

Random Process Background (1C)

Young W Lim

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles, Jr. and B. Shi

Outline

- 1 Open Sets and Neighborhoods
 - Open Set
 - Neighborhood
 - Class
- 2 Filters
 - Filter
 - Proper Filter and Ultra Filter
 - Filter Example
- 3 Topological Space
 - Topological Space
 - A discrete topology
 - Examples of topology

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Open set examples

- The *circle* represents the set of points (x, y) satisfying $x^2 + y^2 = r^2$.
the *circle* set is its **boundary set**
- The *disk* represents the set of points (x, y) satisfying $x^2 + y^2 < r^2$.
The *disk* set is an **open set**
- the **union** of the *circle* and *disk* sets is a **closed set**.

https://en.wikipedia.org/wiki/Open_set

Open set (1)

- an **open set** is a generalization of an **open interval** in the real line.
- a **metric space** is a **set** along with a **distance** defined between any two **points**
- in a **metric space**, an **open set** is a **set** that, along with every **point** P , contains all points that are sufficiently near to P
 - all points whose **distance** to P is less than some value depending on P

https://en.wikipedia.org/wiki/Open_set

Open set (2-1)

- more generally, an **open set** is a **member** of a **given collection** of **subsets** of a **given set**

- a given set
- subsets of a given set
- a given collection of subsets of a given set

https://en.wikipedia.org/wiki/Open_set

Open set (2-2)

- a **collection** has the following property of **containing**

- a **collection** contains
 - every **union** of its **members**
 - every **finite intersection** of its members
 - the **empty set**
 - the **whole set** itself

https://en.wikipedia.org/wiki/Open_set

Open set (3)

- These conditions are very loose, and allow enormous flexibility in the choice of **open sets**.
- For example,
 - every **subset** can be **open** (the **discrete topology**)
 - no **subset** can be **open** (the **indiscrete topology**) except
 - the space itself and
 - the empty set

https://en.wikipedia.org/wiki/Open_set

Open set (4)

- A **set** in which such a **collection** is given is called a **topological space**, and the **collection** is called a **topology**.
 - A **set** is a **collection** of distinct **objects**.
 - Given a **set** A , we say that a is an **element** of A if a is one of the distinct **objects** in A , and we write $a \in A$ to denote this
 - Given two **sets** A and B , we say that A is a **subset** of B if every element of A is also an element of B write $A \subseteq B$ to denote this.

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (5) Open Balls

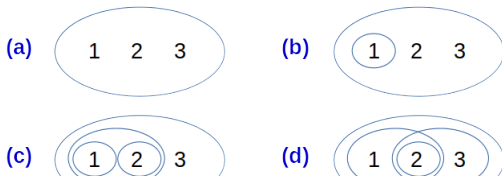
- An **open ball** $B_r(\mathbf{a})$ in \mathbb{R}^n
centered at $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ with radius r
is the set of all points $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$
such that the distance between \mathbf{x} and \mathbf{a} is less than r
- In \mathbb{R}^2 an **open ball** is often called an **open disk**

We give these definitions in general, for when one is working in \mathbb{R}^n
since they are really not all that different to define in \mathbb{R}^n than in \mathbb{R}^2

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

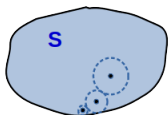
Open set (6) Interior points

- Suppose that $S \subseteq \mathbb{R}^n$
- A point $p \in S$ is an **interior point** of S if there exists an **open ball** $B_r(p) \subseteq S$
- Intuitively, p is an **interior point** of S if we can squeeze an entire open ball centered at p within S

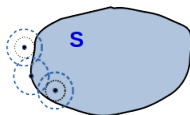


Open set (7) Boundary points

- A point $\mathbf{p} \in \mathbb{R}^n$ is a **boundary point** of S if all **open balls** centered at \mathbf{p} contain both **points** in S and **points** not in S
- The **boundary** of S is the **set** ∂S that consists of all of the **boundary points** of S .



an interior point



a boundary point

Open set (8) Open and Closed Sets

- A set $O \subseteq \mathbb{R}^n$ is **open** if every point in O is an **interior point**.
- A set $C \subseteq \mathbb{R}^n$ is **closed** if it contains all of its **boundary points**.

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (9) Bounded and Unbounded

- A set S is **bounded** if there is an **open ball** $B_M(0)$ such that

$$S \subseteq B.$$

intuitively, this means that we can enclose
all of the **set** S within a large enough **ball**
centered at the origin, $B_M(0)$

- A **set** that is not **bounded** is called **unbounded**

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Family of sets (1)

- a **collection** F of **subsets** of a given **set** S is called a **family** of **subsets** of S , or a **family** of **sets** over S .
- More generally, a **collection** of any **sets** whatsoever is called a **family** of **sets**, **set family**, or a **set system**

https://en.wikipedia.org/wiki/Family_of_sets

Family of sets (2)

- The term "**collection**" is used here because,
 - in some contexts,
a **family** of **sets** may be allowed
to contain repeated copies of any given **member**, and
 - in other contexts
it may form a **proper class** rather than a **set**.

https://en.wikipedia.org/wiki/Family_of_sets

Examples of family of sets (1)

- The **set** of all **subsets** of a given **set** S is called the **power set** of S and is denoted by $\wp(S)$.

