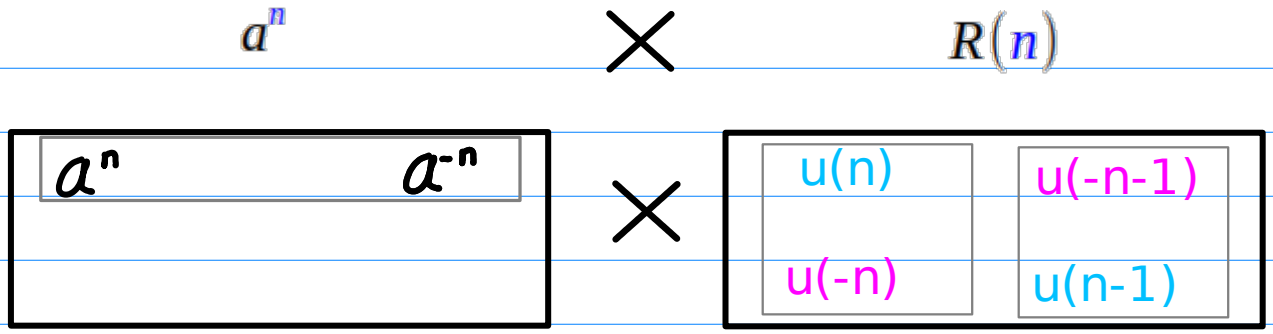
# Laurent Series and z-Transform - Geometric Series Permutations B

20240622 Sat



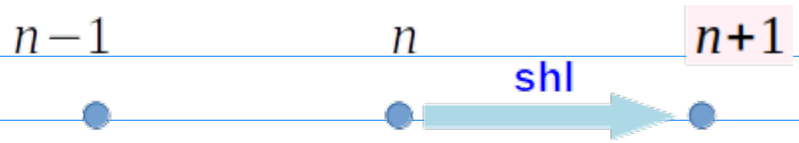

| | | | | |
|---|---|---|---|---|
| (1) | $a^n$ u(n) | $a^{-n}$ u(n) | (2) |
| (3) | $a^n$ u(-n) | $a^{-n}$ u(-n) | (4) |
| (5) | $a^n$ u(-n-1) | $a^{-n}$ u(-n-1) | (6) |
| (7) | $a^n$ u(n-1) | $a^{-n}$ u(n-1) | (8) |

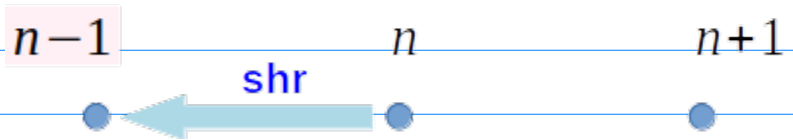
| $b^n$ | $shl(b^n)$ |
|---|---|
| $a^n$ | $a^{(n+1)}$ |
| $a^{-n}$ | $a^{-(n+1)}$ |

| $b^n$ | $shr(b^n)$ |
|---|---|
| $a^n$ | $a^{(n-1)}$ |
| $a^{-n}$ | $a^{-(n-1)}$ |

| $R(n)$ | $shl(R(n))$ |
|---|---|
| $u(n-1)$ | $u(n)$ |
| $u(-n)$ | $u(-(n+1))$ |

| $R(n)$ | $shr(R(n))$ |
|---|---|
| $u(n)$ | $u(n-1)$ |
| $u(-(n+1))$ | $u(-n)$ |

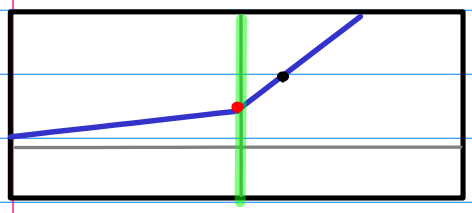


**shl(b^n)**
**shr(b^n)**

**shl(n) = n + 1**
**shr(n) = n - 1**

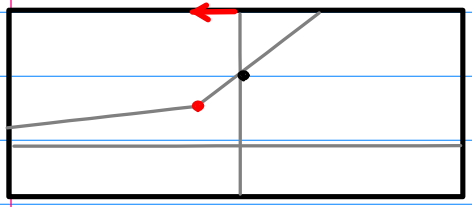
$2^n$



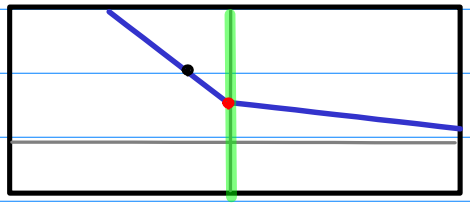
shift left

**n ← n+1**

$2^{n+1}$



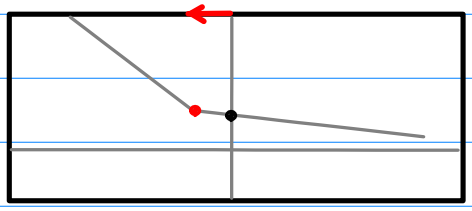
$2^{-n}$



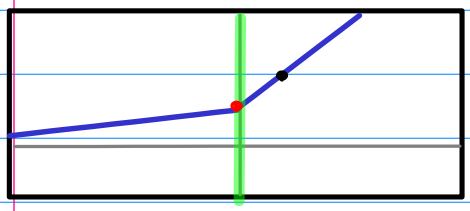
shift left

**n ← n+1**

$2^{-(n+1)}$



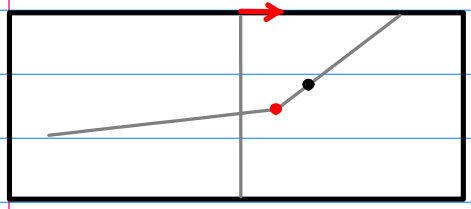
$2^n$



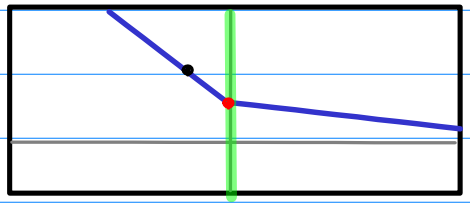
shift right

**n ← n-1**

$2^{n-1}$



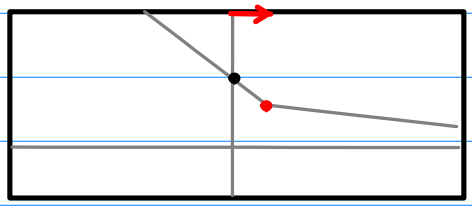
$2^{-n}$



shift right

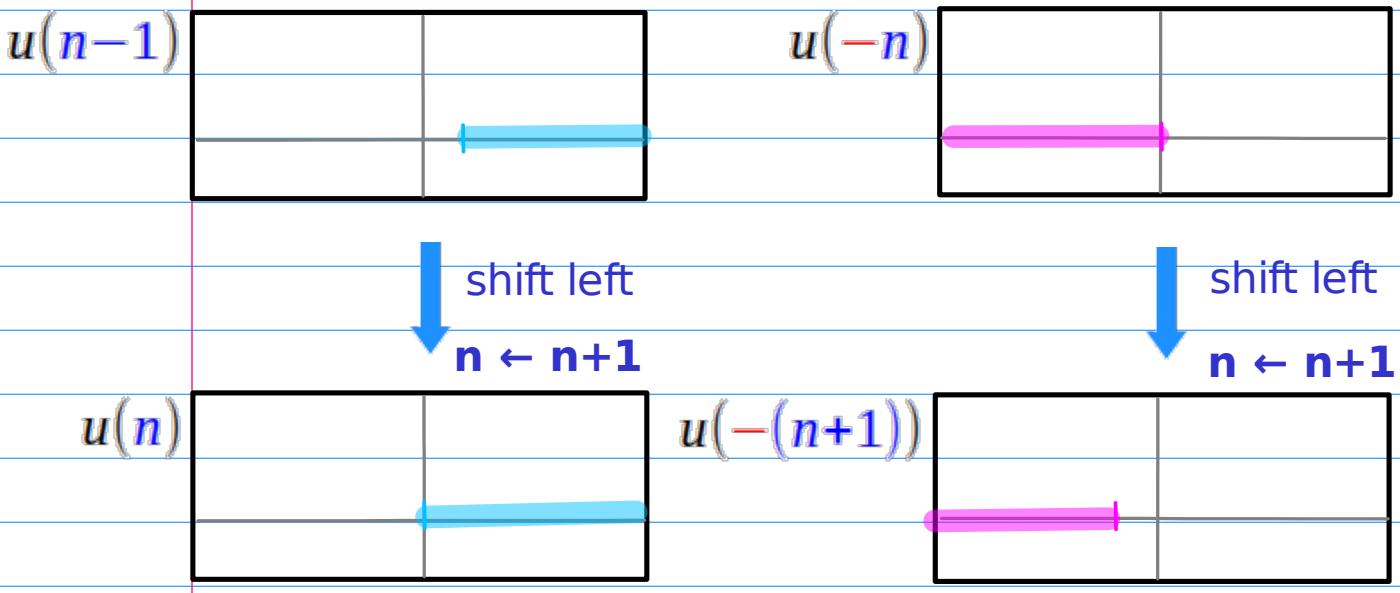
**n ← n-1**

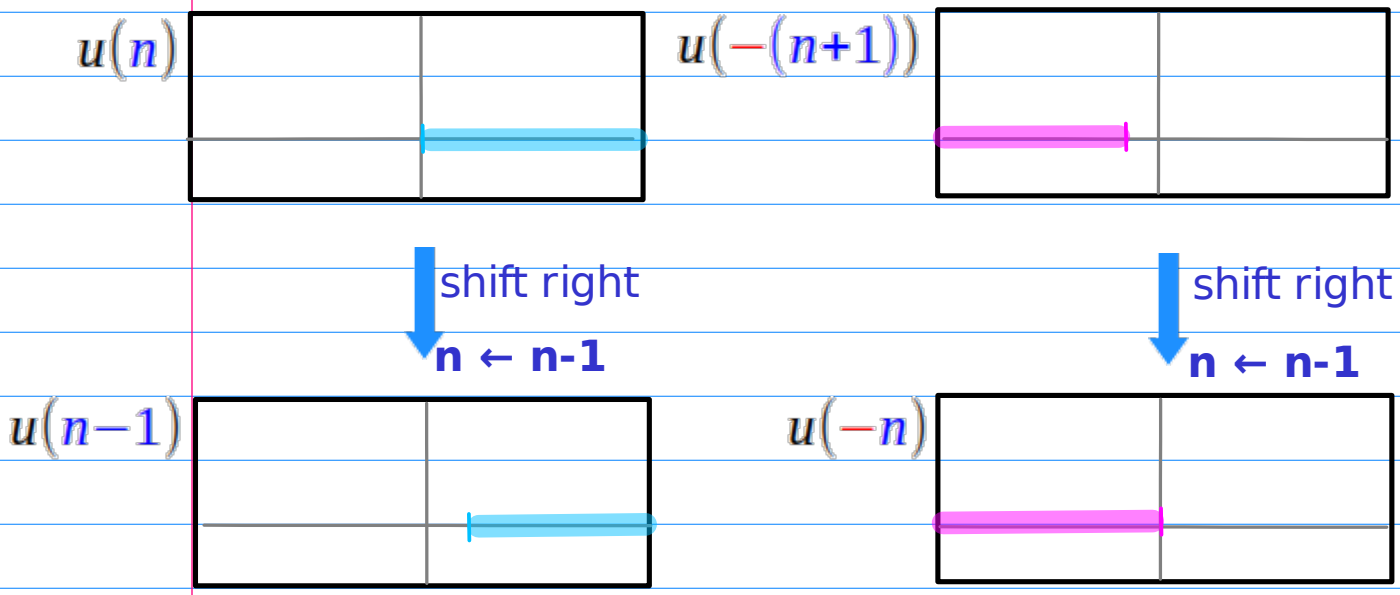
$2^{-(n-1)}$



**shl(R(n))**
**shr(R(n))**

**shl(n) = n + 1**
**shr(n) = n - 1**



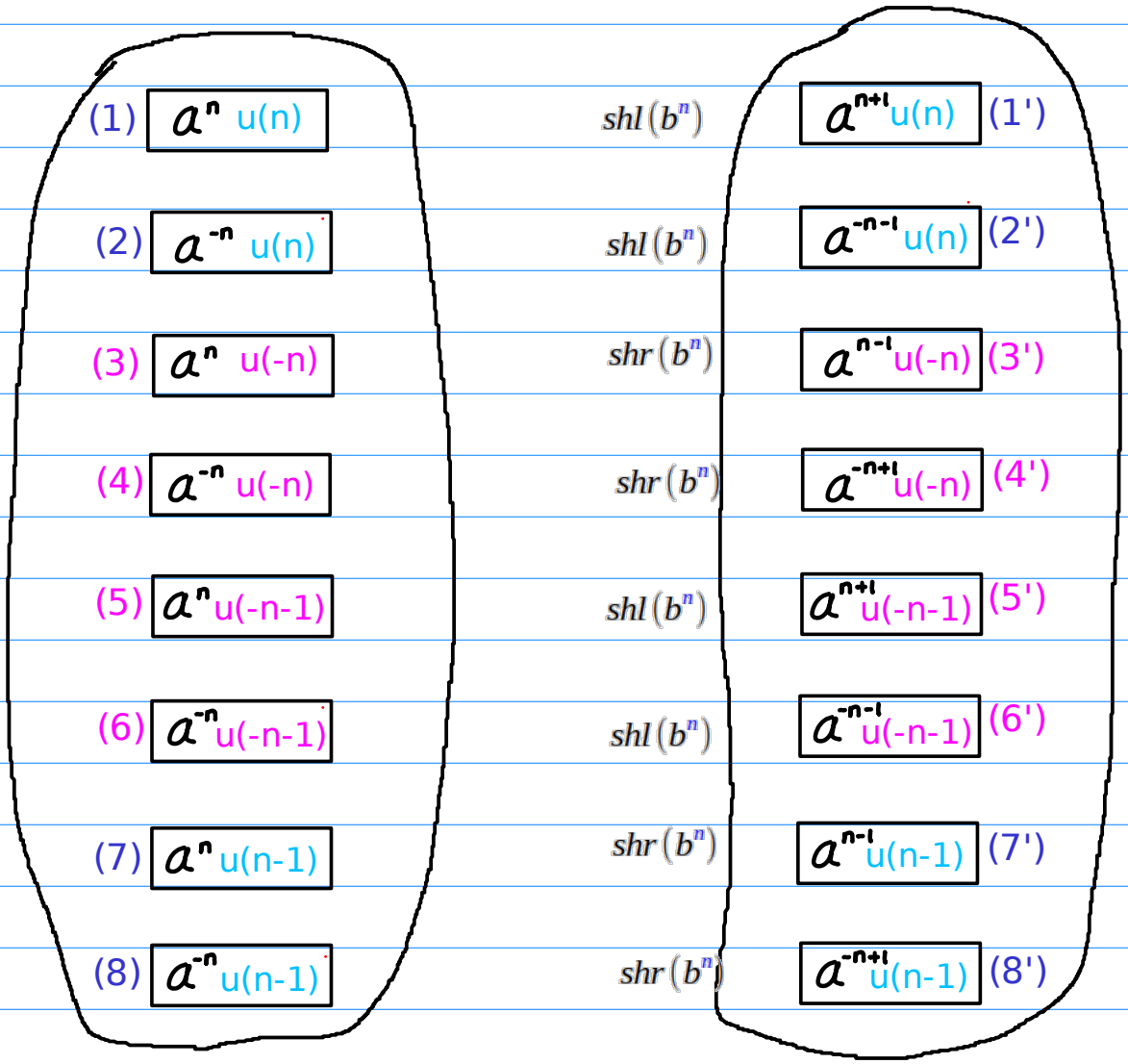



| Unshifted Sequence | | Shifted Sequence 1 | | Shifted Sequence 2 | | |
|---|---|---|---|---|---|---|
| (1) | $a^{n}\ u(n)$ | $shl(b^{n})$ | $a^{n+1}u(n)$ (1') | $shl(b^{-n})$ | $a^{-n-1}u(n)$ (1'') | $n \leftarrow n+1$ |
| (2) | $a^{-n}\ u(n)$ | $shl(b^{n})$ | $a^{-n-1}u(n)$ (2') | $shl(b^{-n})$ | $a^{n+1}u(n)$ (2'') | $n \leftarrow n+1$ |
| (3) | $a^{n}\ u(-n)$ | $shr(b^{n})$ | $a^{n-1}u(-n)$ (3') | $shr(b^{-n})$ | $a^{-n+1}u(-n)$ (3'') | $n \leftarrow n-1$ |
| (4) | $a^{-n}\ u(-n)$ | $shr(b^{n})$ | $a^{-n+1}u(-n)$ (4') | $shr(b^{-n})$ | $a^{n-1}u(-n)$ (4'') | $n \leftarrow n-1$ |
| (5) | $a^{n}u(-n-1)$ | $shl(b^{n})$ | $a^{n+1}u(-n-1)$ (5') | $shl(b^{-n})$ | $a^{-n-1}u(-n-1)$ (5'') | $n \leftarrow n+1$ |
| (6) | $a^{-n}u(-n-1)$ | $shl(b^{n})$ | $a^{-n-1}u(-n-1)$ (6') | $shl(b^{-n})$ | $a^{n+1}u(-n-1)$ (6'') | $n \leftarrow n+1$ |
| (7) | $a^{n}u(n-1)$ | $shr(b^{n})$ | $a^{n-1}u(n-1)$ (7') | $shr(b^{-n})$ | $a^{-n+1}u(n-1)$ (7'') | $n \leftarrow n-1$ |
| (8) | $a^{-n}u(n-1)$ | $shr(b^{n})$ | $a^{-n+1}u(n-1)$ (8') | $shr(b^{-n})$ | $a^{n-1}u(n-1)$ (8'') | $n \leftarrow n-1$ |

