum_{k=1}^{B/3} b_k 2^{-k+1}\right)$$

$$\sin\left(\phi_0 + \sum_{k=1}^{B/3} b_k 2^{-k+1}\right)$$

Lower Half of the 1st Quadrant

- all positive X_k & Y_k
- no need sign extension
- reduce the loads
- high speed

Merging Butterflies

merge m final butterflies

$$\begin{pmatrix} X_k \\ Y_k \end{pmatrix} \longrightarrow \begin{pmatrix} X_{k+m} \\ Y_{k+m} \end{pmatrix} \quad \text{directly}$$

$$X_{k+m} = X_k - Y_k \sum_{i=k}^{k+m-1} r_i \tan 2^{-i}$$

$$Y_{k+m} = Y_k + X_k \sum_{i=k}^{k+m-1} r_i \tan 2^{-i}$$

valid merging $\quad k \geqslant (B-1)/2$

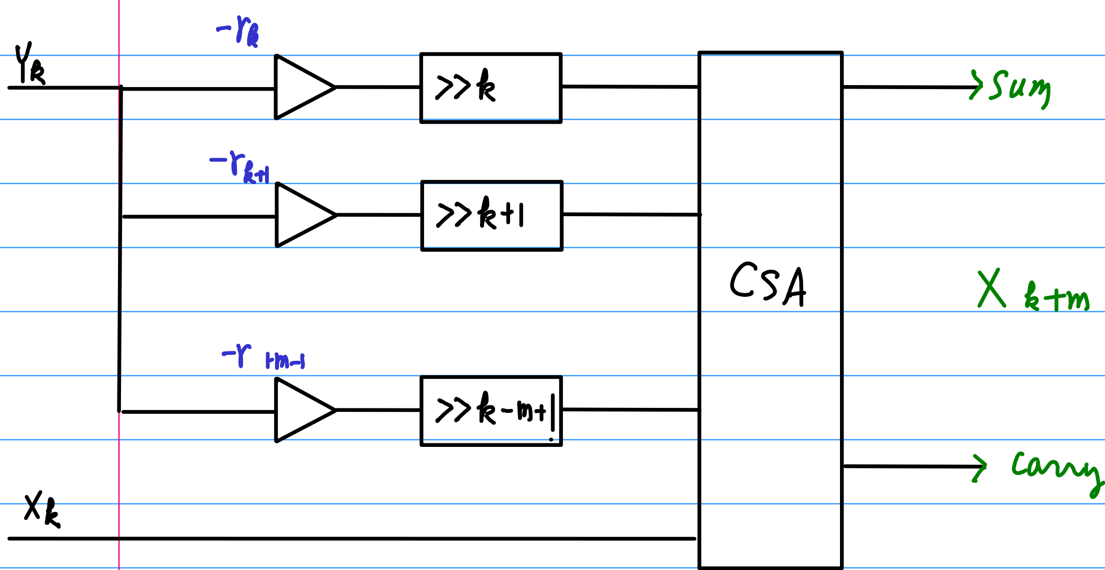
$\tan(2^{-i}) = 2^{-i} \quad k \geqslant B/3$