The **power set** $\wp(S)$ of a given **set** S is a **family** of **sets** over S .

- A **subset** of S having k elements is called a **k -subset** of S .

The **k -subset** $S^{(k)}$ of a set S form a **family** of **sets**.

https://en.wikipedia.org/wiki/Family_of_sets

Examples of family of sets (2)

- Let $S = \{a, b, c, 1, 2\}$.

An example of a **family** of **sets** over S

(in the multiset sense) is given by $F = \{A_1, A_2, A_3, A_4\}$, where

$A_1 = \{a, b, c\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2\}$, and $A_4 = \{a, b, 1\}$.

https://en.wikipedia.org/wiki/Family_of_sets

Neighbourhood basis (1)

- A **neighbourhood basis** or **local basis** (or **neighbourhood base** or **local base**) for a **point** x is a **filter base** of the **neighbourhood filter**;
- this means that it is a **subset** $\mathcal{B} \subseteq \mathcal{N}(x)$ such that for all $V \in \mathcal{N}(x)$, there exists some $B \in \mathcal{B}$ such that $B \subseteq V$.
That is, for any **neighbourhood** V we can find a **neighbourhood** B in the **neighbourhood basis** that is contained in V .

https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis

Neighbourhood basis (2)

- Equivalently, \mathcal{B} is a local basis at x if and only if the neighbourhood filter \mathcal{N} can be recovered from \mathcal{B} in the sense that the following equality holds:

$$\mathcal{N}(x) = \{V \subseteq X : B \subseteq V \text{ for some } B \in \mathcal{B}\}$$

- A family $\mathcal{B} \subseteq \mathcal{N}(x)$ is a neighbourhood basis for x if and only if \mathcal{B} is a cofinal subset of $(\mathcal{N}(x), \supseteq)$ with respect to the partial order \supseteq (importantly, this partial order is the superset relation and not the subset relation).

https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis

A collection of sets around x

- In general, one refers to the family of **sets** containing 0, used to **approximate** 0, as a **neighborhood basis**;
- a **member** of this **neighborhood basis** is referred to as an **open set**.
- In fact, one may generalize these notions to an arbitrary set (X); rather than just the **real numbers**.
- In this case, given a **point** (x) of that **set** (X), one may define a **collection** of **sets** "**around**" (that is, containing) x , used to **approximate** x .

https://en.wikipedia.org/wiki/Open_set

Smaller sets containing x

- Of course, this **collection** would have to *satisfy* certain properties (known as **axioms**) for otherwise we may not have a *well-defined method* to measure **distance**.
- For example, every **point** in X should **approximate** x to some **degree** of **accuracy**.
- Thus X should be in this **family**.
- Once we begin to define "smaller" **sets** containing x , we tend to **approximate** x to a greater **degree** of **accuracy**.
- Bearing this in mind, one may define the remaining **axioms** that the **family** of **sets** about x is required to satisfy.

https://en.wikipedia.org/wiki/Open_set

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Open ball (1)

- a **ball** is the solid figure bounded by a **sphere**;
it is also called a **solid sphere**.
 - a **closed ball**
includes the *boundary points* that constitute the sphere
 - an **open ball**
excludes them

[https://en.wikipedia.org/wiki/Ball_\(mathematics\)](https://en.wikipedia.org/wiki/Ball_(mathematics))

Open ball (2)

- A **ball** in n dimensions is called a **hyperball** or **n-ball** and is bounded by a **hypersphere** or $(n - 1)$ -sphere
- One may talk about **balls** in any **topological space** X , not necessarily induced by a **metric**.
- An n -dimensional **topological ball** of X is any **subset** of X which is **homeomorphic** to an **Euclidean n-ball**.

[https://en.wikipedia.org/wiki/Ball_\(mathematics\)](https://en.wikipedia.org/wiki/Ball_(mathematics))

Neighborhood (1)

- a **neighbourhood** is one of the basic *concepts* in a **topological space**.
- It is closely related to the *concepts* of **open set** and **interior**.
- Intuitively speaking, a **neighbourhood** of a **point** is a **set of points** containing that **point** where one can move some amount in any direction away from that **point** without leaving the **set**.