many possible permuations are possible
but consider these two

**Unshifted Sequence**

**Shifted Sequence 1**



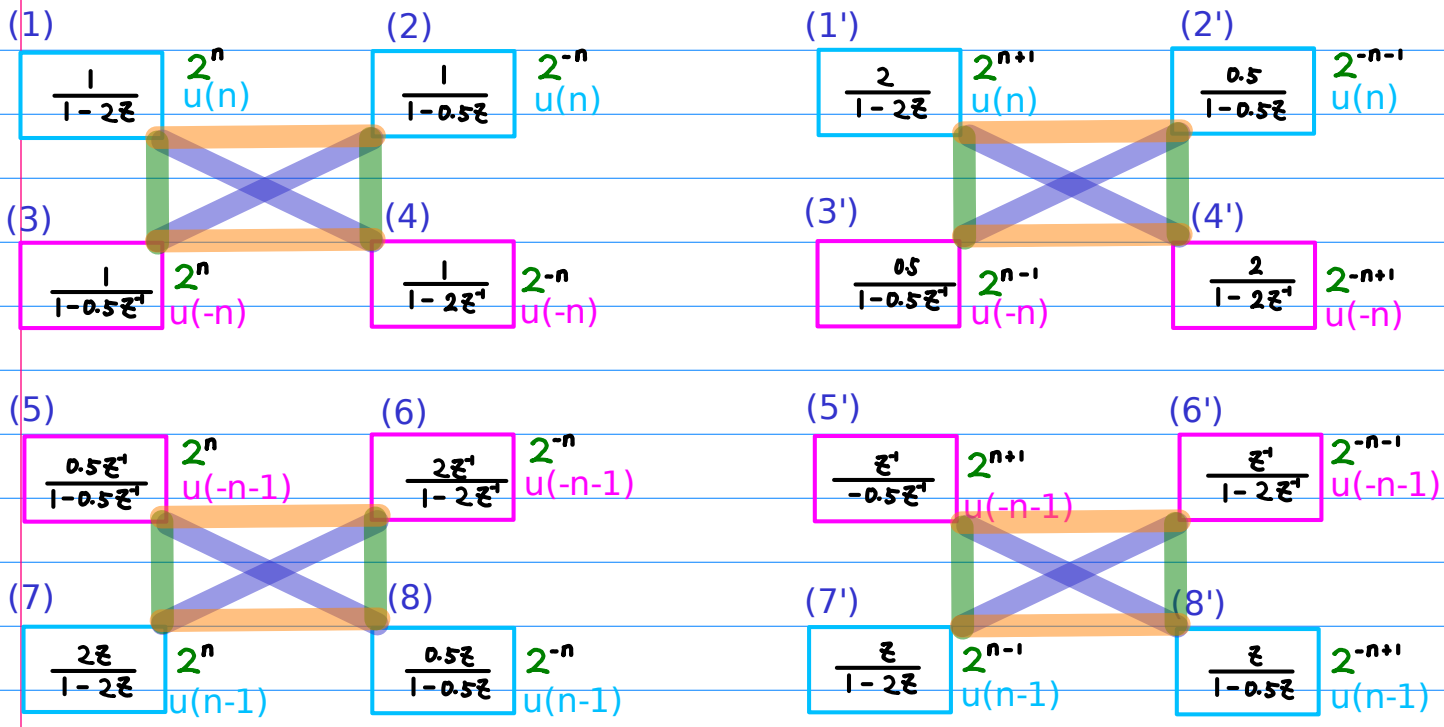

**Inter-permutations over unshifted sequnce and shifted sequence**

**Intra-permutations over unshifted sequence**

**Intra-permutations over shifted sequence**

# Inter-permutation (x) --> (x')

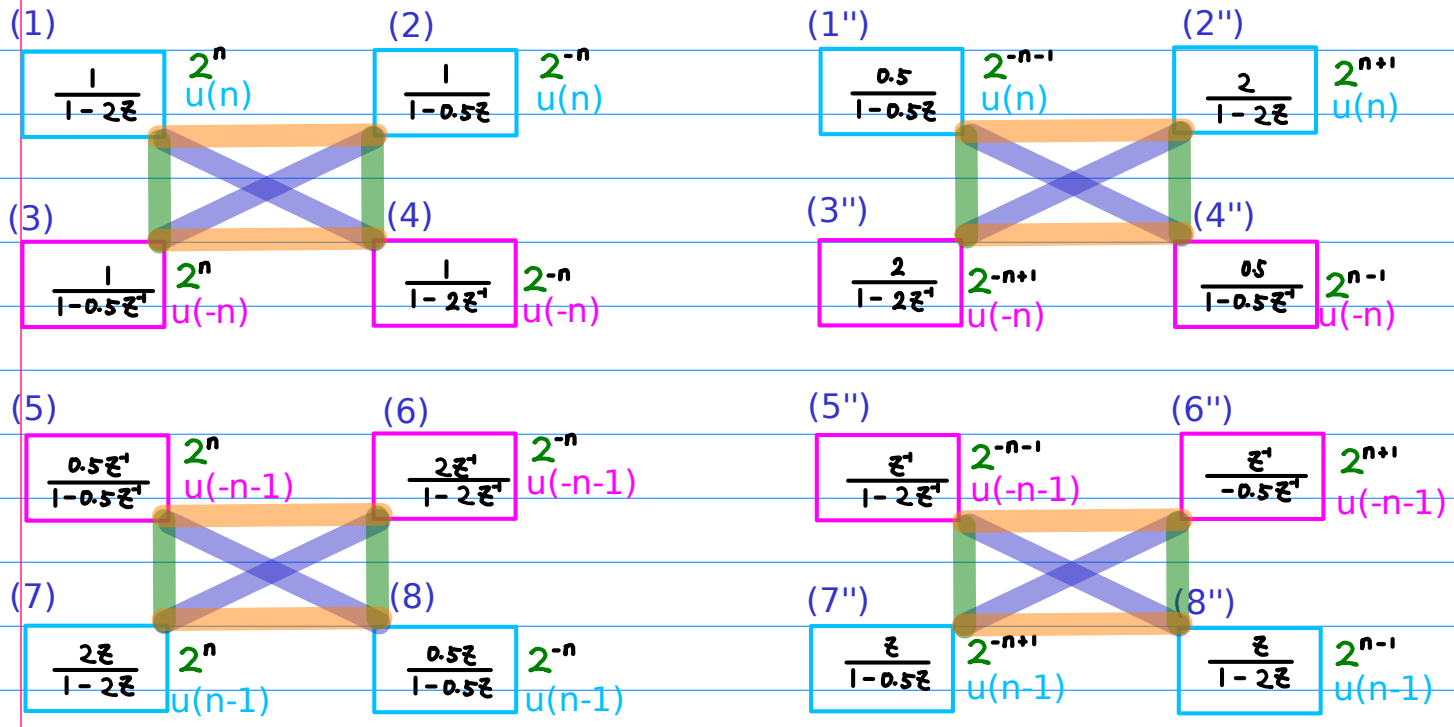
(1)~(8) --> (1')~(8')




| | | | | |
|---|---|---|---|---|
| (1) | $a^{n}$ u(n) | $shl(b^{n})$ | $a^{n+1}$u(n) | (1') |
| (2) | $a^{-n}$ u(n) | $shl(b^{n})$ | $a^{-n-1}$u(n) | (2') |
| (3) | $a^{n}$ u(-n) | $shr(b^{n})$ | $a^{n-1}$u(-n) | (3') |
| (4) | $a^{-n}$ u(-n) | $shr(b^{n})$ | $a^{-n+1}$u(-n) | (4') |
| (5) | $a^{n}$u(-n-1) | $shl(b^{n})$ | $a^{n+1}$u(-n-1) | (5') |
| (6) | $a^{-n}$u(-n-1) | $shl(b^{n})$ | $a^{-n-1}$u(-n-1) | (6') |
| (7) | $a^{n}$u(n-1) | $shr(b^{n})$ | $a^{n-1}$u(n-1) | (7') |
| (8) | $a^{-n}$u(n-1) | $shr(b^{n})$ | $a^{-n+1}$u(n-1) | (8') |

# Inter-permutation (x) --> (x'')

(1)~(8) --> (1'')~(8'')




| | | | | |
|---|---|---|---|---|
| (1) | $a^n$ u(n) | $shl(b^n)$ | $a^{-n-1}$ u(n) | (1'') |
| (2) | $a^{-n}$ u(n) | $shl(b^n)$ | $a^{n+1}$ u(n) | (2'') |
| (3) | $a^n$ u(-n) | $shr(b^n)$ | $a^{-n+1}$ u(-n) | (3'') |
| (4) | $a^{-n}$ u(-n) | $shr(b^n)$ | $a^{n-1}$ u(-n) | (4'') |
| (5) | $a^n$ u(-n-1) | $shl(b^n)$ | $a^{-n-1}$ u(-n-1) | (5'') |
| (6) | $a^{-n}$ u(-n-1) | $shl(b^n)$ | $a^{n+1}$ u(-n-1) | (6'') |
| (7) | $a^n$ u(n-1) | $shr(b^n)$ | $a^{-n+1}$ u(n-1) | (7'') |
| (8) | $a^{-n}$ u(n-1) | $shr(b^n)$ | $a^{n-1}$ u(n-1) | (8'') |

$a^n$ × $R(n)$

| | |
|---|---|
| $a^{n+1}$ | $a^{-n-1}$ |
| $a^{n-1}$ | $a^{-n+1}$ |

×

| | |
|---|---|
| u(n) | u(-n-1) |
| u(-n) | u(n-1) |

| | | | |
|---|---|---|---|
| (1') | $a^{n+1}u(n)$ | $a^{-n-1}u(n)$ | (2') |
| (3') | $a^{n-1}u(-n)$ | $a^{-n+1}u(-n)$ | (4') |
| (5') | $a^{n+1}u(-n-1)$ | $a^{-n-1}u(-n-1)$ | (6') |
| (7') | $a^{n-1}u(n-1)$ | $a^{-n+1}u(n-1)$ | (8') |

| | | | |
|---|---|---|---|
| (1'') | $a^{n-1}u(n)$ | $a^{-n+1}u(n)$ | (2'') |
| (3'') | $a^{n+1}u(-n)$ | $a^{-n-1}u(-n)$ | (4'') |
| (5'') | $a^{n-1}u(-n-1)$ | $a^{-n+1}u(-n-1)$ | (6'') |
| (7'') | $a^{n+1}u(n-1)$ | $a^{-n-1}u(n-1)$ | (8'') |

$a^n$ × $R(n)$

| $a^{n+1}$ | $a^{-n-1}$ |
|---|---|

×

| u(n) | u(-n-1) |
|---|---|
| u(-n) | u(n-1) |

| | | | |
|---|---|---|---|
| (1') | $a^{n+1}u(n)$ | $a^{-n-1}u(n)$ | (2') |
| (3') | $a^{n-1}u(-n)$ | $a^{-n+1}u(-n)$ | (4') |
| (5') | $a^{n+1}u(-n-1)$ | $a^{-n-1}u(-n-1)$ | (6') |
| (7') | $a^{n-1}u(n-1)$ | $a^{-n+1}u(n-1)$ | (8') |

$a^n$ × $R(n)$

| | |
|---|---|
| $a^{n+1}$ | $a^{-n-1}$ |
| $a^{n-1}$ | $a^{-n+1}$ |

×

| | |
|---|---|
| u(n) | u(-n-1) |
| u(-n) | u(n-1) |

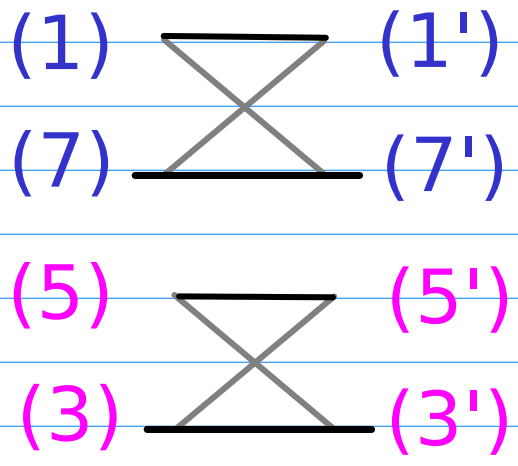
| | | | |
|---|---|---|---|
| (1'') | $a^{n-1}$ u(n) | $a^{-n+1}$ u(n) | (2'') |
| (3'') | $a^{n+1}$ u(-n) | $a^{-n-1}$ u(-n) | (4'') |
| (5'') | $a^{n-1}$ u(-n-1) | $a^{-n+1}$ u(-n-1) | (6'') |
| (7'') | $a^{n+1}$ u(n-1) | $a^{-n-1}$ u(n-1) | (8'') |

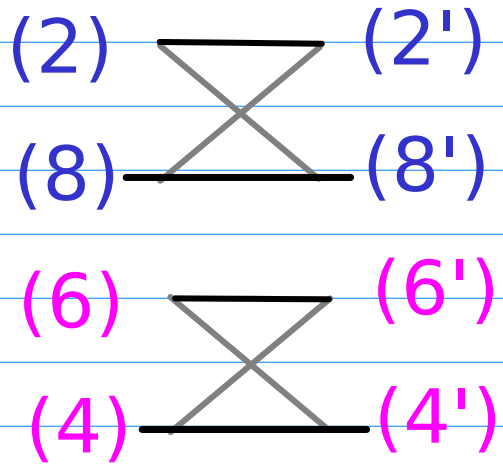
(1) - (1')
(7) - (7')
(5) - (5')
(3) - (3')

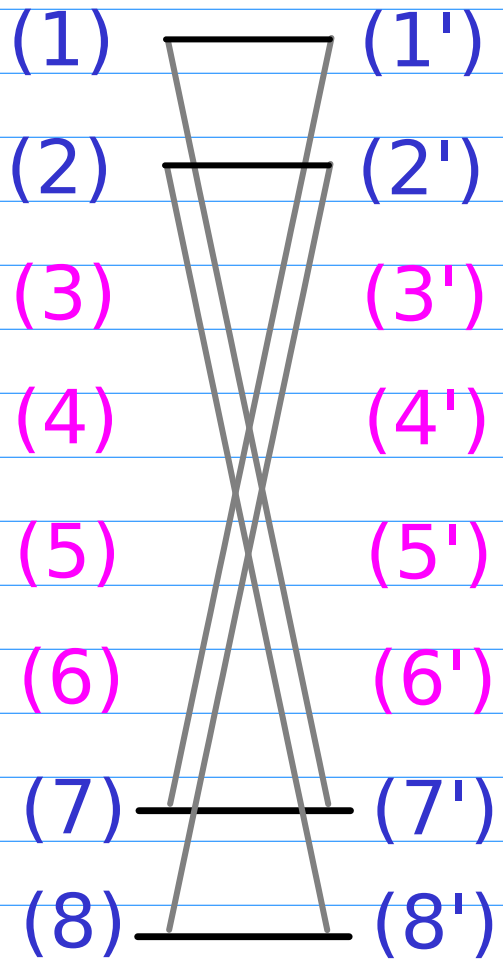
(2) - (2')
(8) - (8')
(6) - (6')
(4) - (4')

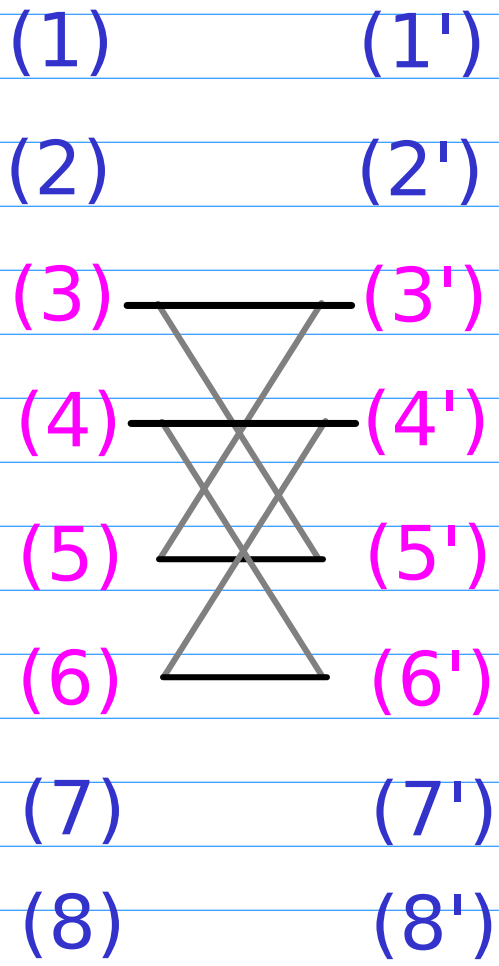
(1) - (1')
(2) - (2')
(3) - (3')
(4) - (4')

(5) - (5')
(6) - (6')
(7) - (7')
(8) - (8')



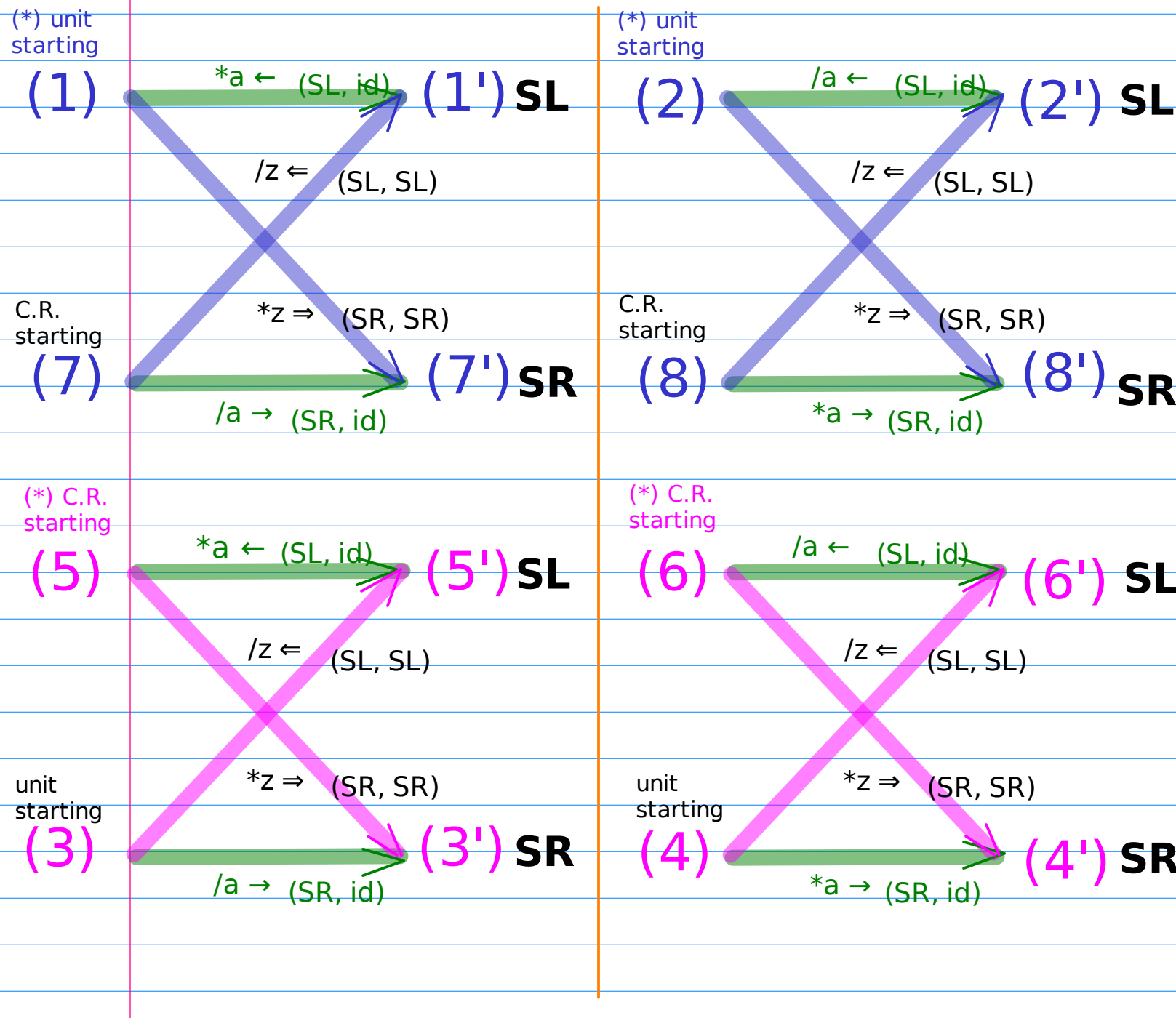



# Butterfly Relations

**(x) --> (x')**

(1)~(8) --> (1')~(8')