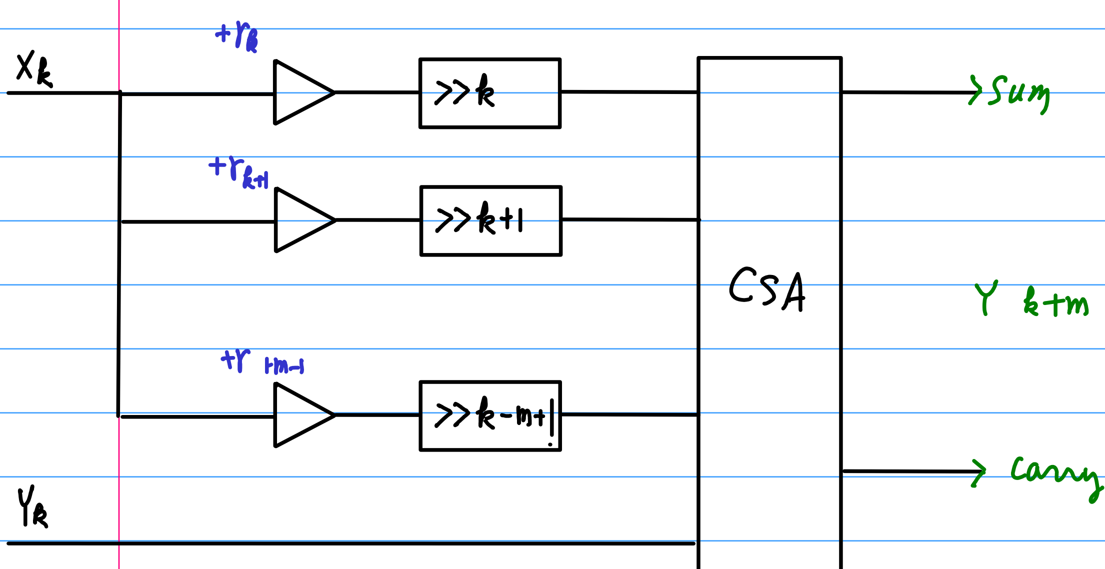
look ahead by m

the individual terms in the summation
can be computed independently
and summed in parallel

$$X_{k+m} = X_k - Y_k \sum_{i=k}^{k+m-1} r_i \tan 2^{-i}$$

$$Y_{k+m} = Y_k + X_k \sum_{i=k}^{k+m-1} r_i \tan 2^{-i}$$





+ reduced latency
+ reduced routing
+ only the half for a single-ended system.

Output Stage

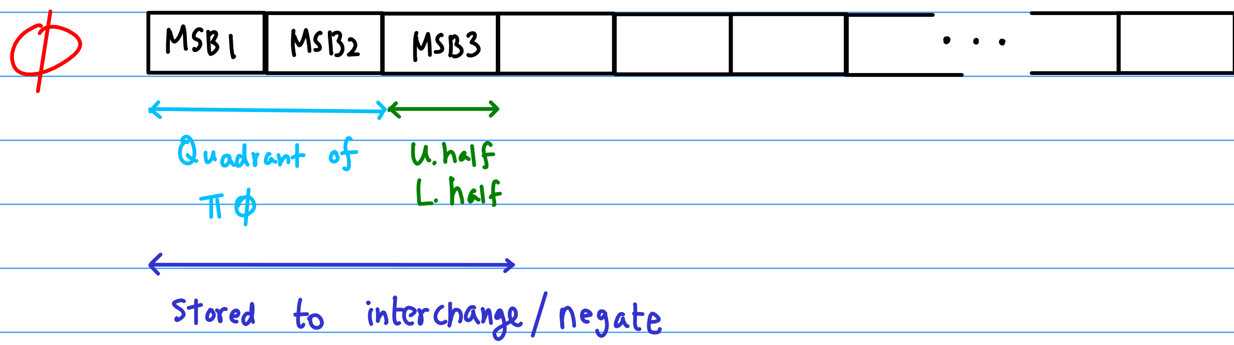
$$\boxed{\begin{matrix}\sin\ \theta\\ \cos\ \theta\end{matrix}} \longrightarrow \boxed{\begin{matrix}\sin \pi\phi\\ \cos \pi\phi\end{matrix}}$$

$$\theta \in [0, \tfrac{\pi}{4}] \longrightarrow \phi \in [-1, +1]$$

$\left\{\begin{array}{l}\text{negation}\\ \text{interchange}\end{array}\right.$

* negation before interchange

normalized angle ϕ



MSB of ϕ	ϕ	X_{inv}	Y_{inv}	Swap	$\cos\pi\phi$	$\sin\pi\phi$
0 0 0	0	0	0	0	$\cos\theta$	$\sin\theta$
0 0 1	1	0	0	1	$\sin\theta$	$\cos\theta$
0 1 0	2	0	1	1	$-\sin\theta$	$\cos\theta$
0 1 1	3	1	0	0	$-\cos\theta$	$\sin\theta$
1 0 0	4	1	1	0	$-\cos\theta$	$-\sin\theta$
1 0 1	5	1	1	1	$-\sin\theta$	$-\cos\theta$
1 1 0	6	1	0	1	$\sin\theta$	$-\cos\theta$
1 1 1	7	0	1	0	$\cos\theta$	$-\sin\theta$