[https://en.wikipedia.org/wiki/Neighbourhood_\(mathematics\)](https://en.wikipedia.org/wiki/Neighbourhood_(mathematics))

Interior

- the **interior** of a **subset** S of a **topological space** X is the **union** of *all* **subsets** of S that are **open** in X .
- A **point** that is in the **interior** of S is an **interior point** of S .
- The **interior** of S is the **complement** of the **closure** of the complement of S .
the closure of (boundary + exterior)
- In this sense, **interior** and **closure** are dual notions.

[https://en.wikipedia.org/wiki/Interior_\(topology\)#Interior_point](https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point)

Exterior

- The **exterior** of a set S is the **complement** of the **closure** of S ; the closure of $S = \text{boundary} + \text{interior}$
- it consists of the **points** that are in neither the **set** nor its **boundary**.
- The **interior**, **boundary**, and **exterior** of a **subset** together partition the whole **space** into three **blocks**
- fewer when one or more of these is empty

[https://en.wikipedia.org/wiki/Interior_\(topology\)#Interior_point](https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point)

Interior Point (1)

- If S is a **subset** of a **Euclidean space**, then x is an **interior point** of S if there exists an **open ball** centered at x which is completely contained in S .
- This definition generalizes to any **subset** S of a **metric space** X with **metric** d :
 x is an **interior point** of S if there exists a real number $r > 0$, such that y is in S whenever the distance $d(x, y) < r$.

[https://en.wikipedia.org/wiki/Interior_\(topology\)#Interior_point](https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point)

Interior Point (2)

- This definition generalizes to **topological spaces** by replacing "**open ball**" with "**open set**".
 - if there exists an *open ball* centered at x which is completely contained in S .
 - if x is contained in an *open subset* of X that is completely contained in S .

[https://en.wikipedia.org/wiki/Interior_\(topology\)#Interior_point](https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point)

Interior Point (3)

- If S is a **subset** of a **topological space** X then x is an **interior point** of S in X if x is contained in an **open subset** of X that is completely contained in S .
- Equivalently, x is an **interior point** of S if S is a **neighbourhood** of x .

[https://en.wikipedia.org/wiki/Interior_\(topology\)#Interior_point](https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point)

Interior of a Set (1)

- The **interior** of a **subset** S of a **topological space** X , can be defined in any of the following equivalent ways:
 - the largest **open subset** of X contained in S .
 - the union of all **open sets** of X contained in S .
 - the **set** of all **interior points** of S .

[https://en.wikipedia.org/wiki/Interior_\(topology\)#Interior_point](https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point)

Interior of a Set (2)

- The **interior** of a **subset** S of a **topological space** X , denoted by $\mathit{int}_X S$ or $\mathit{int} S$ or S°
- If the **space** X is understood from **context** then the shorter notation $\mathit{int} S$ is usually preferred to $\mathit{int}_X S$.

[https://en.wikipedia.org/wiki/Interior_\(topology\)#Interior_point](https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point)

Neighborhood of a point (1-1)

- If X is a **topological space** and p is a **point** in X , then a **neighbourhood** of p is a **subset** V of X that includes an **open set** U containing p ,

$$p \in U \subseteq V \subseteq X.$$

- X : a **topological space**
- V : a **subset** of X
- U : an **open set** containing p
- p : a **point** in X
- V : a **neighbourhood** of p

[https://en.wikipedia.org/wiki/Neighbourhood_\(mathematics\)](https://en.wikipedia.org/wiki/Neighbourhood_(mathematics))

Neighborhood of a point (1-2)

- This is also equivalent to the **point** $p \in X$ belonging to the **topological interior** of V in X .
- The **neighbourhood** V need not be an **open subset** of X , but when V is **open** in X then it is called an **open neighbourhood**.
- Some authors have been known to require **neighbourhoods** to be **open**, so it is important to note conventions.

[https://en.wikipedia.org/wiki/Neighbourhood_\(mathematics\)](https://en.wikipedia.org/wiki/Neighbourhood_(mathematics))

Neighborhood of a point (2)

- A set that is a neighbourhood of each of its points is open since it can be expressed as the union of open sets containing each of its points.
- A closed rectangle, as illustrated in the figure, is not a neighbourhood of all its points;
 - points on the edges or corners of the rectangle are not contained in any open set that is contained within the rectangle.
- The collection of all neighbourhoods of a point is called the neighbourhood system at the point.

[https://en.wikipedia.org/wiki/Neighbourhood_\(mathematics\)](https://en.wikipedia.org/wiki/Neighbourhood_(mathematics))

Neighborhood of a set (1-1)

- If S is a **subset** of a **topological space** X , then a **neighbourhood** of S is a **set** V that includes an **open set** U containing S ,

$$S \subseteq U \subseteq V \subseteq X.$$

- It follows that a **set** V is a **neighbourhood** of S if and only if it is a **neighbourhood** of all the **points** in S .
- Furthermore, V is a **neighbourhood** of S if and only if S is a **subset** of the **interior** of V .