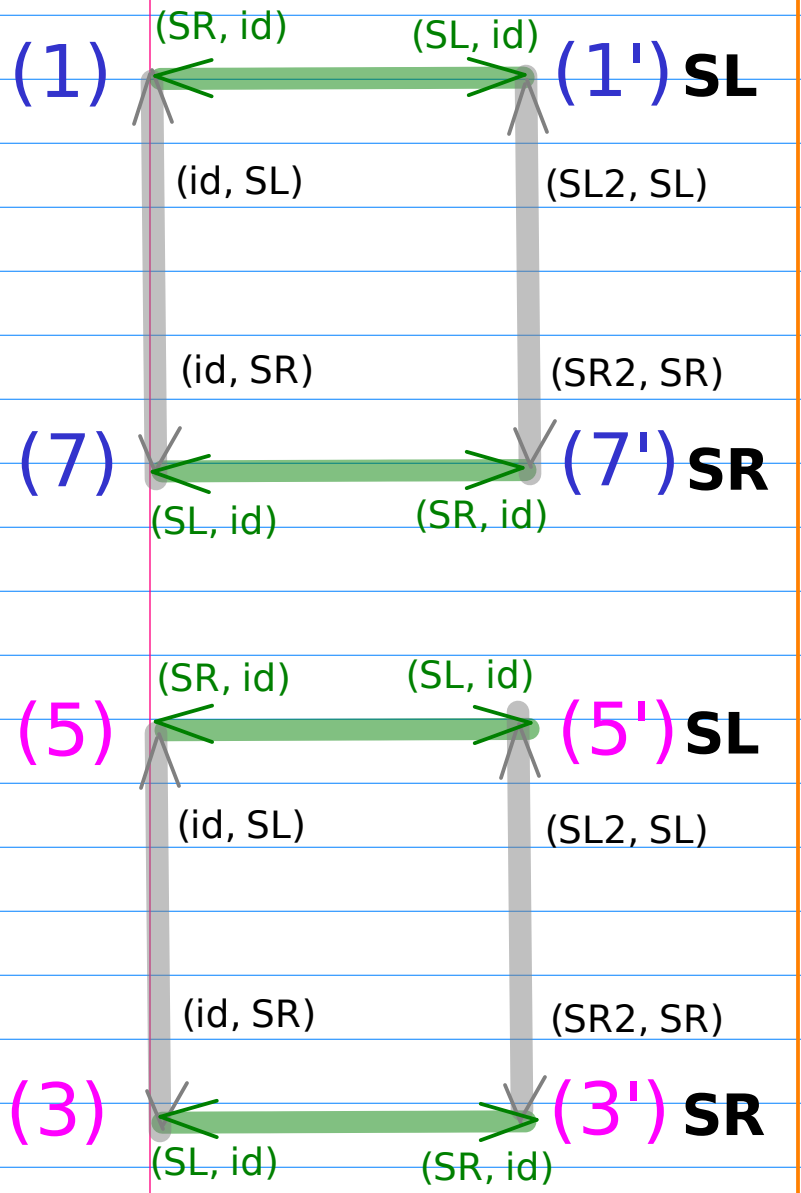


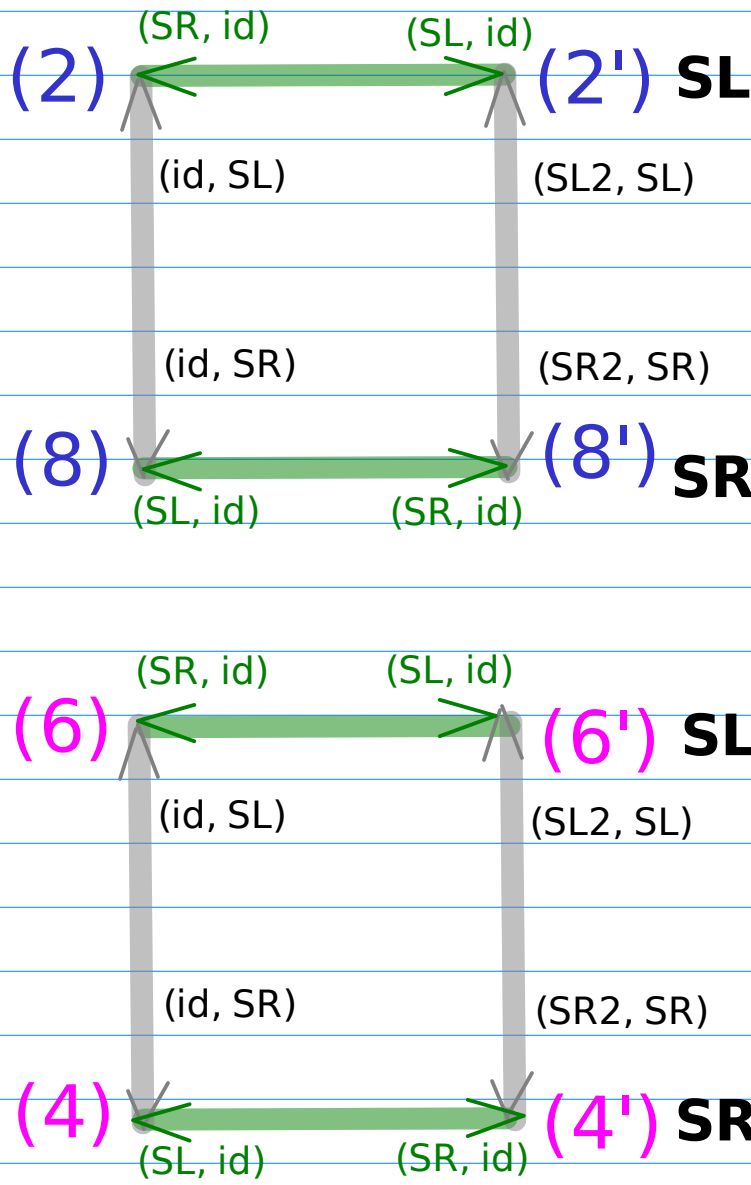

**(Exp Shift, Range Shift)**

# Butterfly Relations

**(x) --> (x')**

(1)~(8) --> (1')~(8')





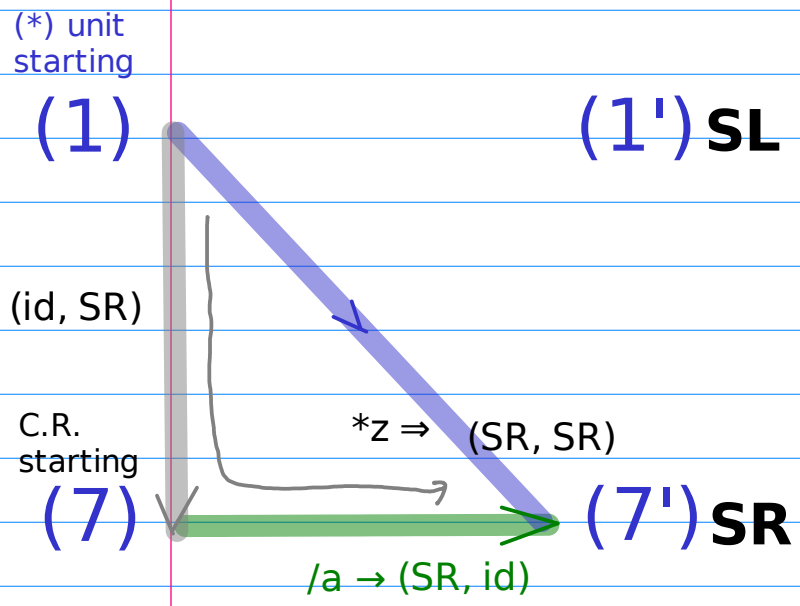


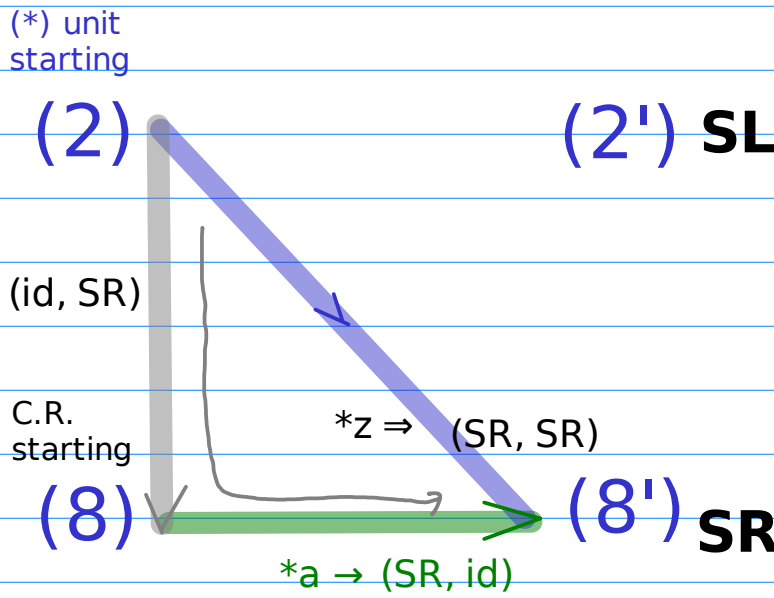


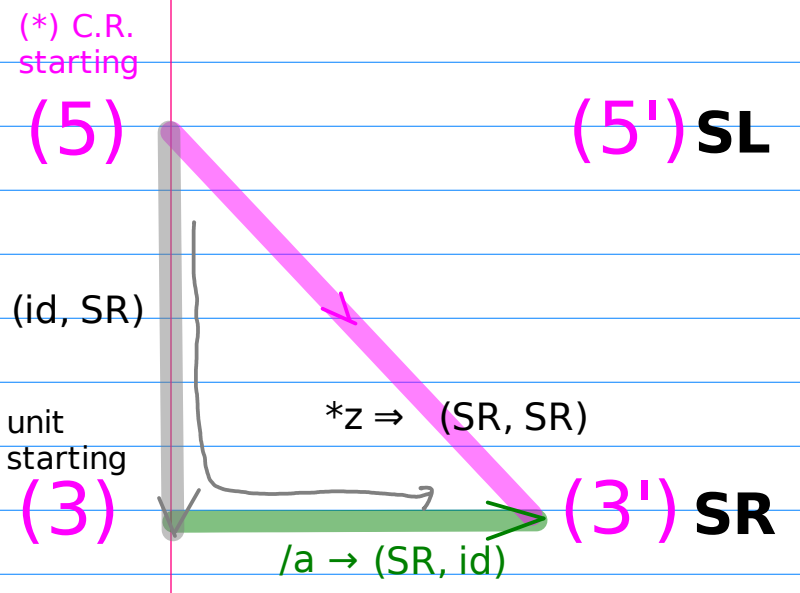

**(Exp Shift, Range Shift)**

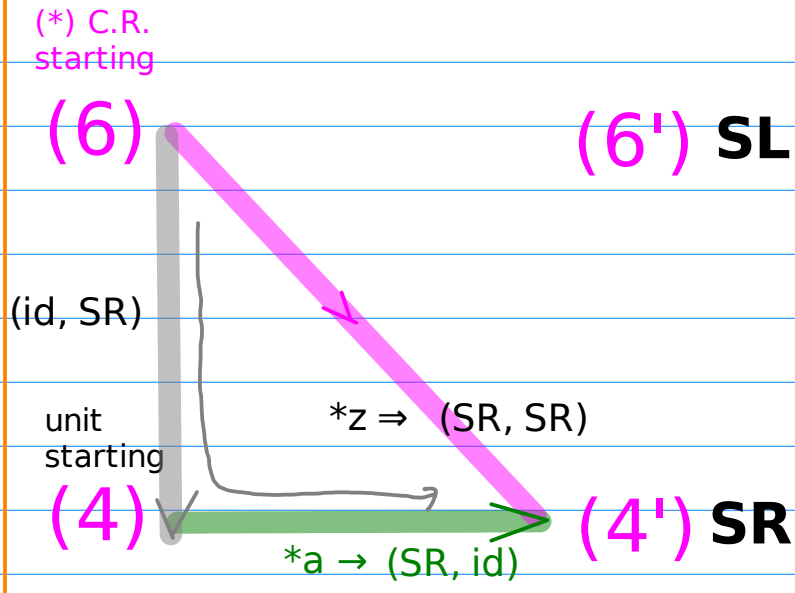
# Decomposition of Exp and Rng Shifts (1)

**(id, SR) + (SR, id) = (SR, SR)**



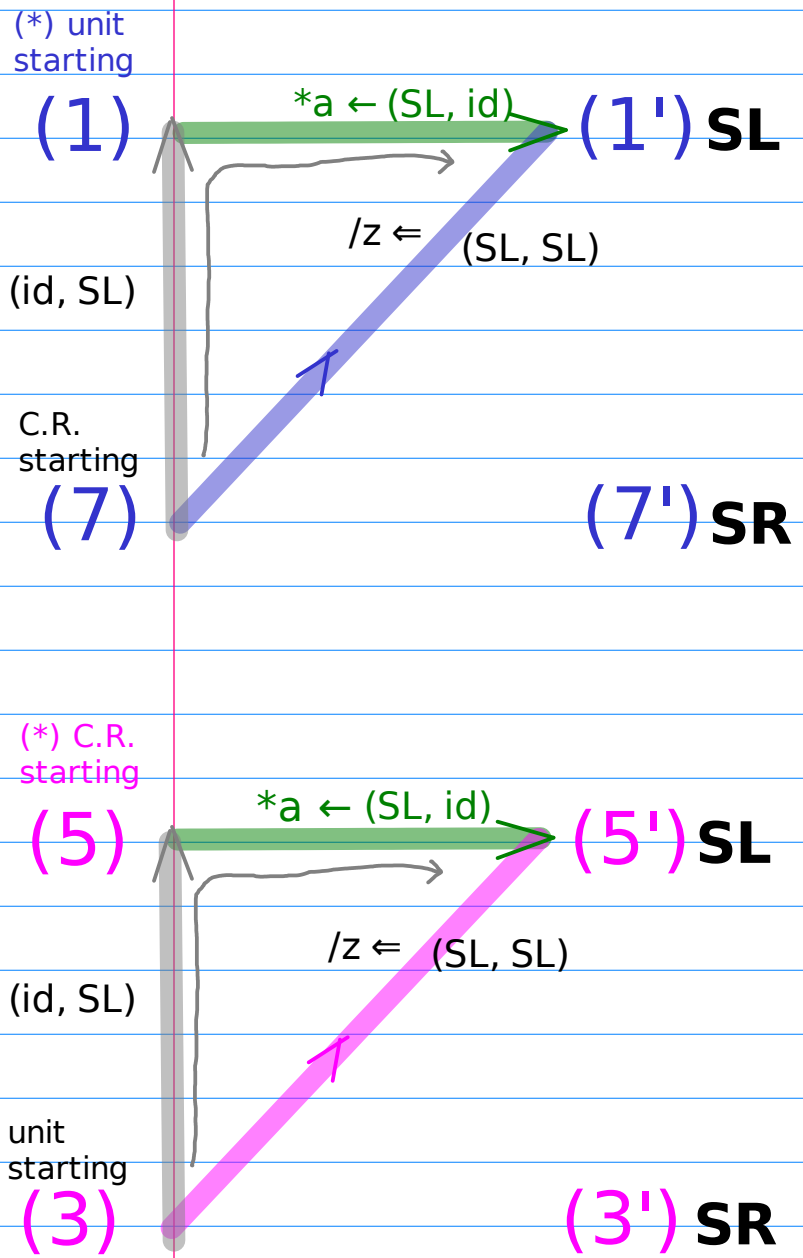



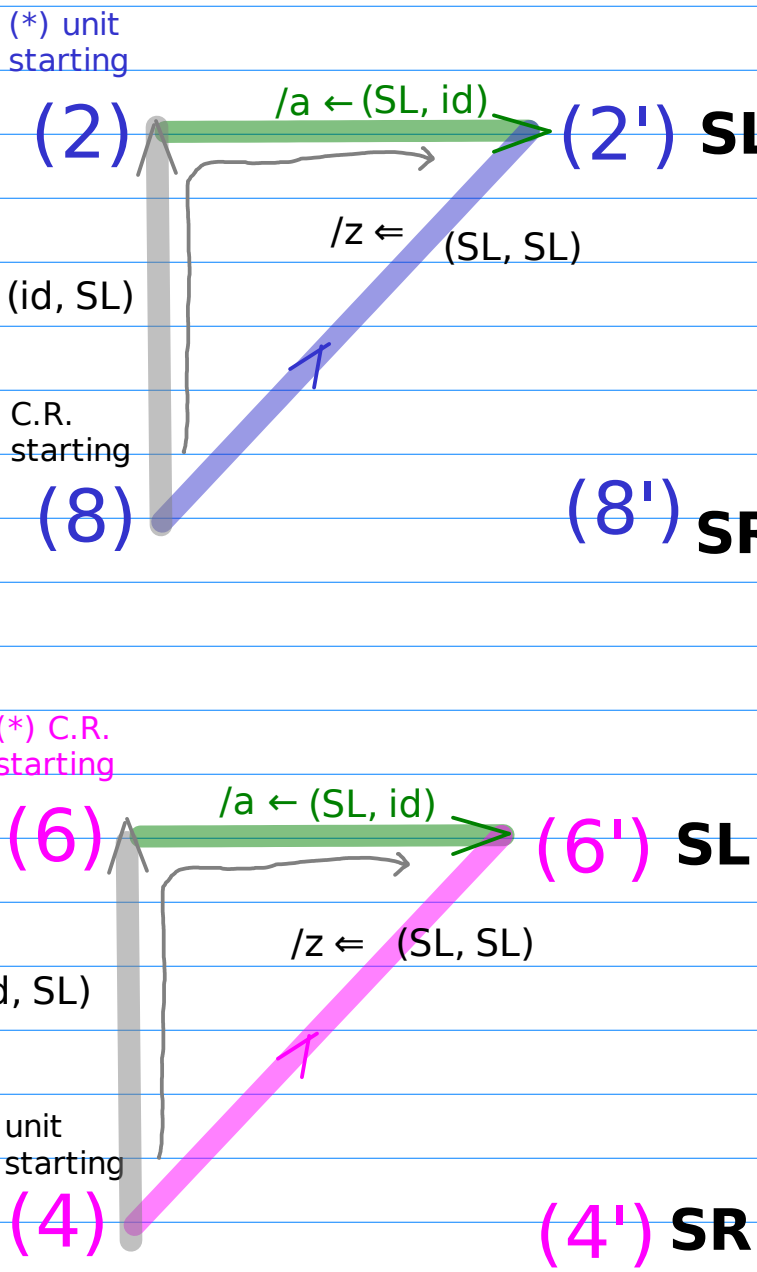



# Decomposition of Exp and Rng Shifts (2)

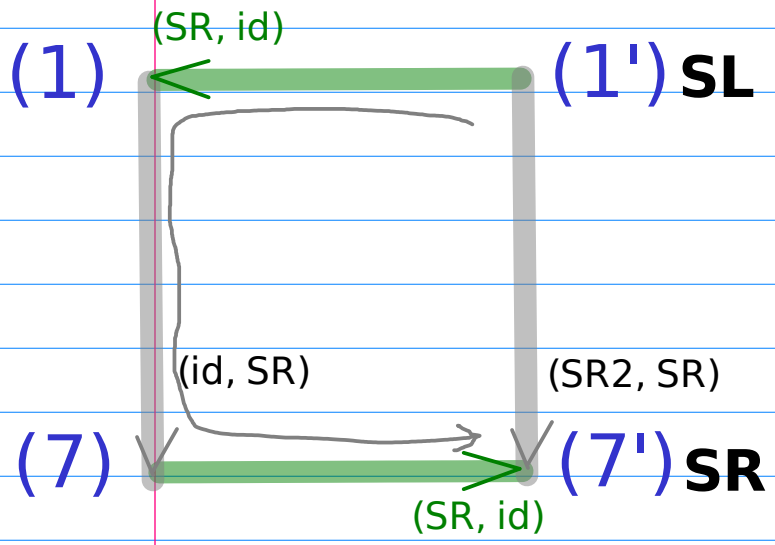
**(id, SL) + (SL, id) = (SL, SL)**

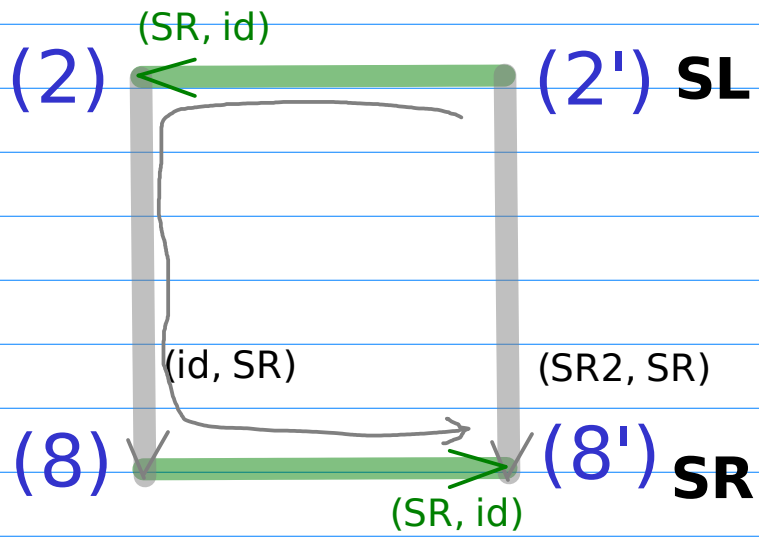


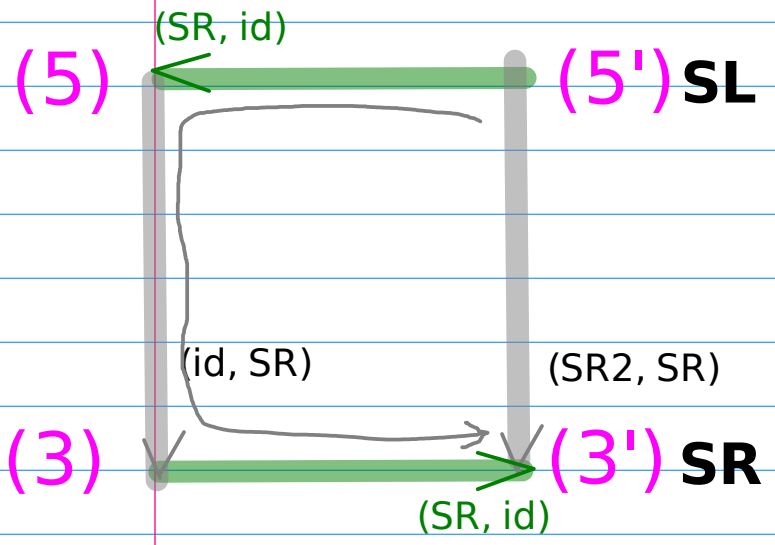



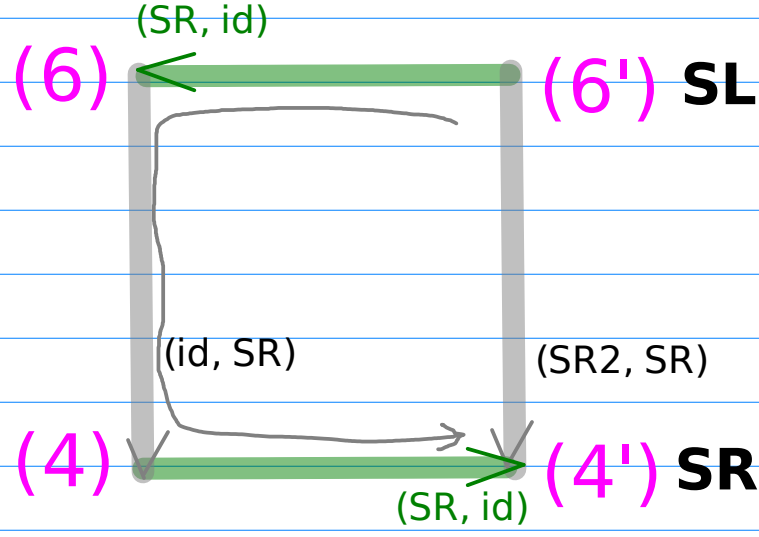
# Decomposition of Exp and Rng Shifts (3)