[https://en.wikipedia.org/wiki/Neighbourhood_\(mathematics\)](https://en.wikipedia.org/wiki/Neighbourhood_(mathematics))

Neighborhood of a set (1-2)

- A **neighbourhood** of S that is also an **open subset** of X is called an **open neighbourhood** of S .
- The **neighbourhood** of a **point** is just a special case of this definition.

[https://en.wikipedia.org/wiki/Neighbourhood_\(mathematics\)](https://en.wikipedia.org/wiki/Neighbourhood_(mathematics))

Neighborhood definition (1)

- the **open set axioms** are often taken as the definition of a **topology**, when they are quite *unintuitive*, though extremely useful in the long run.
- the **neighbourhood** definition, while somewhat *cumbersome*, has the advantage of being closely related to ideas from **analysis**, and has a *historical basis*

<https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology>

Neighborhood definition (2-1)

- A **neighbourhood topology** on a set X assigns to each element $x \in X$ a non empty set $N(x)$ of **subsets** of X , called **neighbourhoods** of x
- with the following properties:

<https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology>

Neighborhood definition (2-2)

- the properties of a **neighbourhood topology**:
 - If N is a **neighbourhood** of x then $x \in X$
 - If M is a **neighbourhood** of x and $M \subseteq N \subseteq X$, then N is a **neighbourhood** of x
 - The **intersection** of two **neighbourhoods** of x is a **neighbourhood** of x
 - If N is a **neighbourhood** of x , then N contains a **neighbourhood** M of x such that N is a **neighbourhood** of each **point** of M .

<https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology>

Neighborhood definition (3-1)

- Then one says a **function** $f : X \rightarrow Y$ is **continuous** wrt **neighbourhoods** on X and Y if for each $x \in X$ and **neighbourhood** N of $f(x)$ there is a **neighbourhood** M of x such that $f(M) \subseteq N$.
- The **open set** definition of **continuity** is then justified as being equivalent to this definition in terms of **neighbourhoods**.

<https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology>

Neighborhood definition (3-2)

- One also says a set U in X is **open** if U is a **neighbourhood** of all of its **points**. THEN one can develop the **open set axioms** and show that one can recover the **neighbourhoods**.

<https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology>

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Class (1)

- a **class** is a **collection** of **sets**
(or sometimes other **mathematical objects**)
that can be unambiguously defined
by a **property** that all its members share.
- **Classes** act as a way to have **set-like collections**
while **differing** from **sets** so as to **avoid Russell's paradox**

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Class (2)

- A class *that is not a set* is called a **proper class**, and
- a class *that is a set* is sometimes called a **small class**.
- the followings are **proper classes** in many formal systems
 - the **class** of all sets
 - the **class** of all ordinal numbers
 - the **class** of all cardinal numbers

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Class (3)

- consider "the **set** of all **sets** with **property** X ."
- especially when dealing with **categories**, since the **objects** of a **concrete category** are all **sets** with certain additional **structure**.
- However, **if** we're not *careful* about this we can get into serious *trouble* –

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

Class (4)

- let X be the **set** of all **sets** which do not contain *themselves*
- Since X is a **set**, we can ask whether X is an element of *itself*.
- But then we run into a **paradox** – **if** X contains *itself* as an element, **then** it does not, and vice versa.

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

Class (5)

- In order to avoid this **paradox**, we cannot consider the **collection** of all **sets** to be itself a **set**.
- This means we have to *throw out* the whole "the **set** of all **sets** with **property** X" construction. But we wanted that.
- So the way we get around it is to say that there's something called a **class**, which is like a **set** but not a **set**.

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

Class (6)

- Then we can talk about "the class X of all sets with property Y ."
- Since X is not a set, it can't be an element of itself, and we're fine.
- Of course, if we need to talk about the collection of all classes, we need to create another name that goes another step back, and so forth.

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

Class Examples (1)

- The **collection** of all **algebraic structures** of a given type will usually be a **proper class**.
(a **class** *that is not a set* is called a **proper class**)
 - the **class** of all **groups**
 - the **class** of all **vector spaces**
 - and many others.
- Within set theory, many **collections** of **sets** turn out to be **proper classes**.

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Class Examples (2)

- One way to *prove* that a **class** is **proper** is to place it in **bijection** with the **class** of all ordinal numbers.
 - **Cardinal numbers** indicate an amount how many of something we have: one, two, three, four, five.
 - **Ordinal numbers** indicate position in a series: first, second, third, fourth, fifth.

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))
<https://editarians.com/cardinals-ordinals/>

Class Paradoxes (1)

- The **paradoxes** of naive set theory can be explained in terms of the *inconsistent tacit assumption* that "all **classes** are **sets**".
- These **paradoxes** do not arise with **classes** because there is no notion of **classes** containing **classes**.
- Otherwise, one could, for example, define a **class** of all **classes** that do not contain themselves, which would lead to a **Russell paradox** for classes.

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Class Paradoxes (2)

- With a rigorous foundation, these **paradoxes** instead *suggest proofs* that certain **classes** are **proper** (i.e., that they are not **sets**).
 - **Russell's paradox** *suggests a proof* that the **class** of all **sets** which do not contain themselves is **proper**
 - the **Burali-Forti paradox** *suggests* that the **class** of all ordinal numbers is **proper**.

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

Russell's Paradox (1)

- According to the unrestricted comprehension principle, for any sufficiently well-defined **property**, there is the **set** of **all** and only the **objects** that have that **property**.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (2)

- Let R be the **set of all sets** ($R = \{x \mid x \notin x\}$)
that are not members of themselves ($R \notin R$).
 - **if** R is not a **member** of itself ($R \notin R$),
then its definition (the **set of all sets**) entails
that it is a **member** of itself ($R \in R$);
 - yet, **if** it is a **member** of itself ($R \in R$),
then it is not a **member** of itself ($R \notin R$),
since it is the **set of all sets**
that are not members of themselves ($R \notin R$)
- the resulting **contradiction** is **Russell's paradox**.
- Let $R = \{x \mid x \notin x\}$, then $R \in R \iff R \notin R$

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (3)

- Most *sets* commonly encountered are not *members* of themselves.
- For example, consider the *set* of all squares in a plane.
- This *set* is not itself a square in the plane, thus it is not a *member* of itself.
- Let us call a *set* "**normal**" if it is not a *member* of itself, and "**abnormal**" if it is a *member* of itself.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (4)

- Clearly every **set** must be either **normal** or **abnormal**.
- The **set** of squares in the plane is **normal**.
- In contrast, the **complementary set** that contains everything which is not a square in the plane is itself not a square in the plane, and so it is one of its own **members** and is therefore **abnormal**.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (5)

- Now we consider the **set** of all **normal sets**, R , and try to determine whether R is **normal** or **abnormal**.
 - *If* R were **normal**, it would be contained in the **set** of all **normal sets** (itself), and therefore be **abnormal**;
 - on the other hand *if* R were **abnormal**, it would not be contained in the **set** of all **normal sets** (itself), and therefore be **normal**.
- This leads to the conclusion that R is neither **normal** nor **abnormal**: **Russell's paradox**.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Outline

- 1 Open Sets and Neighborhoods
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 - **Filter**
 - Proper Filter and Ultra Filter
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Binary Relation (1)

- a **binary relation** associates **elements** of one **set**, called the **domain**, with **elements** of another set, called the **codomain**.
- A **binary relation** over sets X and Y is a new set of **ordered pairs** (x,y) consisting of **elements** x from X and y from Y .

https://en.wikipedia.org/wiki/Binary_relation

Binary Relation (2)

- It is a generalization of a unary function.
- It encodes the common concept of relation:
- an element x is related to an element y ,
if and only if the pair (x, y) belongs
to the set of ordered pairs that defines the binary relation.
- A binary relation is the most studied special case $n = 2$
of an n -ary relation over sets X_1, \dots, X_n ,
which is a subset of the Cartesian product $X_1 \times \dots \times X_n$.

https://en.wikipedia.org/wiki/Binary_relation

Homogeneous Relation

- a **homogeneous relation** (also called endorelation) on a set X is a **binary relation** between X and itself, i.e. it is a **subset** of the **Cartesian product** $X \times X$.
- This is commonly phrased as "a **relation** on X " or "a **(binary) relation** over X ".
- An example of a **homogeneous relation** is the relation of **kinship**, where the relation is between people.

https://en.wikipedia.org/wiki/Homogeneous_relation

Partially Ordered Set (1-1)

- a **partial order** on a **set** is an arrangement such that, for certain **pairs** of elements, one precedes the other.
- The word **partial** is used to indicate that not every **pair** of elements needs to be comparable; that is, there may be **pairs** for which neither element precedes the other.
- **Partial orders** thus generalize **total orders**, in which every pair is comparable.

https://en.wikipedia.org/wiki/Partially_ordered_set

Partially Ordered Set (1-2)

- Formally, a **partial order** is a **homogeneous binary relation** that is **reflexive**, **transitive** and **antisymmetric**.
- A **partially ordered set** (**poset** for short) is a set on which a **partial order** is defined.
- A **reflexive**, **weak**, or **non-strict partial order**, commonly referred to simply as a **partial order**, is a **homogeneous relation** \leq on a **set** P that is **reflexive**, **antisymmetric**, and **transitive**.

https://en.wikipedia.org/wiki/Partially_ordered_set

Partially Ordered Set (2)