**(SR, id) + (id, SR) + (SR, id) = (SR2, SR)**



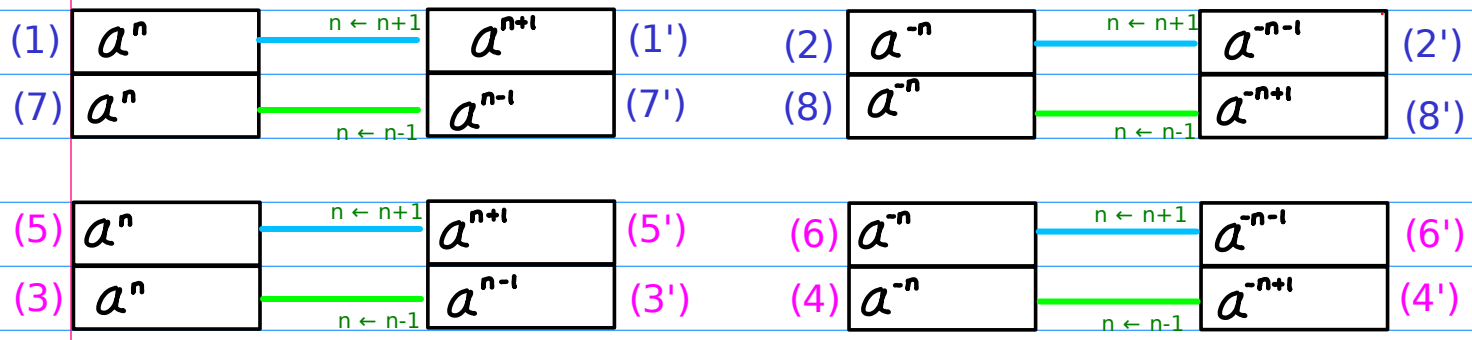



**(id, SR) + (SR, id) = (SR, SR)**
**(id, SL) + (SL, id) = (SL, SL)**

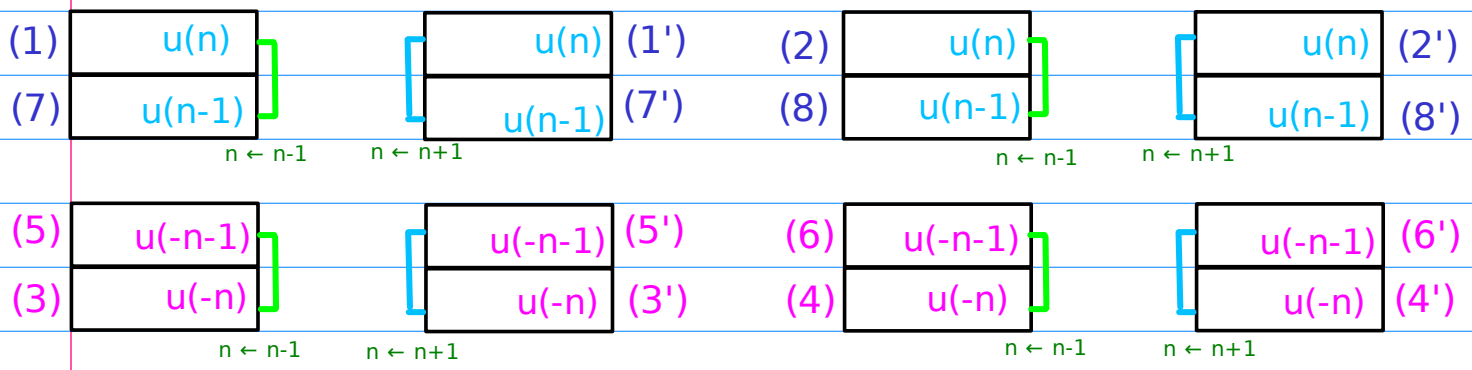
**(x) --> (x')**

(1)~(8) --> (1')~(8')

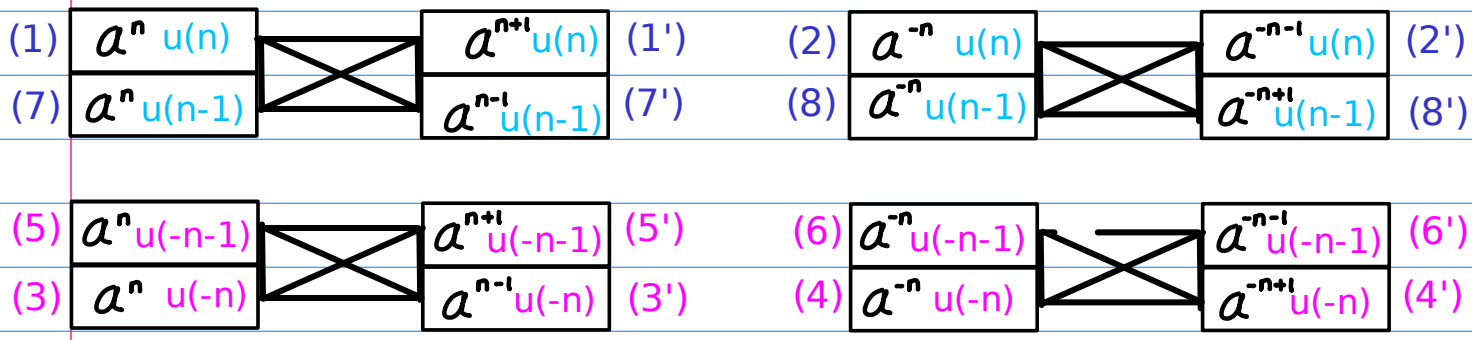
## Exponent Shifts




## Range Shifts




## Exponent & Range Permutations

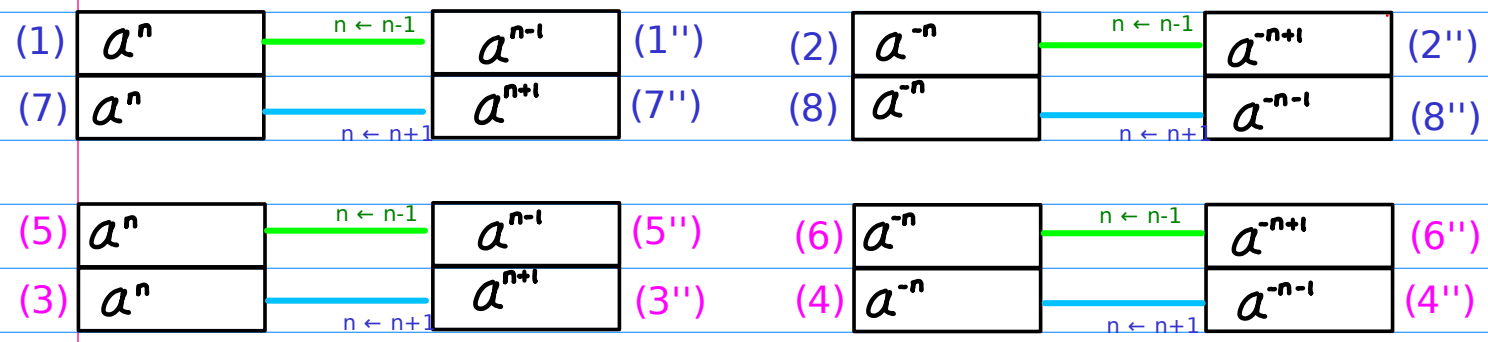


**(id, SR) + (SL, id) = (SL, SR)**
**(id, SL) + (SR, id) = (SR, SL)**

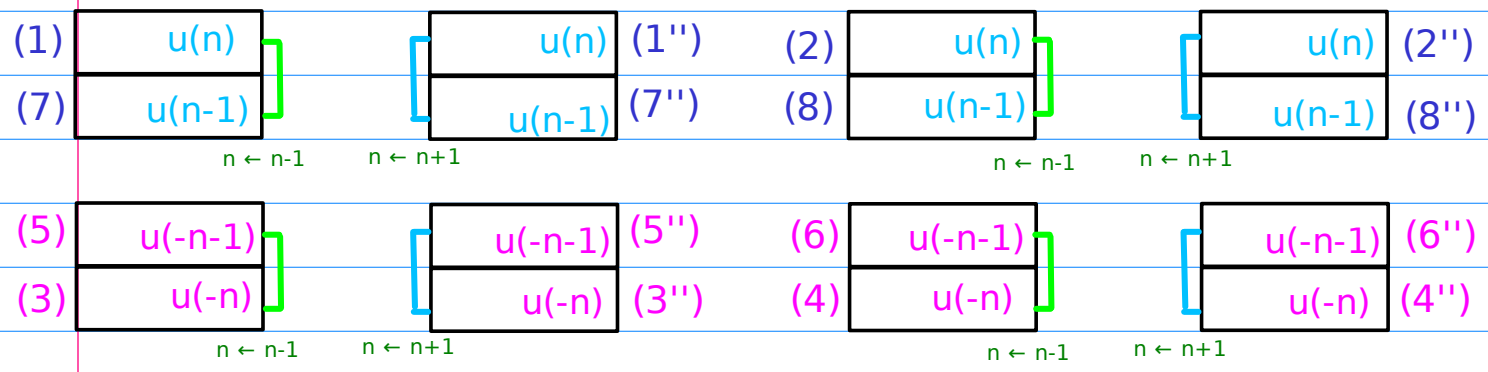
**(x) --> (x'')**

(1)~(8) --> (1'')~(8'')

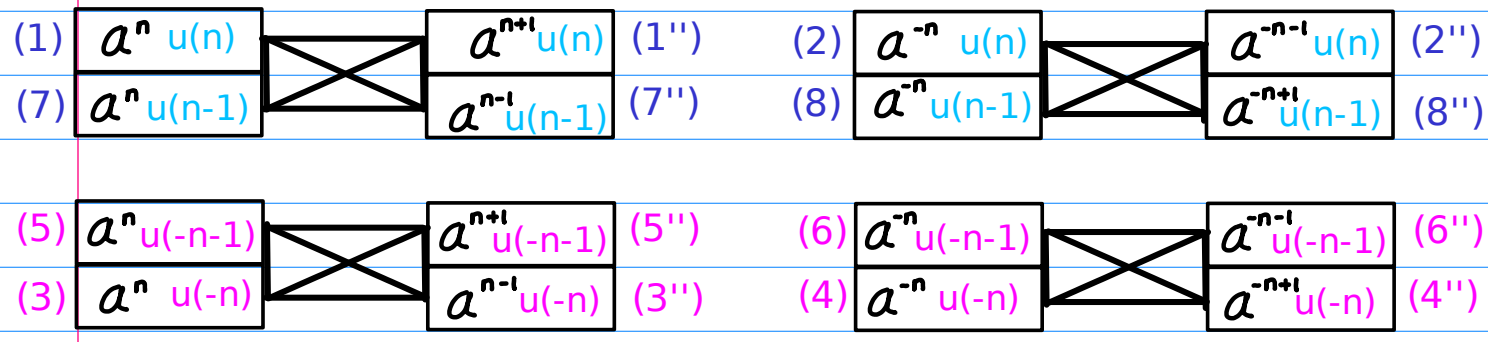
## Exponent Shifts




## Range Shifts



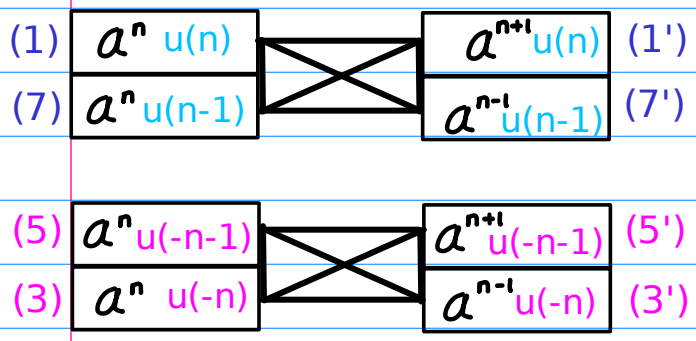

## Exponent & Range Permutations

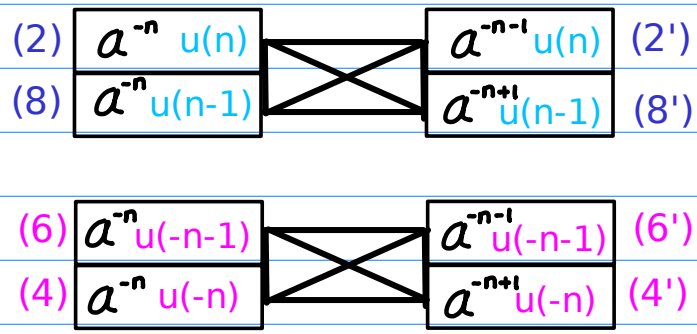


**(id, SR) + (SR, id) = (SR, SR)**
**(id, SL) + (SL, id) = (SL, SL)**

**(x) --> (x')**

(1)~(8) --> (1')~(8')





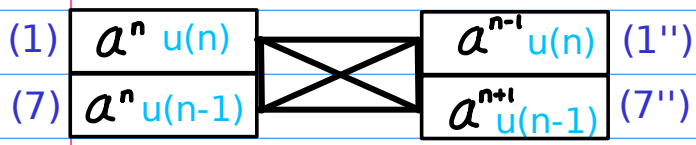


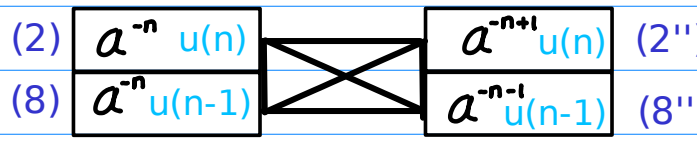


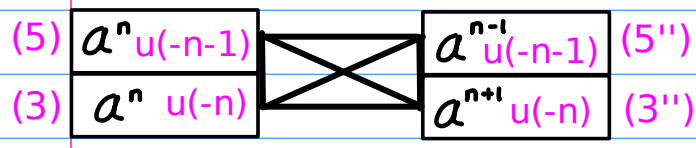

**(id, SR) + (SL, id) = (SL, SR)**
**(id, SL) + (SR, id) = (SR, SL)**

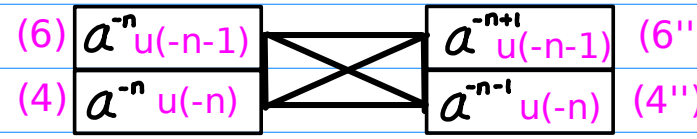
**(x) --> (x'')**

(1)~(8) --> (1'')~(8'')

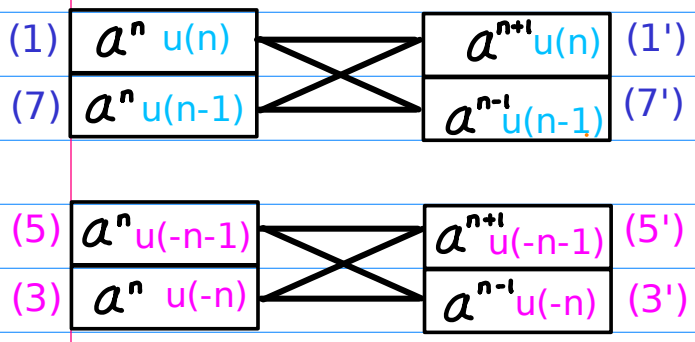


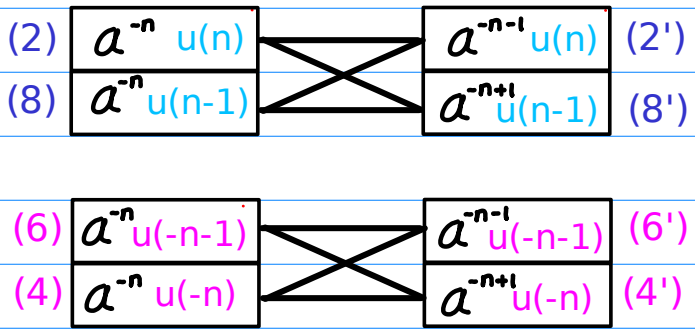



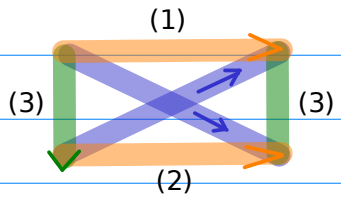



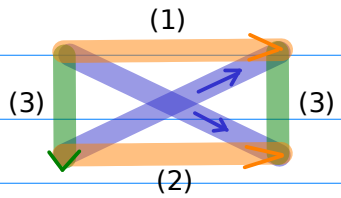



# Exponent Permutations




**(1) shift left exponent**

**(2) shift right exponent**

**(3) shift right range**

**(1) shift left exponent**

**(2) shift right exponent**

**(3) shift right range**

(SR, id) shift right exponent

(id, SR) shift right range

(SR, SR)

(SR, id) shift right exponent

(id, SR) shift right range

(SR, SR)

(SL, id) shift left exponent

(id, SL) shift left range

(SL, SL)

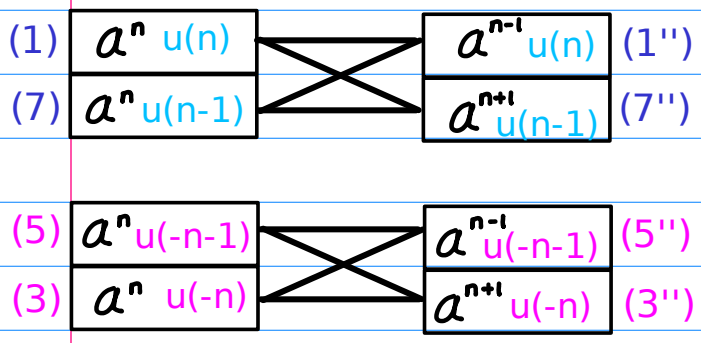
(SL, id) shift left exponent

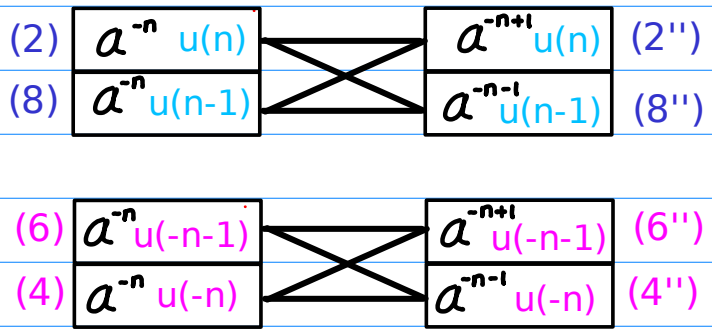
(id, SL) shift left range

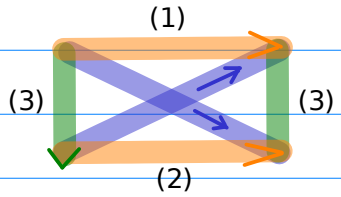
(SL, SL)