- a **homogeneous relation** \leq on a **set** P that is **reflexive**, **antisymmetric**, and **transitive**.
- That is, for all $a, b, c \in P$, it must satisfy:
 - **Reflexivity**:
 $a \leq a$, i.e. every element is related to itself.
 - **Antisymmetry**:
if $a \leq b$ and $b \leq a$ then $a = b$,
i.e. no two distinct elements precede each other.
 - **Transitivity**:
if $a \leq b$ and $b \leq c$ then $a \leq c$.
- A **non-strict partial order** is also known as an **antisymmetric preorder**.

https://en.wikipedia.org/wiki/Partially_ordered_set

Filter in Set Theory (1-1)

- A **filter** on a **set** may be thought of as representing a "**collection** of *large subsets*", one intuitive example being the **neighborhood filter**.
- keep *large* grains excluding *small* impurities

[https://en.wikipedia.org/wiki/Filter_\(set_theory\)#filter_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)

Filter in Set Theory (1-2)

- When you put a filter in your sink, the idea is that you filter out the *big* chunks of food, and let the water and the *smaller* chunks go through (which can, in principle, be washed through the pipes) .
- You filter out the *larger parts*.
- A filter filters out the *larger sets*.
- It is a way to say "these *sets* are '*large*'"

[https://en.wikipedia.org/wiki/Filter_\(set_theory\)#filter_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)

Filter in Set Theory (1-3)

- a **filter** on a set X is a family \mathcal{B} of subsets such that:

- 1 $X \in \mathcal{B}$ and $\emptyset \notin \mathcal{B}$
- 2 if $A \in \mathcal{B}$ and $B \in \mathcal{B}$,
then $A \cap B \in \mathcal{B}$
- 3 If $A, B \subset X, A \in \mathcal{B}$, and $A \subset B$,
then $B \in \mathcal{B}$

[https://en.wikipedia.org/wiki/Filter_\(set_theory\)#filter_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)

Filter in Set Theory (1-4)

- The set of "everything" is definitely *large*

$$X \in \mathcal{B}$$

- and "nothing" is definitely not;

$$\emptyset \notin \mathcal{B}$$

- if something is *larger* than a *large set*, then it is also *large*;

$$\text{If } A, B \subset X, A \in \mathcal{B}, \text{ and } A \subset B, \text{ then } B \in \mathcal{B}$$

- and two *large sets intersect* on a *large set*.

$$\text{If } A \in \mathcal{B} \text{ and } B \in \mathcal{B}, \text{ then } A \cap B \in \mathcal{B}$$

[https://en.wikipedia.org/wiki/Filter_\(set_theory\)#filter_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)



Filter in Set Theory (1-5)

- you can think about this as
 - being **co-finite**,
 - or being of **measure 1** on the **unit interval**,
 - or having a **dense open subset** (again on the unit interval).
- These are examples of ways where a [set](#) can be thought of as "almost everything". and that is the idea behind a filter.

[https://en.wikipedia.org/wiki/Filter_\(set_theory\)#filter_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)

Co-finite

- a **cofinite subset** of a set X is a subset A whose complement in X is a finite set.
- a subset A contains all but *finitely many* elements of X
- If the complement is not finite, but is countable, then one says the set is **countable**.
- These arise naturally when generalizing structures on finite sets to infinite sets, particularly on infinite products, as in the **product topology** or direct sum.
- This use of the prefix "co" to describe a property possessed by a set's complement is *consistent* with its use in other terms such as "comeagre set".

<https://en.wikipedia.org/wiki/Cofiniteness>

Unit interval

- the **unit interval** is the **closed interval** $[0,1]$, that is, the set of all real numbers that are greater than or equal to 0 and less than or equal to 1.
- It is often denoted I (capital letter I).
- In addition to its role in **real analysis**, the **unit interval** is used to study **homotopy theory** in the field of **topology**.
- the term "**unit interval**" is sometimes applied to the other shapes that an interval from 0 to 1 could take: $(0,1]$, $[0,1)$, and $(0,1)$.
- However, the notation I is most commonly reserved for the **closed interval** $[0,1]$.

Dense set

- In **topology**, a **subset** A of a topological space X is said to be **dense** in X if every **point** of X either belongs to A or else is arbitrarily "close" to a **member** of A
 - for instance, the **rational numbers** are a **dense** subset of the **real numbers** because every **real number** either is a **rational number** or has a rational number arbitrarily close to it (see **Diophantine approximation**).
- Formally, A is **dense** in X if the *smallest* **closed subset** of X containing A is X itself.
- The **density** of a **topological space** X is the **least cardinality** of a **dense subset** of X .

https://en.wikipedia.org/wiki/Dense_set

Proper Subset

- a **set** A is a **subset** of a set B
if all **elements** of A are also **elements** of B ;
- B is then a **superset** of A .
- It is possible for A and B to be equal;
- if they are unequal, then A is a **proper subset** of B .
- The relationship of one **set** being a **subset** of another is called **inclusion** (or sometimes **containment**).
- A is a **subset** of B may also be expressed as B includes (or contains) A or A is included (or contained) in B .
- A **k -subset** is a **subset** with k **elements**.

<https://en.wikipedia.org/wiki/Subset>

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Proper Filter (1-1)

- Fix a **partially ordered set (poset)** P .
- Intuitively, a **filter** F is a **subset** of P whose members are **elements large enough** to satisfy some *criterion*.
- For instance, if $x \in P$, then the **set of elements above** x is a **filter**, called the **principal filter** at x .

[https://en.wikipedia.org/wiki/Filter_\(mathematics\)](https://en.wikipedia.org/wiki/Filter_(mathematics))

Proper Filter (1-2)

- If x and y are **incomparable elements** of P , then neither the **principal filter** at x nor y is contained in the other
 - two **elements** x and y of a set P are said to be **comparable** with respect to a **binary relation** \leq if at least one of $x \leq y$ or $y \leq x$ is **true**. They are called **incomparable** if they are not **comparable**.
 - Hasse diagram of the natural numbers, partially ordered by " $x \leq y$ if x divides y ". The numbers 4 and 6 are **incomparable**, since neither divides the other.

[https://en.wikipedia.org/wiki/Filter_\(mathematics\)](https://en.wikipedia.org/wiki/Filter_(mathematics))
<https://en.wikipedia.org/wiki/Comparability>

Proper Filter (1-3)

- Similarly, a **filter** on a **set** S contains those **subsets** that are sufficiently large to contain some given *thing*.
- For example, if S is the *real line* and $x \in S$, then the **family** of **sets** including x in their interior is a **filter**, called the **neighborhood filter** at x .
- The *thing* in this case is slightly larger than x , but it still does not contain any other specific point of the line.

[https://en.wikipedia.org/wiki/Filter_\(mathematics\)](https://en.wikipedia.org/wiki/Filter_(mathematics))

Proper Filter (2)

- The above considerations motivate the **upward closure** requirement in the definition below: "large enough" **objects** can always be made larger.
- To understand the other two conditions, reverse the roles and instead consider F as a "locating scheme" to find x .
- In this interpretation, one searches in some **space** X , and expects F to describe those **subsets** of X that contain the **goal**.
- The **goal** must be located somewhere; thus the empty set \emptyset can never be in F .
- And if two **subsets** both contain the **goal**, then should "zoom in" to their common region.

[https://en.wikipedia.org/wiki/Filter_\(mathematics\)](https://en.wikipedia.org/wiki/Filter_(mathematics))

Proper Filter (3)

- An **ultrafilter** describes a "perfect locating scheme" where each scheme component gives new information (either "look here" or "look elsewhere").
- **Compactness** is the property that "every search is fruitful," or, to put it another way, "every locating scheme ends in a search result."
- A common use for a **filter** is to define properties that are satisfied by "generic" elements of some topological space.
- This application generalizes the "locating scheme" to find **points** that might be hard to write down explicitly.

[https://en.wikipedia.org/wiki/Filter_\(mathematics\)](https://en.wikipedia.org/wiki/Filter_(mathematics))

Neighborhood Filter (1-1)

- Let X be a **set**;
- the **elements** of X are usually called **points**
- We allow X to be **empty**.

- Let \mathcal{N} be a **function**
assigning to each x (**point**) in X
a non-empty **collection** $\mathcal{N}(x)$ of **subsets** of X .

- The **elements** of $\mathcal{N}(x)$ will be called
neighbourhoods of x with respect to \mathcal{N}
(or, simply, **neighbourhoods** of x).

https://en.wikipedia.org/wiki/Topological_space

Neighborhood Filter (1-2)

- Let X be a [set](#);
- \mathcal{N} : a [function](#) assigning to each [point](#) x in X
- $\mathcal{N}(x)$: a non-empty [collection](#) of [subsets](#) of X .
- The [elements](#) of $\mathcal{N}(x)$
 - [subsets](#) of X
 - [neighbourhoods](#) of x with respect to \mathcal{N}

https://en.wikipedia.org/wiki/Topological_space

Neighborhood Filter (1-3)

- The function \mathcal{N} is called a neighbourhood topology if *some axioms* are satisfied;
- then X with \mathcal{N} is called a topological space – (X, \mathcal{N})

https://en.wikipedia.org/wiki/Topological_space

Neighborhood Filter (1-4)

- If (X, \mathcal{T}) is a **topological space** and $p \in X$, a **neighbourhood** of p is a **subset** V of X , in which $p \in U \subseteq V$, and U is open.
- We say that V is a \mathcal{T} - **neighbourhood** of $x \in X$ or that V is a **neighborhood** of x
- The **set** of all **neighbourhoods** of $x \in X$, denoted \mathcal{N}_x is called the **neighbourhood filter** of x