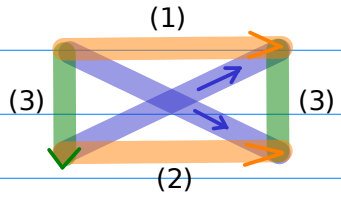
**(SR, id) + (id, SR) = (SR, SR)**
**(SL, id) + (id, SL) = (SL, SL)**

**(SR, id) + (id, SR) = (SR**
**(SL, id) + (id, SL) = (SL,**




**(1) shift right exponent**

**(2) shift left exponent**

**(3) shift right range**

**(1) shift right exponent**

**(2) shift left exponent**

**(3) shift right range**

(SL, id) shift right exponent

(id, SR) shift right range

(SL SR)

(SL, id) shift right exponent

(id, SR) shift right range

(SL, SR)

(SR, id) shift left exponent

(id, SL) shift left range

(SR, SL)

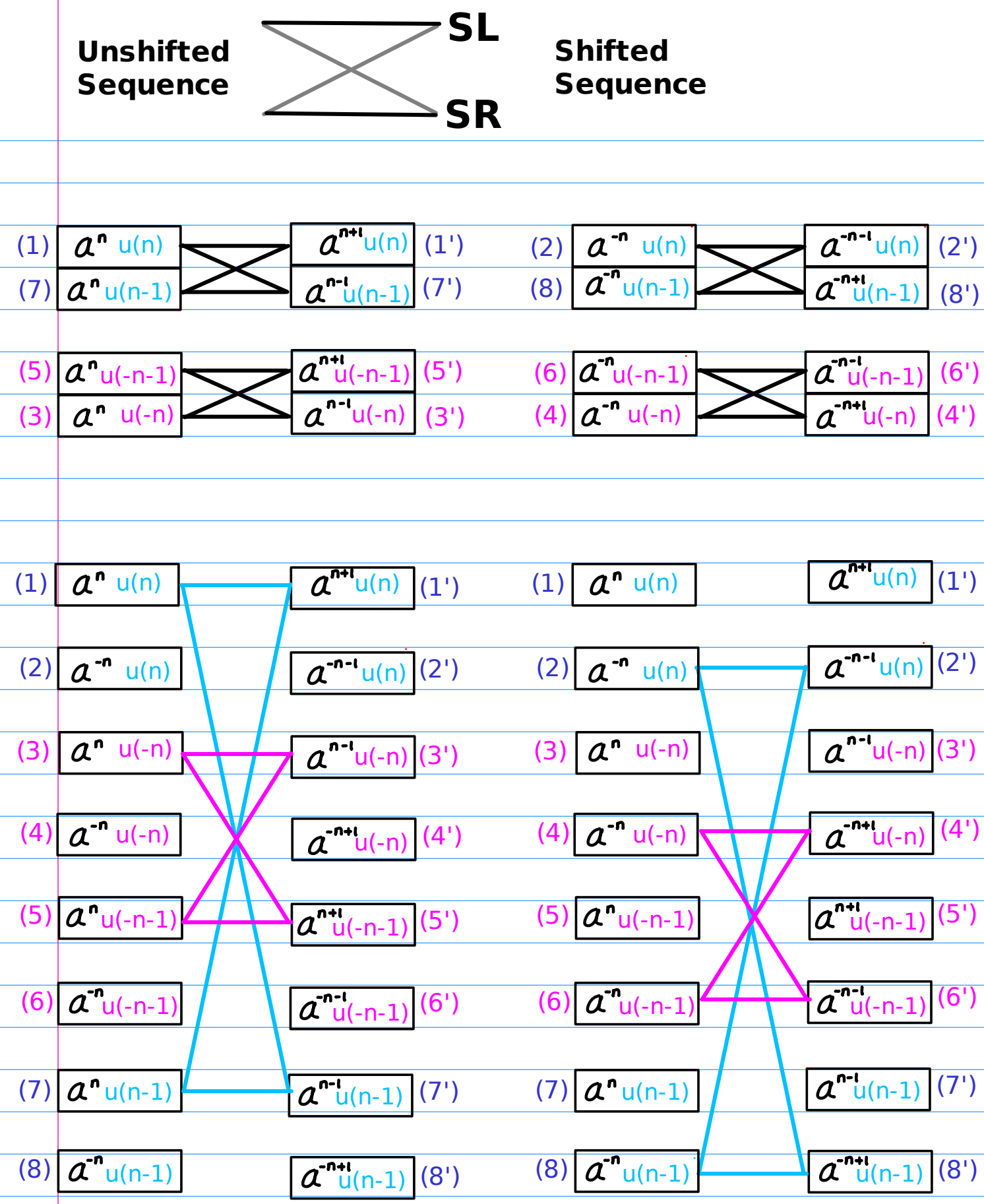
(SR, id) shift left exponent

(id, SL) shift left range

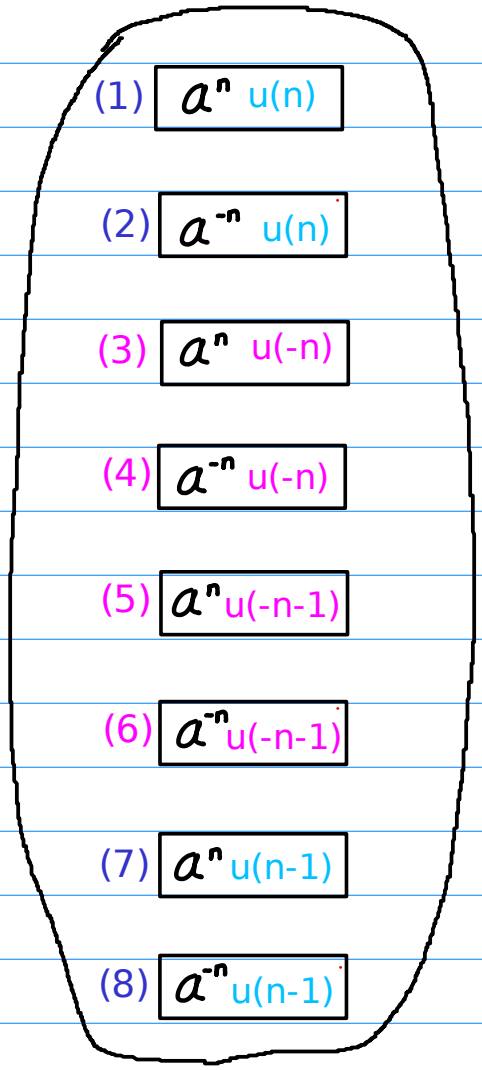
(SR, SL)

**(SL, id) + (id, SR) = (SL, SR)**
**(SR, id) + (id, SL) = (SR, SL)**

**(SL, id) + (id, SR) = (SL**
**(SR, id) + (id, SL) = (SL**


Unshifted Sequence
SL
SR
Shifted Sequence
(1) $a^{n}$ u(n)
(7) $a^{n}$ u(n-1)
$a^{n+l}$u(n) (1')
$a^{n-l}$u(n-1) (7')
(5) $a^{n}$u(-n-1)
(3) $a^{n}$ u(-n)
$a^{n+l}$u(-n-1) (5')
$a^{n-l}$u(-n) (3')
(2) $a^{-n}$ u(n)
(8) $a^{-n}$u(n-1)
$a^{-n-l}$u(n) (2')
$a^{-n+l}$u(n-1) (8')
(6) $a^{-n}$u(-n-1)
(4) $a^{-n}$ u(-n)
$a^{-n-l}$u(-n-1) (6')
$a^{-n+l}$u(-n) (4')
(1) $a^{n}$ u(n)
(2) $a^{-n}$ u(n)
(3) $a^{n}$ u(-n)
(4) $a^{-n}$ u(-n)
(5) $a^{n}$u(-n-1)
(6) $a^{-n}$u(-n-1)
(7) $a^{n}$u(n-1)
(8) $a^{-n}$u(n-1)
$a^{n+l}$u(n) (1')
$a^{-n-l}$u(n) (2')
$a^{n-l}$u(-n) (3')
$a^{-n+l}$u(-n) (4')
$a^{n+l}$u(-n-1) (5')
$a^{-n-l}$u(-n-1) (6')
$a^{n-l}$u(n-1) (7')
$a^{-n+l}$u(n-1) (8')
(1) $a^{n}$ u(n)
(2) $a^{-n}$ u(n)
(3) $a^{n}$ u(-n)
(4) $a^{-n}$ u(-n)
(5) $a^{n}$u(-n-1)
(6) $a^{-n}$u(-n-1)
(7) $a^{n}$u(n-1)
(8) $a^{-n}$u(n-1)
$a^{n+l}$u(n) (1')
$a^{-n-l}$u(n) (2')
$a^{n-l}$u(-n) (3')
$a^{-n+l}$u(-n) (4')
$a^{n+l}$u(-n-1) (5')
$a^{-n-l}$u(-n-1) (6')
$a^{n-l}$u(n-1) (7')
$a^{-n+l}$u(n-1) (8')

# Intra-Permuations over (1) ~ (8)




$a^{n+1}$u(n) (1')

$a^{-n-1}$u(n) (2')

$a^{n-1}$u(-n) (3')

$a^{-n+1}$u(-n) (4')

$a^{n+1}$u(-n-1) (5')

$a^{-n-1}$u(-n-1) (6')

$a^{n-1}$u(n-1) (7')

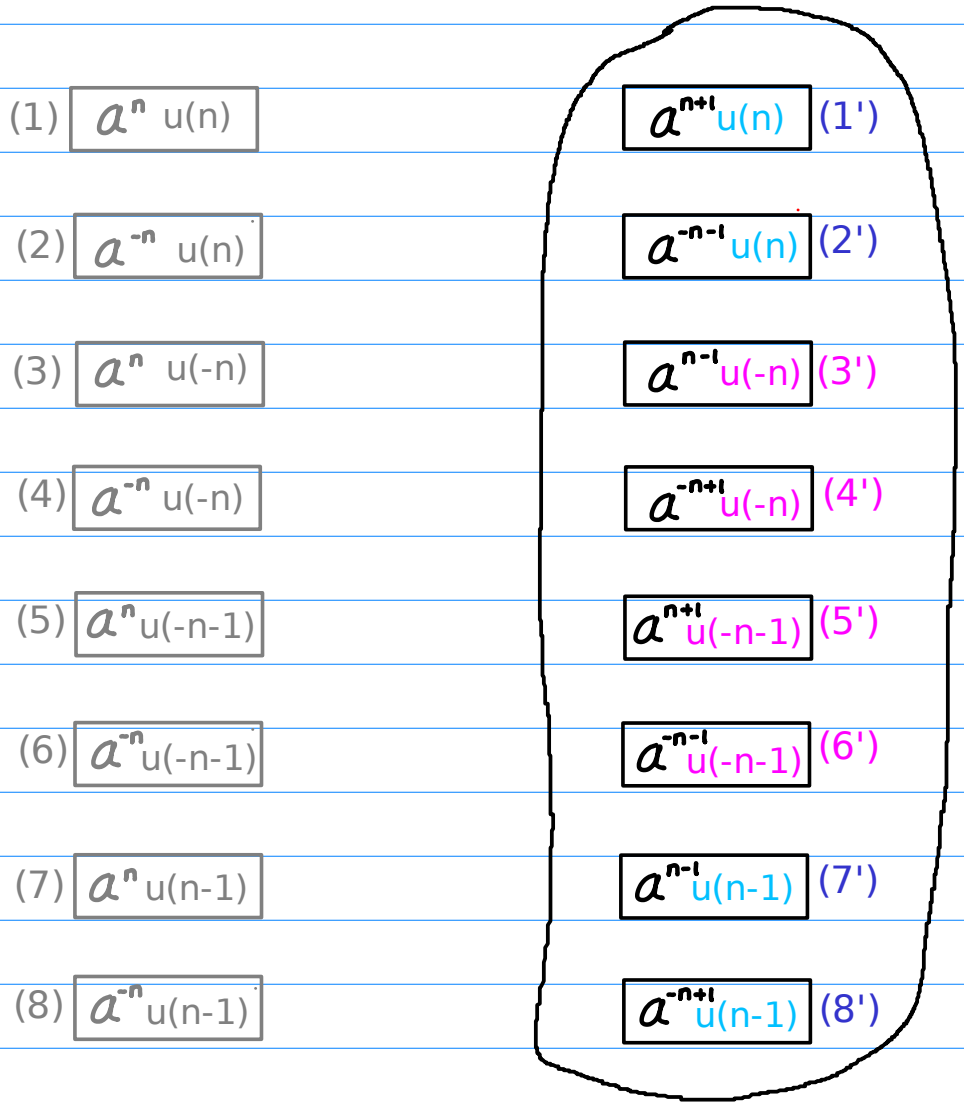
$a^{-n+1}$u(n-1) (8')

**A. Flipping**
- **Base Inverting**
- **Range Flipping**

**B. Range Shifting = Range Flipping + Range Complementing**
- **Range Flipping**
- **Range Complementing**

**C. Complementary Inverting**
- **Base Inverting**
- **Range Complementing**

# Intra-Permuations over (1') ~ (8')




**D. Flipping2**

**Base Inverting**

**Shifted Range Flipping = Exponent Shifting2 + Range Flipppi**

**E. Shifting2 = Exponent Shifting2 + Range Shifting**

**Shifted Range Flipping = Exponent Shifting2 + Range Flipping**

**Range Complementing**

**F. Complementary Inverting**

**Base Inverting**

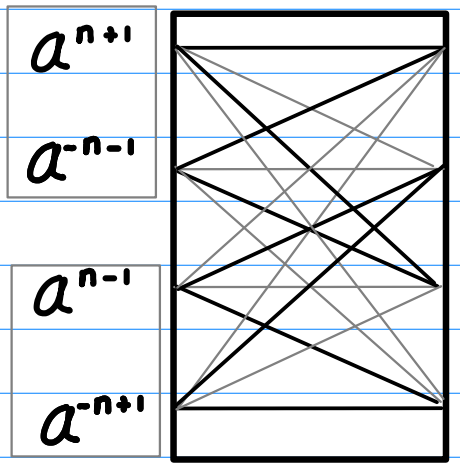
**Range Complementing**

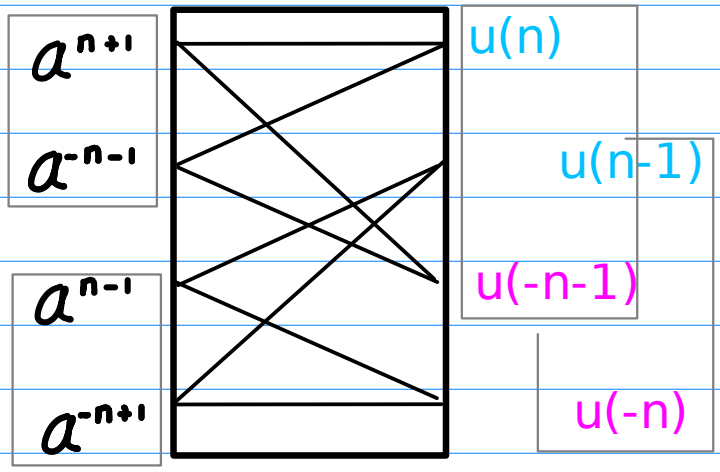
$a^n$ × $R(n)$

| $a^{n+1}$ | $a^{-n-1}$ |
|---|---|
| $a^{n-1}$ | $a^{-n+1}$ |

×

| u(n) | u(-n-1) |
|---|---|
| u(n-1) | u(-n) |




n ← n+2 or
n ← n - 2

| | | | |
|---|---|---|---|
| (1') $a^{n+1}u(n)$ | | $a^{-n-1}u(n)$ | (2') |
| (3') $a^{n-1}u(-n)$ | | $a^{-n+1}u(-n)$ | (4') |
| (5') $a^{n+1}u(-n-1)$ | | $a^{-n-1}u(-n-1)$ | (6') |
| (7') $a^{n-1}u(n-1)$ | | $a^{-n+1}u(n-1)$ | (8') |

| | | | |
|---|---|---|---|
| (1'') $a^{n-1}u(n)$ | | $a^{-n+1}u(n)$ | (2'') |
| (3'') $a^{n+1}u(-n)$ | | $a^{-n-1}u(-n)$ | (4'') |
| (5'') $a^{n-1}u(-n-1)$ | | $a^{-n+1}u(-n-1)$ | (6'') |
| (7'') $a^{n+1}u(n-1)$ | | $a^{-n-1}u(n-1)$ | (8'') |

## permutation over (1) ~ (8)

A. Flipping
    Base Inverting
    Range Flipping

B. Range Shifting = Range Flipping + Range Complementing
    Range Flipping
    Range Complementing

C. Complementary Inverting
    Base Inverting
    Range Complementing

## permutation over (1') ~ (8')

D. Flipping2
    Base Inverting
    Shifted Range Flipping = Exponent Shifting2 + Range Flippping

E. Shifting2 = Exponent Shifting2 + Range Shifting
    Shifted Range Flipping = Exponent Shifting2 + Range Flipping
    Range Complementing

F. Complementary Inverting
    Base Inverting
    Range Complementing

Range Shifting = Range Flipping + Range Complementing
Shifted Range Flipping = Exponent Shifting2 + Range Flippping

Shifting2 = Shifted Range Flipping + Range Complementing
    = Exponent Shifting2 + Range (Flippping+Complementing)
    = Exponent Shfiting2 + Range Shifting

## permutation over (1) ~ (8)

**A. Flipping**
- **Base Inverting**
- **Range Flipping**

(1) - (4) (5) - (8)
(2) - (3) (6) - (7)
(3) - (2) (7) - (6)
(4) - (1) (8) - (5)

**B. Range Shifting**
- **Range Flipping**
- **Range Complementing**

(1) - (7) (5) - (3)
(2) - (8) (6) - (4)
(3) - (5) (7) - (1)
(4) - (6) (8) - (2)

**C. Complementary Inverting**
- **Base Inverting**
- **Range Complementing**

(1) - (6) (5) - (2)
(6) - (1) (6) - (1)
(2) - (5) (7) - (4)
(5) - (2) (8) - (3)

## permutation over (1') ~ (8')

**D. Flipping2**
- **Base Inverting**
- **Shifted Range Flipping**

(1') - (4') (5') - (8')
(2') - (3') (6') - (7')
(3') - (2') (7') - (6')
(4') - (1') (8') - (5')

**E. Shifting2**
- **Shifted Range Flipping**
- **Range Complementing**

(1') - (7') (5') - (3')
(2') - (8') (6') - (4')
(3') - (5') (7') - (1')
(4') - (6') (8') - (2')

**F. Complementary Inverting**
- **Base Inverting**
- **Range Complementing**

(1') - (6') (5') - (2')
(6') - (1') (6') - (1')
(2') - (5') (7') - (4')
(5') - (2') (8') - (3')

## permutation over (1) ~ (8)

| | A | B | C |
|---|---|---|---|
| Base Inverting | x | | x |
| Range Flipping | x | x | |
| Range Complementing | | x | x |

## permutation over (1') ~ (8')

| | D | E | F |
|---|---|---|---|
| Base Inverting | x | | x |
| Shifted Range Flipping | x | x | |
| Range Complementing | | x | x |

Range Shifting = Range Flipping + Range Complementing
Shifted Range Flipping = Exponent Shifting2 + Range Flippping

Shifting2 = Shifted Range Flipping + Range Complementing
= Exponent Shifting2 + Range (Flippping+Complementing)
= Exponent Shfiting2 + Range Shifting

**Over (1) ~ (8)**

**Base Inverting** $a^n \Longleftrightarrow a^{-n}$

**Range Flipping** $R(n) \Longleftrightarrow R(-n)$

**Range Complementing** $R(n) \Longleftrightarrow \overline{R(n)}$

**Over (1') ~ (8')**

**Base Inverting** $a^n \Longleftrightarrow a^{-n}$

**Shifted Range Flipping** $a^n R(n) \Longleftrightarrow a^{sh\,2(n)} R(-n)$

**Range Complementing** $R(n) \Longleftrightarrow \overline{R(n)}$

**A.I Flipping**
**Base Inverting**
**Range Flipping**

$$a^n \Longleftrightarrow a^{-n}$$
$$R(n) \Longleftrightarrow R(-n)$$
$$a^n R(n) \Longleftrightarrow a^{-n} R(-n)$$

**B.I Range Shifting**
**Range Flipping**
**Range Complementing**

$$R(n) \Longleftrightarrow R(-n)$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$R(n) \Longleftrightarrow \overline{R(-n)}$$

**C.I Complementary Inverting**
**Base Inverting**
**Range Complementing**

$$a^n \Longleftrightarrow a^{-n}$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{-n} \overline{R(n)}$$

**D.I Flipping2**
**Base Inverting**
**Shifted Range Flipping**

$$a^n \Longleftrightarrow a^{-n}$$
$$a^n R(n) \Longleftrightarrow a^{sh2(n)} R(-n)$$
$$a^n R(n) \Longleftrightarrow a^{-sh2(n)} R(-n)$$

**E.I Shifting2**
**Shifted Range Flipping**
**Range Complementing**

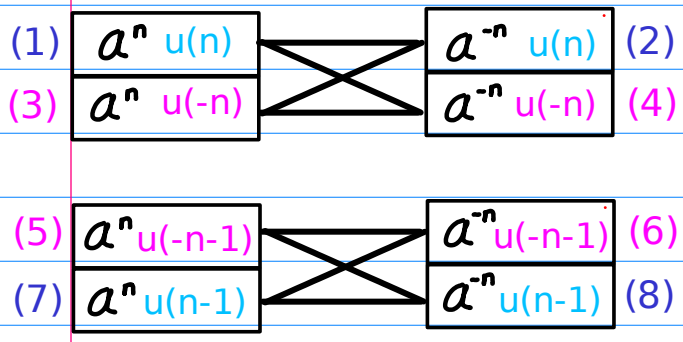
$$a^n R(n) \Longleftrightarrow a^{sh2(n)} R(-n)$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

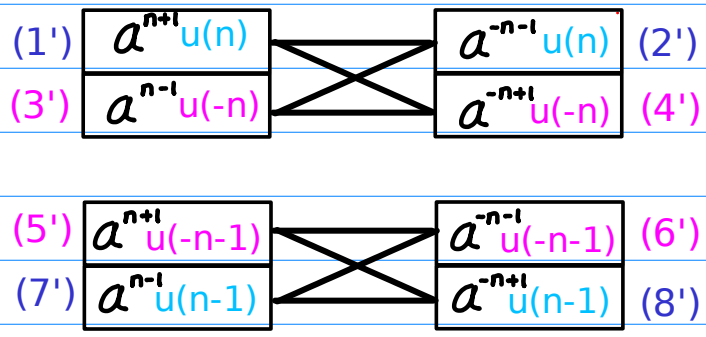
**F.I Complementary Inverting**
**Base Inverting**
**Range Complementing**

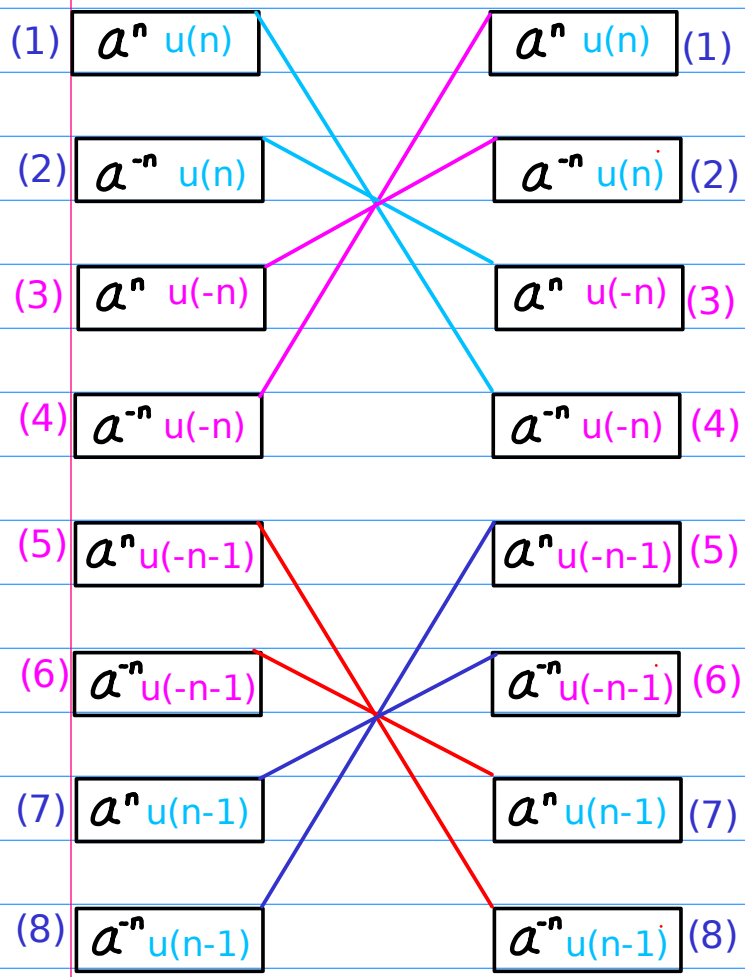
$$a^n \Longleftrightarrow a^{-n}$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{-n} \overline{R(n)}$$

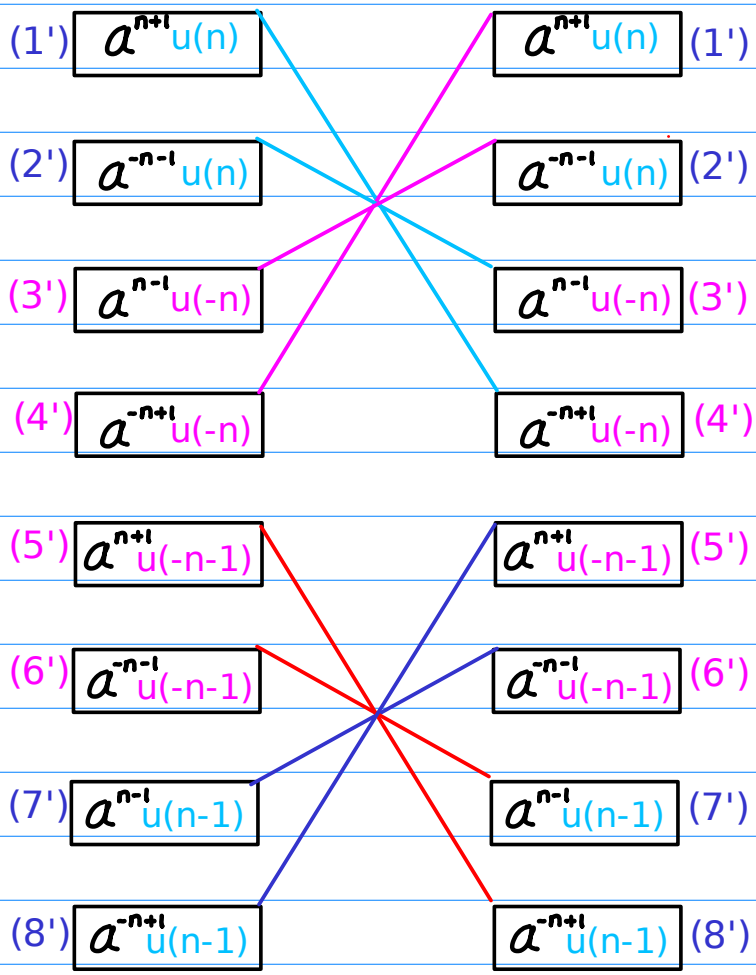
## A.I Flipping
### Base Inverting
### Range Flipping

## D.I Flipping2
### Base Inverting
### Shifted Range Flipping



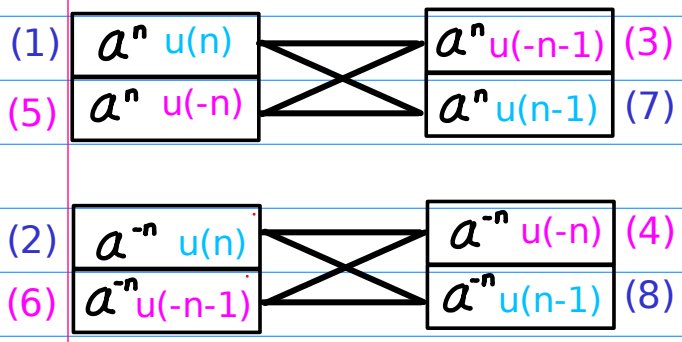

(1) - (4)
(2) - (3)
(3) - (2)
(4) - (1)

(5) - (8)
(6) - (7)
(7) - (6)
(8) - (5)

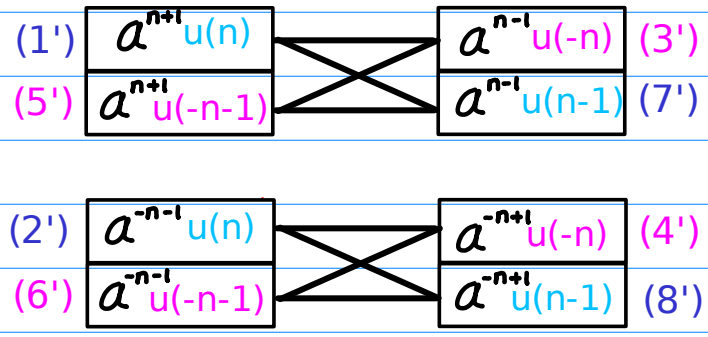
(1') - (4')
(2') - (3')
(3') - (2')
(4') - (1')

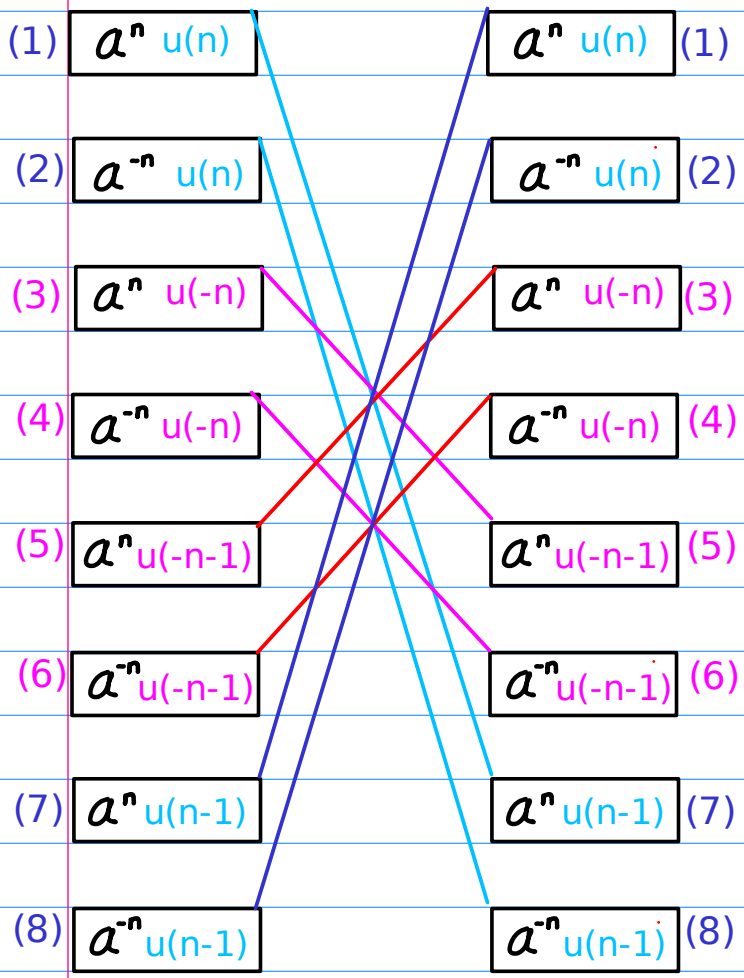
(5') - (8')
(6') - (7')
(7') - (6')
(8') - (5')

## B.I Range Shifting
## Range Flipping
## Range Complementing



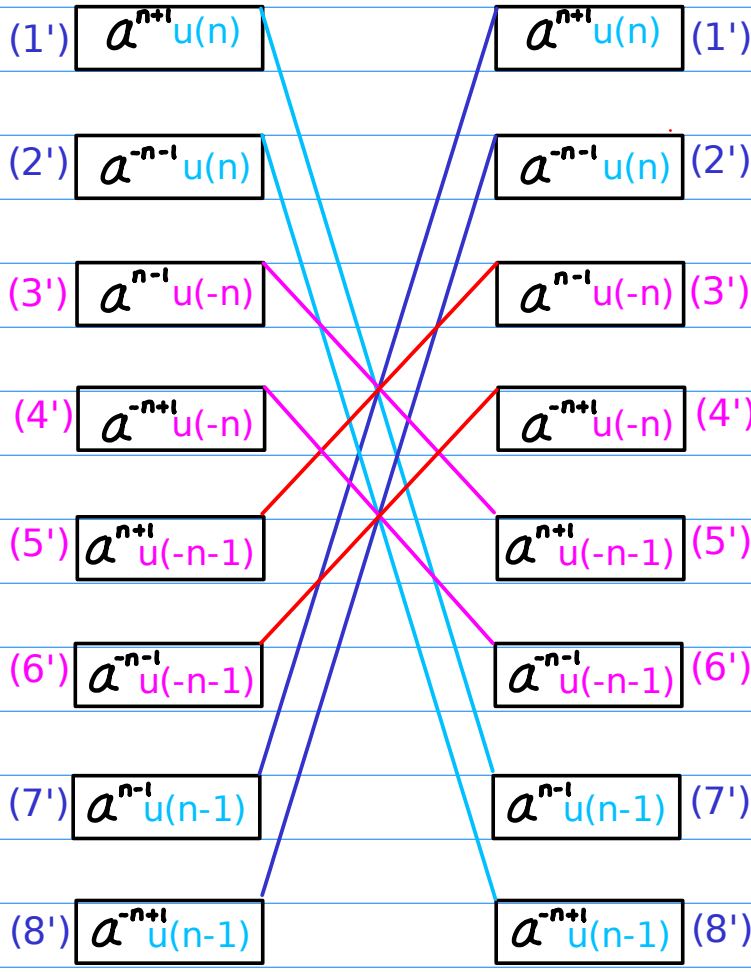

## E.I Shifting2
## Shifted Range Flipping
## Range Complementing




(1) - (7)
(2) - (8)
(3) - (5)
(4) - (6)

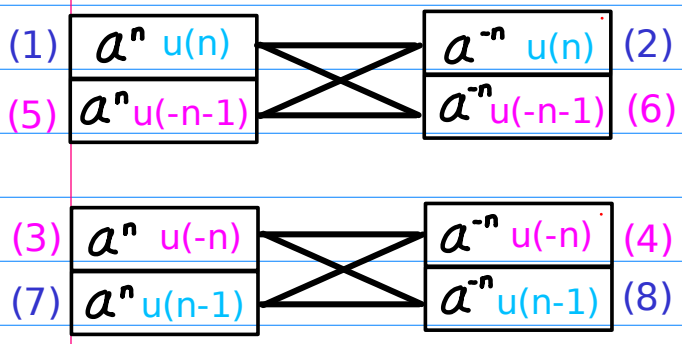
(5) - (3)
(6) - (4)
(7) - (1)
(8) - (2)



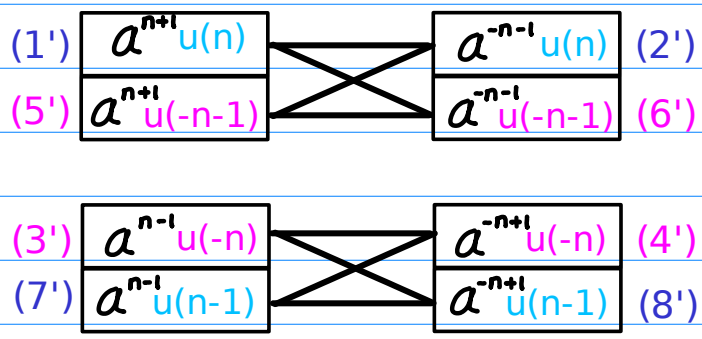

(1') - (7')
(2') - (8')
(3') - (5')
(4') - (6')

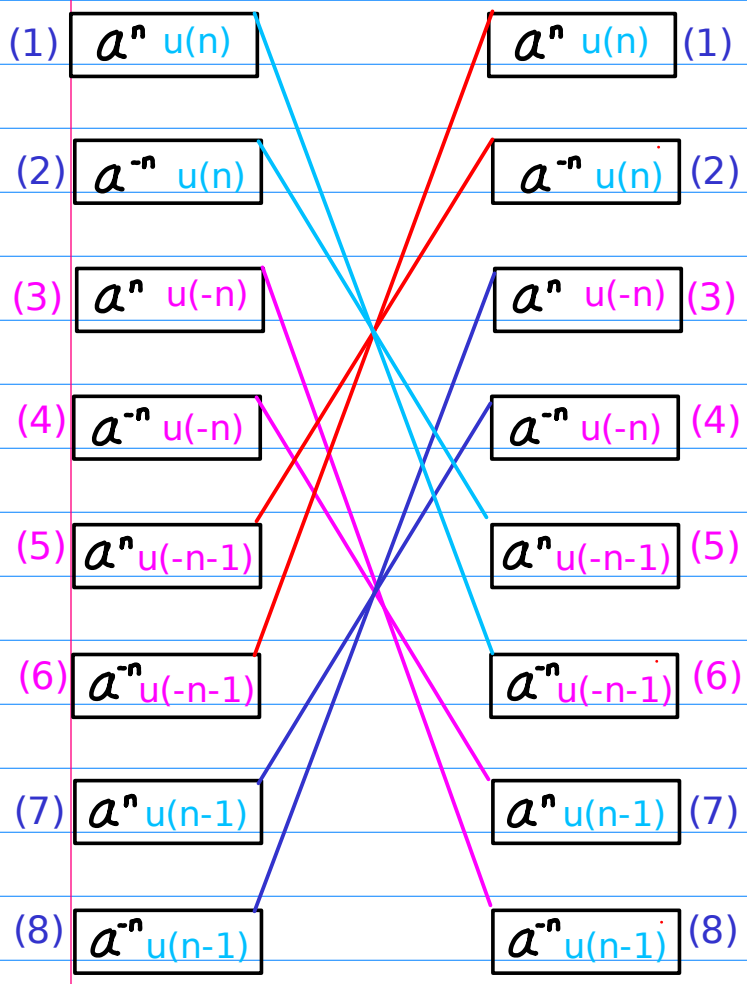
(5') - (3')
(6') - (4')
(7') - (1')
(8') - (2')

## C.I Complementary Inverting
## Base Inverting
## Range Complementing



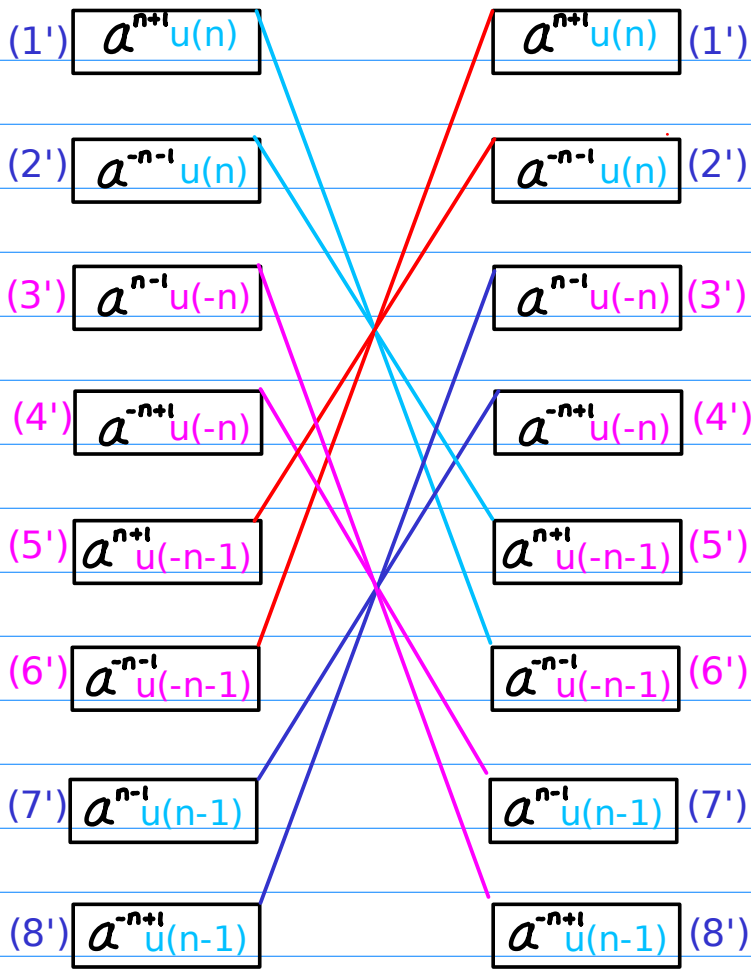

## F.I Complementary Inverting
## Base Inverting
## Range Complementing




(1) - (6)
(6) - (1)
(2) - (5)
(5) - (2)

(5) - (2)
(6) - (1)
(7) - (4)
(8) - (3)

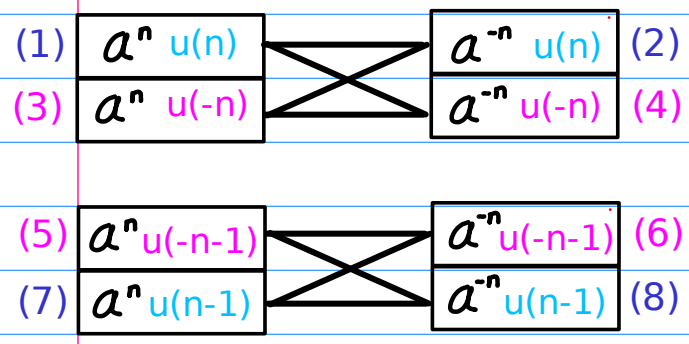


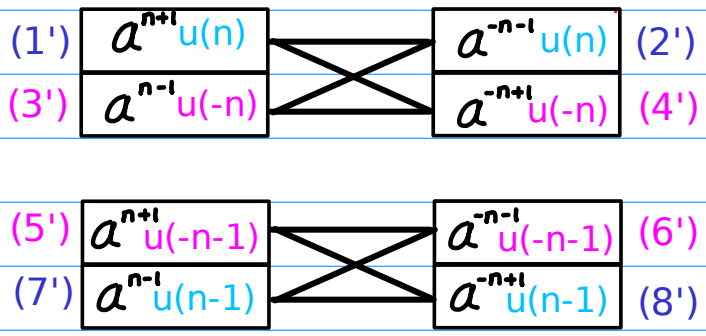

(1') - (6')
(6') - (1')
(2') - (5')
(5') - (2')

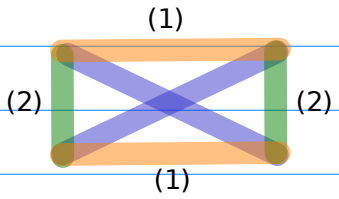
(5') - (2')
(6') - (1')
(7') - (4')
(8') - (3')

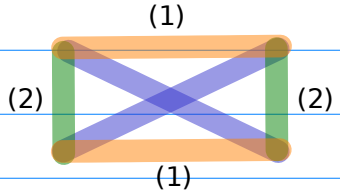
## A.I Flipping
### (1) Base Inverting
### (2) Range Flipping

## D.I Flipping2
### (1) Base Inverting
### (2) Shifted Range Flipping




$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$R(n) \overset{(2)}{\Longleftrightarrow} R(-n)$$
$$a^n R(n) \Longleftrightarrow a^{-n} R(-n)$$

$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$a^n R(n) \overset{(2)}{\Longleftrightarrow} a^{sh2(n)} R(-n)$$
$$a^n R(n) \Longleftrightarrow a^{-sh2(n)} R(-n)$$

| $b^n$ | $b^{-n}$ |
|---|---|
| $a^n$ | $a^{-n}$ |
| $a^{-n}$ | $a^n$ |

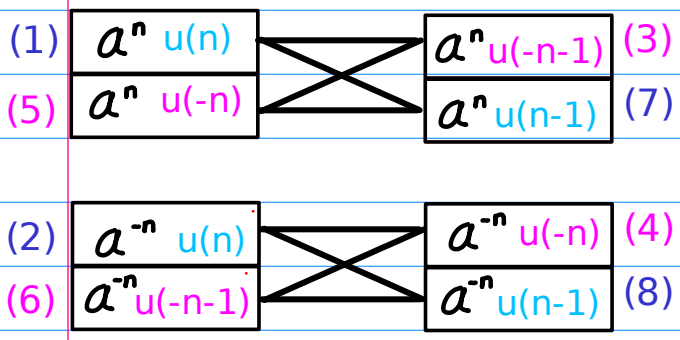
| $b^n$ | $b^{-sh2(n)}$ |
|---|---|
| $a^{(n+1)}$ | $a^{-(n-1)}$ |
| $a^{-(n+1)}$ | $a^{(n-1)}$ |
| $a^{(n-1)}$ | $a^{-(n+1)}$ |
| $a^{-(n-1)}$ | $a^{(n+1)}$ |

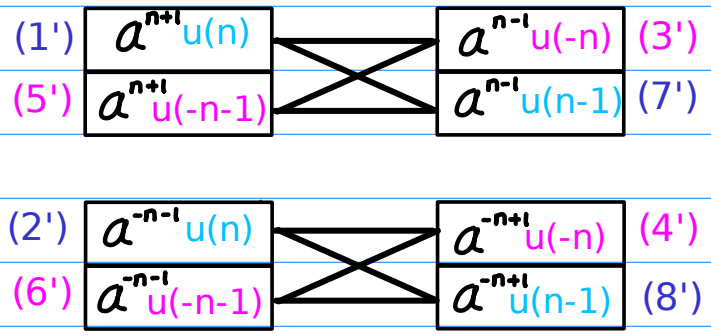
| $R(n)$ | $R(-n)$ |
|---|---|
| $u(n)$ | $u(-n)$ |
| $u(n-1)$ | $u(-n-1)$ |
| $u(-n)$ | $u(n)$ |
| $u(-n-1)$ | $u(n-1)$ |

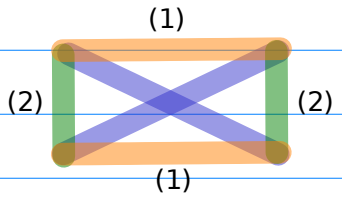
| $R(n)$ | $R(-n)$ |
|---|---|
| $u(n)$ | $u(-n)$ |
| $u(n-1)$ | $u(-n-1)$ |
| $u(-n)$ | $u(n)$ |
| $u(-n-1)$ | $u(n-1)$ |

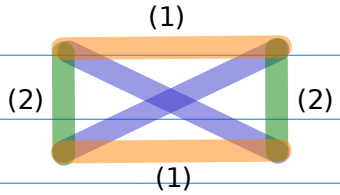
## B.I Range Shifting
### (1) Range Complementing
### (2) Range Flipping

## E.I Shifting2
### (1) Shifted Range Flippin
### (2) Range Complementing




$$R(n) \overset{(1)}{\Longleftrightarrow} R(-n)$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$R(n) \Longleftrightarrow \overline{R(-n)}$$

$$a^n R(n) \overset{(1)}{\Longleftrightarrow} a^{sh2(n)} R(-n)$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

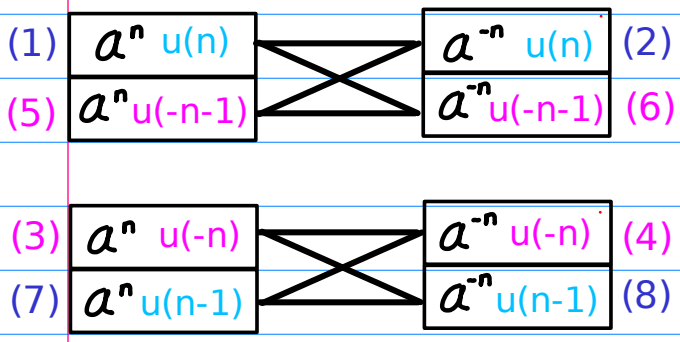
| $b^n$ | $b^{sh2(n)}$ |
|---|---|
| $a^{(n+1)}$ | $a^{(n-1)}$ |
| $a^{-(n+1)}$ | $a^{-(n-1)}$ |
| $a^{(n-1)}$ | $a^{(n+1)}$ |
| $a^{-(n-1)}$ | $a^{-(n+1)}$ |

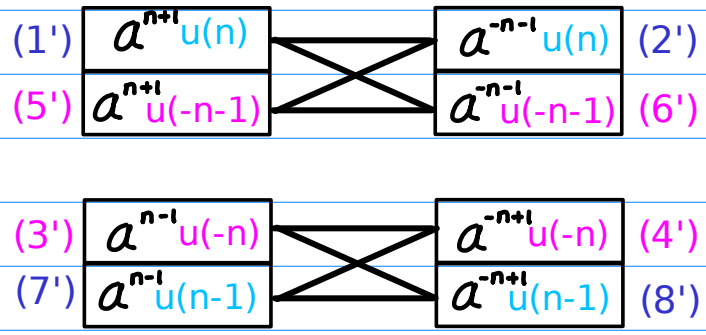
| $R(n)$ | $\overline{R(-n)}$ |
|---|---|
| $u(n)$ | $u(n-1)$ |
| $u(n-1)$ | $u(n)$ |
| $u(-n)$ | $u(-n-1)$ |
| $u(-n-1)$ | $u(-n)$ |

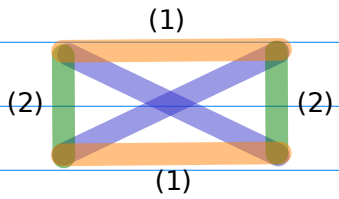
| $R(n)$ | $\overline{R(-n)}$ |
|---|---|
| $u(n)$ | $u(n-1)$ |
| $u(n-1)$ | $u(n)$ |
| $u(-n)$ | $u(-n-1)$ |
| $u(-n-1)$ | $u(-n)$ |

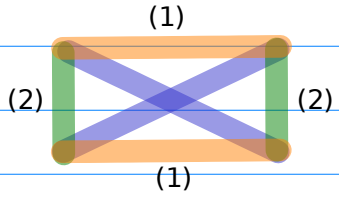
## C.I Complementary Inverting
### (1) Base Inverting
### (2) Range Complementing

## F.I Complementary Inverting
### (1) Base Inverting
### (2) Range Complementing




$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

| $b^n$ | $b^{-n}$ |
|---|---|
| $a^n$ | $a^{-n}$ |
| $a^{-n}$ | $a^n$ |

| $b^n$ | $b^{-n}$ |
|---|---|
| $a^{(n+1)}$ | $a^{-(n+1)}$ |
| $a^{-(n+1)}$ | $a^{(n+1)}$ |
| $a^{(n-1)}$ | $a^{-(n-1)}$ |
| $a^{-(n-1)}$ | $a^{(n-1)}$ |

| $R(n)$ | $\overline{R(n)}$ |
|---|---|
| $u(n)$ | $u(-n-1)$ |
| $u(n-1)$ | $u(-n)$ |
| $u(-n)$ | $u(n)$ |
| $u(-n-1)$ | $u(n-1)$ |

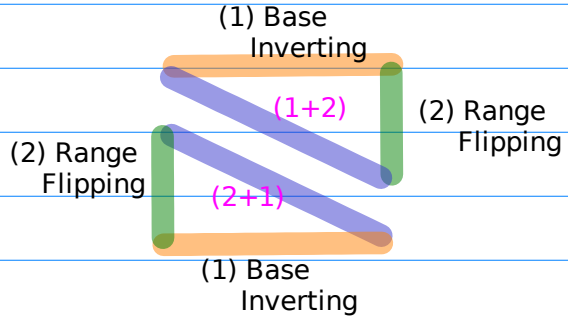
| $R(n)$ | $\overline{R(n)}$ |
|---|---|
| $u(n)$ | $u(-n-1)$ |
| $u(n-1)$ | $u(-n)$ |
| $u(-n)$ | $u(n)$ |
| $u(-n-1)$ | $u(n-1)$ |

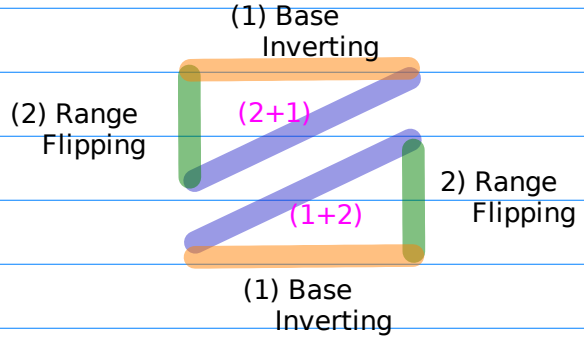
## A.I Flipping
### (1) Base Inverting
### (2) Range Flipping

$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$R(n) \overset{(2)}{\Longleftrightarrow} R(-n)$$
$$a^n R(n) \Longleftrightarrow a^{-n} R(-n)$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$
$$\xrightarrow{(2)} a^{-n} R(-n)$$

$$a^n R(n) \xrightarrow{(2)} a^n R(-n)$$
$$\xrightarrow{(1)} a^{-n} R(-n)$$



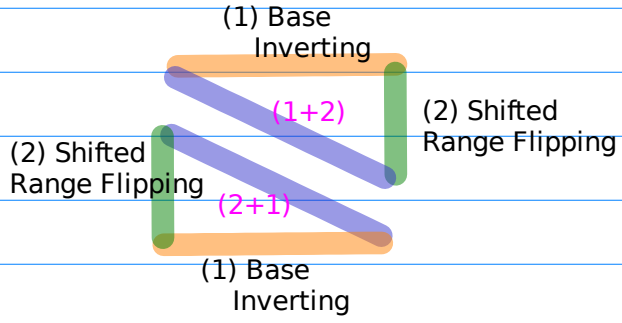



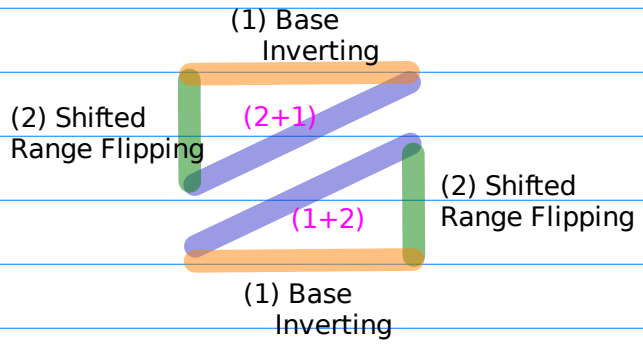

## D.I Flipping2
### (1) Base Inverting
### (2) Shifted Range Flippin

$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$a^n R(n) \overset{(2)}{\Longleftrightarrow} a^{sh2(n)} R(-n)$$
$$a^n R(n) \Longleftrightarrow a^{-sh2(n)} R(-n)$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$
$$\xrightarrow{(2)} a^{-sh2(n)} R(-n)$$

$$a^n R(n) \xrightarrow{(2)} a^{sh2(n)} R(-n)$$
$$\xrightarrow{(1)} a^{-sh2(n)} R(-n)$$



## B.I Range Shifting
## (1) Range Complementing
## (2) Range Flipping

$$R(n) \overset{(1)}{\Longleftrightarrow} R(-n)$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$R(n) \Longleftrightarrow \overline{R(-n)}$$

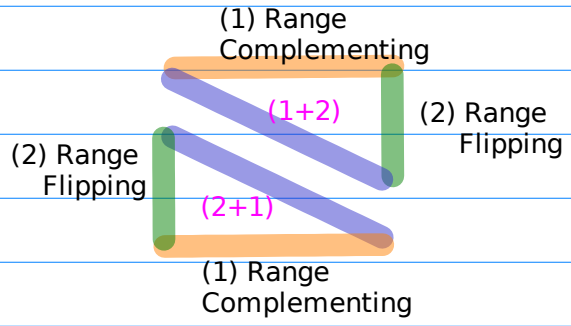
$$a^n R(n) \overset{(1)}{\Longleftrightarrow} a^{sh2(n)} R(-n)$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

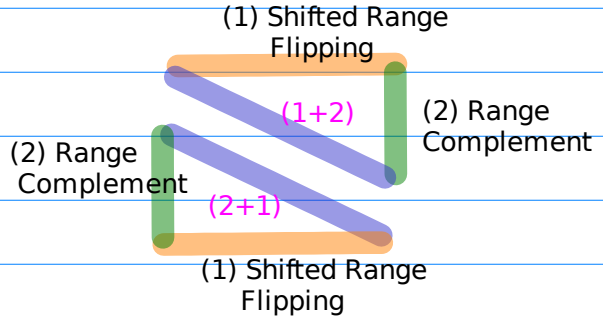
$$a^n R(n) \overset{(1)}{\rightarrow} a^{-n} R(n)$$
$$\overset{(2)}{\rightarrow} a^{-n} \overline{R(n)}$$

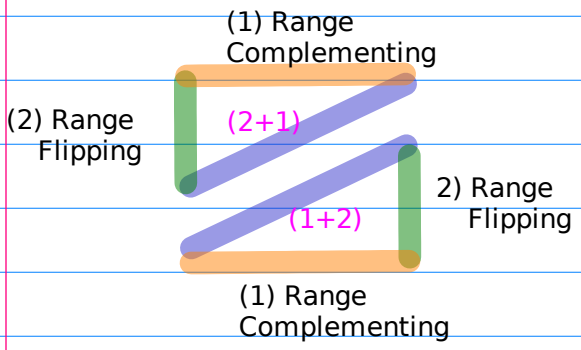
$$a^n R(n) \overset{(1)}{\rightarrow} a^{-n} R(n)$$
$$\overset{(2)}{\rightarrow} a^{-sh2(n)} R(-n)$$

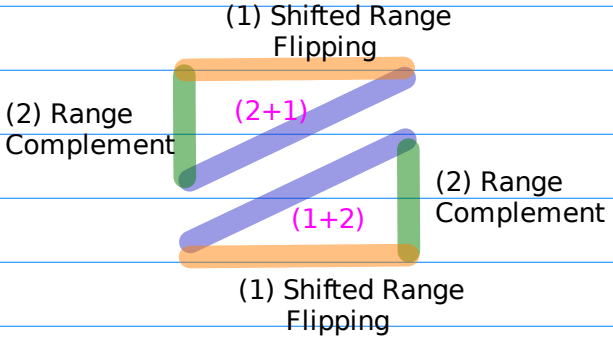
$$a^n rng(n) \overset{(2)}{\rightarrow} a^n \overline{R(n)}$$
$$\overset{(1)}{\rightarrow} a^{-n} \overline{R(n)}$$

$$a^n R(n) \overset{(2)}{\rightarrow} a^{sh2(n)} R(-n)$$
$$\overset{(1)}{\rightarrow} a^{-sh2(n)} R(-n)$$



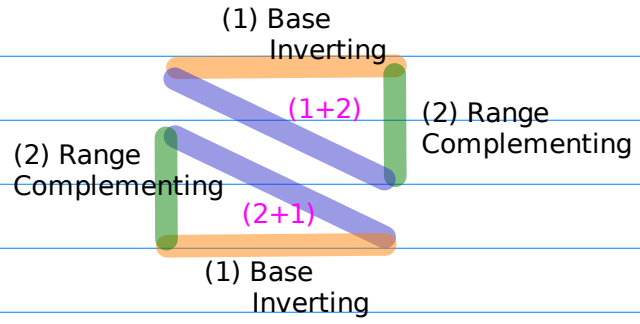

## E.I Shifting2
## (1) Shifted Range Flippin
## (2) Range Complementing

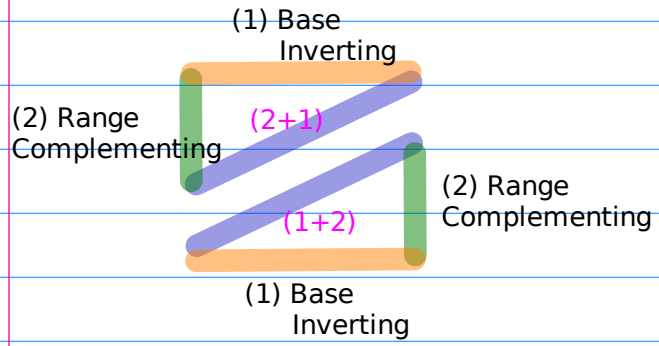
## C.I Complementary Inverting
### (1) Base Inverting
### (2) Range Complementing

$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

$$a^n R(n) \overset{(1)}{\Longrightarrow} a^{-n} R(n)$$
$$\overset{(2)}{\Longrightarrow} a^{-n}\overline{R(n)}$$

$$a^n R(n) \overset{(2)}{\Longrightarrow} a^n \overline{R(n)}$$
$$\overset{(1)}{\Longrightarrow} a^{-n}\overline{R(n)}$$



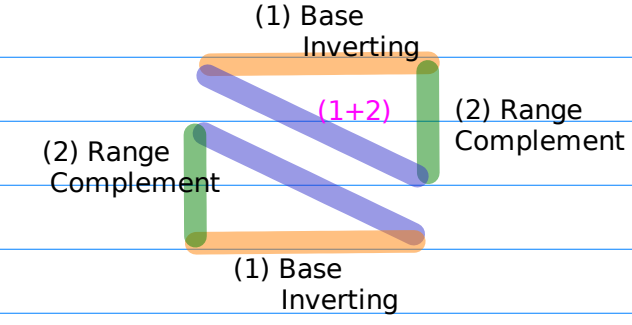



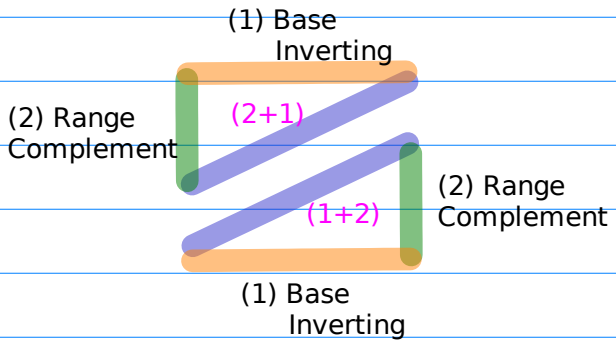

## F.I Complementary Inverting
### (1) Base Inverting
### (2) Range Complementing

$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

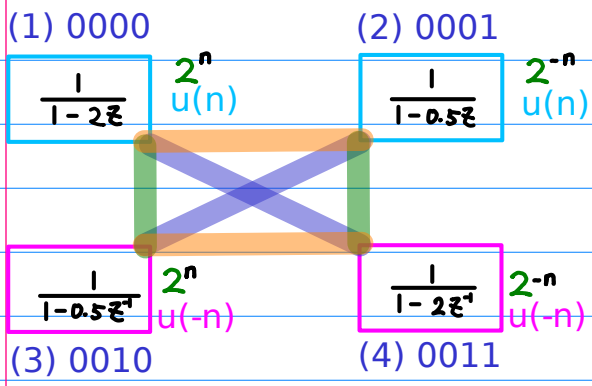
$$a^n R(n) \overset{(1)}{\Longrightarrow} a^{-n} R(n)$$
$$\overset{(2)}{\Longrightarrow} a^{-n}\overline{R(n)}$$

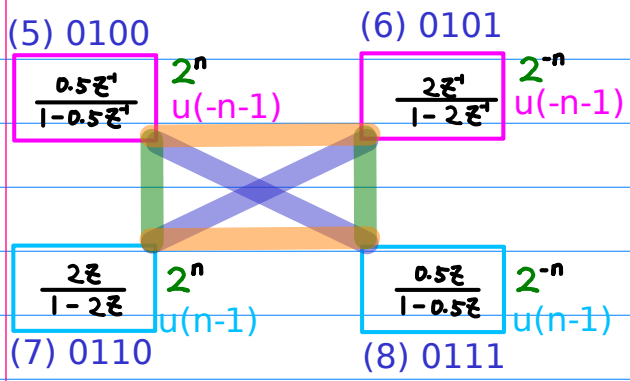
$$a^n R(n) \overset{(2)}{\Longrightarrow} a^n \overline{R(n)}$$
$$\overset{(1)}{\Longrightarrow} a^{-n}\overline{R(n)}$$



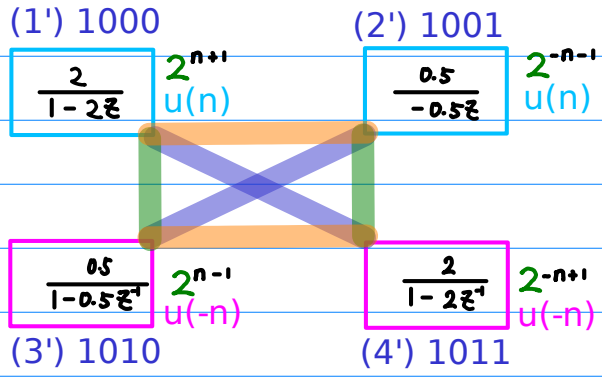



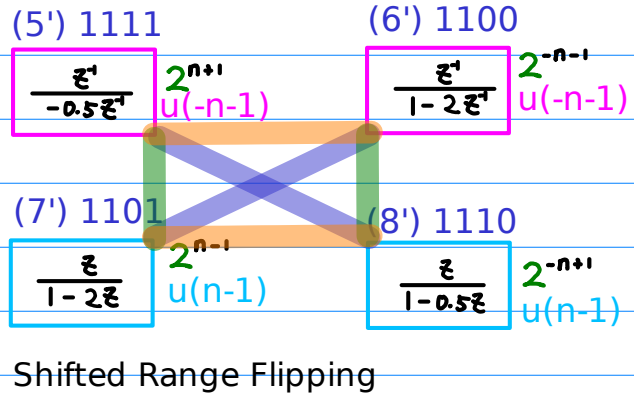
## A.I Flipping
### Base Inverting
### Range Flipping




## D.I Flipping2
### Base Inverting
### Shifted Range Flipping



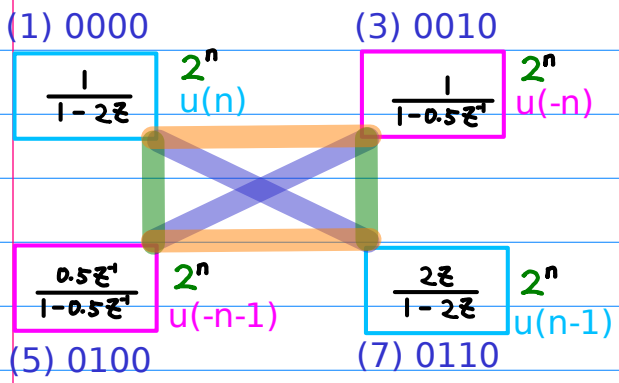



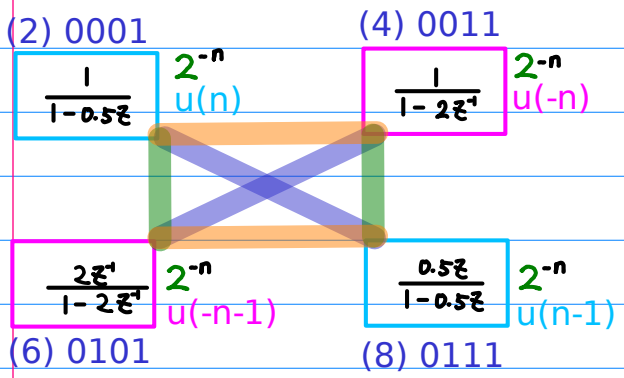

Shifted Range Flipping
= Exponent Shifting2
+ Range Flippping

$$a^n \quad \Longleftrightarrow \quad a^{-n}$$
$$R(n) \quad \Longleftrightarrow \quad R(-n)$$
$$a^n R(n) \quad \Longleftrightarrow \quad a^{-n} R(-n)$$

$$a^n \quad \Longleftrightarrow \quad a^{-n}$$
$$a^n R(n) \quad \Longleftrightarrow \quad a^{sh2(n)} R(-n)$$
$$a^n R(n) \quad \Longleftrightarrow \quad a^{-sh2(n)} R(-n)$$

## B.I Range Shifting
## Range Flipping
## Range Complementing

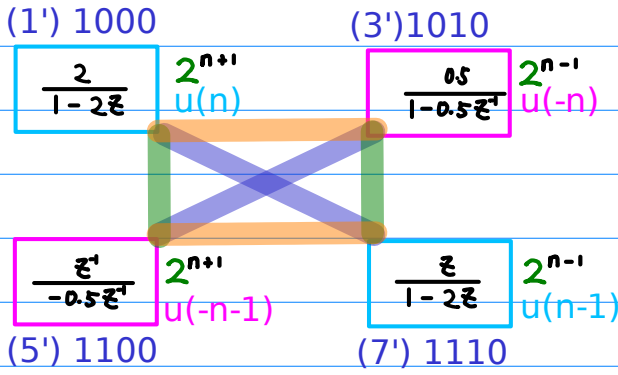


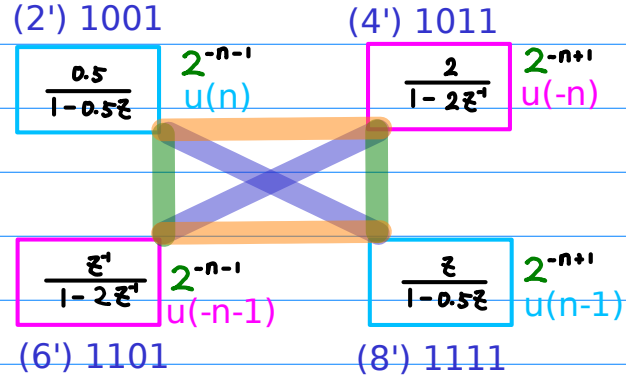




Range Shifting
= Range Flipping
+ Range Complementing

$$R(n) \Longleftrightarrow R(-n)$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$R(n) \Longleftrightarrow \overline{R(-n)}$$

## E.I Shifting2
## Shifted Range Flipping
## Range Complementing




Shifted Range Flipping
= Exponent Shifting2
+ Range Flippping

$$a^n R(n) \Longleftrightarrow a^{sh2(n)} R(-n)$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

## C.I Complementary Inverting
## Base Inverting
## Range Complementing

(1) 0000 $\frac{1}{1-2z}$ $2^{n}$ u(n)

(2) 0001 $\frac{1}{1-0.5z}$ $2^{-n}$ u(n)

(5) 0100 $\frac{0.5z^{-1}}{1-0.5z^{-1}}$ $2^{n}$ u(-n-1)

(6) 0101 $\frac{2z^{-1}}{1-2z^{-1}}$ $2^{-n}$ u(-n-1)

(3) 0010 $\frac{1}{1-0.5z^{-1}}$ $2^{n}$ u(-n)

(4) 0011 $\frac{1}{1-2z^{-1}}$ $2^{-n}$ u(-n)

(7) 0110 $\frac{2z}{1-2z}$ $2^{n}$ u(n-1)

(8) 0111 $\frac{0.5z}{1-0.5z}$ $2^{-n}$ u(n-1)

$$a^{n} \Longleftrightarrow a^{-n}$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$a^{n}R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

## F.I Complementary Inverting
## Base Inverting
## Range Complementing

(1') 1000 $\frac{2}{1-2z}$ $2^{n+1}$ u(n)

(2') 1001 $\frac{0.5}{1-0.5z}$ $2^{-n-1}$ u(n)

(5') 1100 $\frac{z^{-1}}{1-0.5z^{-1}}$ $2^{n+1}$ u(-n-1)

(6') 1101 $\frac{z^{-1}}{1-2z^{-1}}$ $2^{-n-1}$ u(-n-1)

(3') 1010 $\frac{0.5}{1-0.5z^{-1}}$ $2^{n-1}$ u(-n)

(4') 1011 $\frac{2}{1-2z^{-1}}$ $2^{-n+1}$ u(-n)

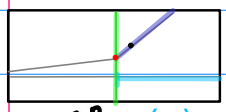
(7') 1110 $\frac{z}{1-2z}$ $2^{n-1}$ u(n-1)

(8') 1111 $\frac{z}{1-0.5z}$ $2^{-n+1}$ u(n-1)

$$a^{n} \Longleftrightarrow a^{-n}$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$a^{n}R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

## A.II Flipping
### Base Inverting
### Range Flipping

(1) 0000



$a^n$ u(n)

(4) 0011

$a^{-n}$ u(-n)

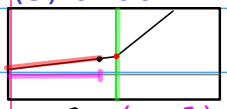
(3) 0010

$a^n$ u(-n)

(2) 0001

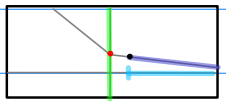
$a^{-n}$ u(n)

(5) 0100



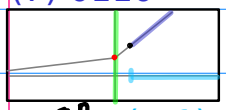
$a^n$ u(-n-1)

(8) 0111



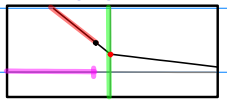
$a^{-n}$ u(n-1)

(7) 0110



$a^n$ u(n-1)

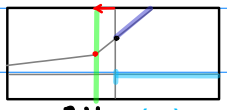
(6) 0101



$a^{-n}$ u(-n-1)

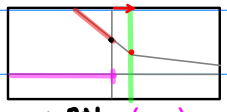
## D.II Flipping2
### Base Inverting
### Shifted Range Flipping

(1') 1000



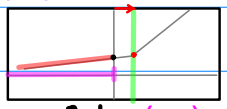
$a^{n+1}$ u(n)

(4') 1011



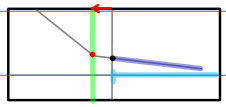
$a^{-n+1}$ u(-n)

(3') 1010



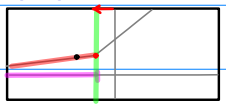
$a^{n-1}$ u(-n)

(2') 1001



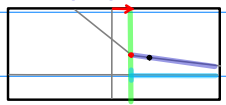
$a^{-n-1}$ u(n)

(5') 1100



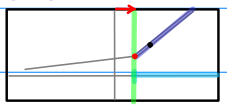
$a^{n+1}$ u(-n-1)

(8') 1111



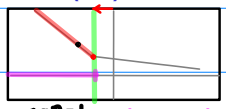
$a^{-n+1}$ u(n-1)

(7') 1110



$a^{n-1}$ u(n-1)

(6') 1101



$a^{-n-1}$ u(-n-1)

Shifted Range Flipping
= Exponent Shifting2
+ Range Flippping

$$a^n R(n) \Longleftrightarrow a^{-n} R(-n)$$

$$a^n \Longleftrightarrow a^{-n}$$

$$R(n) \Longleftrightarrow R(-n)$$

$$a^n R(n) \Longleftrightarrow a^{-sh2(n)} R(-n)$$

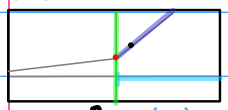
$$a^n \Longleftrightarrow a^{-n}$$

$$a^n R(n) \Longleftrightarrow a^{sh2(n)} R(-n)$$

## B.II Range Shifting
## Range Flipping
## Range Complementing

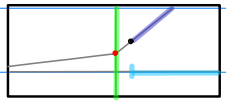
## E.II Shifting2
## Shifted Range Flipping
## Range Complementing

(1) 0000



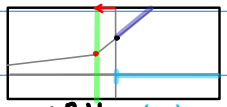
$a^n$ u(n)

(7) 0110



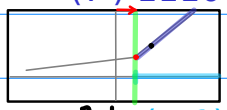
$a^n$ u(n-1)

(1') 1000



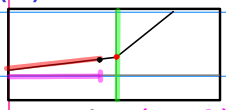
$a^{n+1}$ u(n)

(7') 1110



$a^{n-1}$ u(n-1)

(5) 0100

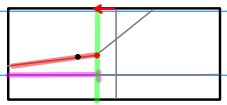


$a^n$ u(-n-1)

(3) 0010

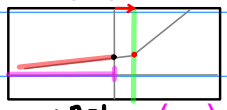
$a^n$ u(-n)

(5') 1100



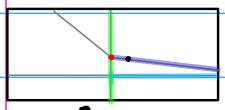
$a^{n+1}$ u(-n-1)

(3') 1010



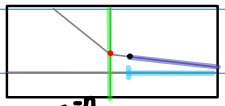
$a^{n-1}$ u(-n)

(2) 0001



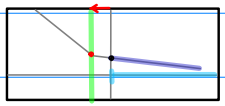
$a^{-n}$ u(n)

(8) 0111



$a^{-n}$ u(n-1)

(2') 1001

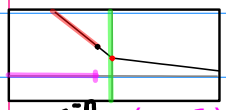


$a^{-n-1}$ u(n)

(8') 1111

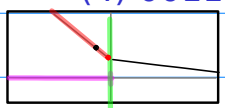
$a^{-n+1}$ u(n-1)

(6) 0101



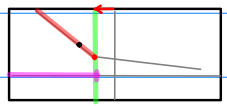
$a^{-n}$ u(-n-1)

(4) 0011



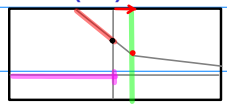
$a^{-n}$ u(-n)

(6') 1101



$a^{-n-1}$ u(-n-1)

(4') 1011



$a^{-n+1}$ u(-n)

Range Shifting
= Range Flipping
+ Range Complementing

$$R(n) \Longleftrightarrow \overline{R(-n)}$$

$$R(n) \Longleftrightarrow R(-n)$$

$$R(n) \Longleftrightarrow \overline{R(n)}$$

Shifted Range Flipping
= Exponent Shifting2
+ Range Flippping

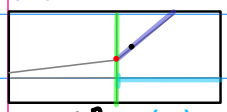
$$a^n R(n) \Longleftrightarrow a^{sh\,2(n)} \overline{R(-n)}$$

$$a^n R(n) \Longleftrightarrow a^{sh\,2(n)} R(-n)$$

$$R(n) \Longleftrightarrow \overline{R(n)}$$

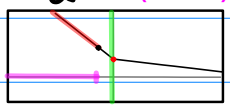
## C.II Complementary Inverting
## Base Inverting
## Range Complementing

(1) 0000



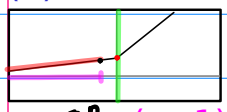
$a^{n}$ u(n)

$a^{-n}$ u(-n-1)



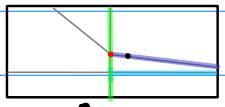
(6) 0101

(5) 0100



$a^{n}$ u(-n-1)

(2) 0001

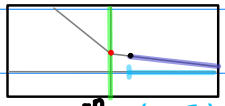


$a^{-n}$ u(n)

(3) 0010

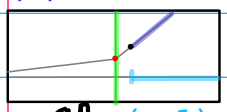
$a^{n}$ u(-n)

(8) 0111



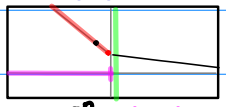
$a^{-n}$ u(n-1)

(7) 0110



$a^{n}$ u(n-1)

(4) 0011



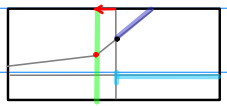
$a^{-n}$ u(-n)

$$a^{n}R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

$$a^{n} \Longleftrightarrow a^{-n}$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$

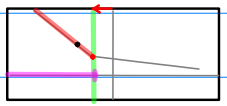
## F.II Complementary Inverting
## Base Inverting
## Range Complementing

(1') 1000



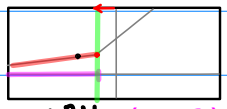
$a^{n+1}$ u(n)

$a^{-n-1}$ u(-n-1)



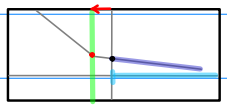
(6') 1101

(5') 1100



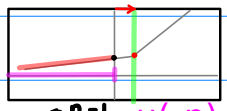
$a^{n+1}$ u(-n-1)

(2') 1001



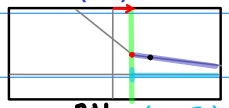
$a^{-n-1}$ u(n)

(3') 1010



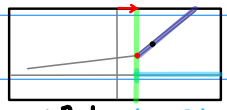
$a^{n-1}$ u(-n)

(8') 1111



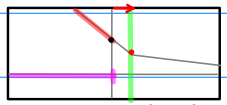
$a^{-n+1}$ u(n-1)

(7') 1110



$a^{n-1}$ u(n-1)

(4') 1011



$a^{-n+1}$ u(-n)

$$a^{n}R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

$$a^{n} \Longleftrightarrow a^{-n}$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$