<https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter>

Neighborhood Filter (1-4)

- An example of **Neighborhood Filters** on a **Topological space**.
- Let $X = \{a, b, c\}$ and let $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{b, c\}, \{a, b\}, X\}$
- Let
$$\mathcal{N}_a = \{\{a\}, \{a, b\}, \{a, c\}, X\}$$
$$\mathcal{N}_b = \{\{b\}, \{a, b\}, \{b, c\}, X\}$$
$$\mathcal{N}_c = \{\{b, c\}, X\}.$$
- In this example a, c is a **neighborhood** of a but not of c .
- Thus a **set** does not have to be a **neighborhood** of all of its points.

<https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter>

Neighborhood Filter (2)

- One can specify a **topology** in more than four different ways.
 - ① The standard definition specifies the **open sets**, what we usually call a "**topology**."
 - ② to specify the **close sets** - this is of course only a *trivial difference*.
 - ③ to specify a **closure operation** on **subsets** of your **space**
 - ④ to specify a **neighborhood filter** for every **point** satisfying the **natural axiom** that every **neighborhood** of x is a **neighborhood** of every **point** of one of its **subsets**
- So in this sense **neighborhood filters** tell you everything they possibly could about a **topological space**.

<https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter>

Neighborhood Filter (3-1)

- Probably the best way to think about the **neighborhood filter** of x : is that it contains all information regarding **convergence** to x .
- In the first **topological spaces** one encounters, **convergence** is usually of **sequences**.
- But this isn't enough to describe the **topology** in arbitrary **spaces**, for instance the infinite-dimensional **spaces** of **functional analysis**.
- It becomes important to speak of **convergence** of **nets**, or of **filters**.

<https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter>

Neighborhood Filter (3-2)

- A **filter** on X is just a nontrivial subset of the **powerset** of X **closed** under finite intersection and **superset**, and a **filter converges** to a **point** x if and only if it contains the **neighborhood filter** of x .
- In contrast to the case with **sequences**, this is enough to specify a **topology**: in fact it's enough to describe how **ultrafilters**, that is, **maximal filters**, **converge**.
- So in this sense the **neighborhood filter** encapsulates the viewpoint that **topology generalizes** the study of **convergent sequences**.

<https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter>

Neighborhood Filter (4-1)

- in a sense the **neighborhood filter** describes the smallest **neighborhood** of a **point**
 - except that there is no **smallest neighborhood**!
- That's true, at least, in many of the most interesting **spaces**, and is the main reason to worry about a whole **filter** of **neighborhoods**
 - if there were a smallest **neighborhood** then in any hypothesis requiring something to hold on a sufficiently small **neighborhood** of x we could just pick the smallest **neighborhood**.

<https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter>

Neighborhood Filter (4-2)

- But the smallest neighborhood of a point must be contained in the intersection of all its neighborhoods, and in, say, a Hausdorff space the intersection of all neighborhoods of x is x , which is not a neighborhood of x when x is not isolated.
- So the filter functions as a virtual smallest neighborhood of x : it doesn't converge to a neighborhood of x , so we can't think about its limit, but functionally we do just that.

<https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter>

Ultrafilter (1)

- an **ultrafilter** on a given **partially ordered set** (or "**poset**") P is a certain **subset** of P , namely a **maximal filter** on P ; that is, a **proper filter** on P that cannot be enlarged to a bigger **proper filter** on P .
- If X is an arbitrary **set**, its **power set** $\mathcal{P}(X)$, ordered by **set inclusion**, is always a Boolean algebra and hence a **poset**, and **ultrafilters** on $\mathcal{P}(X)$ are usually called **ultrafilter** on the **set** X .

<https://en.wikipedia.org/wiki/Ultrafilter>

Ultrafilter (2)

- In **order theory**, an **ultrafilter** is a **subset** of a **partially ordered set** that is **maximal** among all proper filters.
- This implies that any **filter** that properly contains an **ultrafilter** has to be equal to the whole **poset**.

<https://en.wikipedia.org/wiki/Ultrafilter>

Ultrafilter (3)

- An **ultrafilter** on a **set** X may be considered as a **finitely additive measure** on X .
- In this view, every **subset** of X is either considered "*almost everything*" (has measure 1) or "*almost nothing*" (has measure 0), depending on whether it belongs to the given **ultrafilter** or not

<https://en.wikipedia.org/wiki/Ultrafilter>

Ultrafilter (4)

- Formally, if P is a **set**, **partially ordered** by \leq then
- a **subset** $F \subseteq P$ is called a **filter** on P if F is nonempty, for every $x, y \in F$, there exists some **element** $z \in F$ such that $z \leq x$ and $z \leq y$, and for every $x \in F$ and $y \in P$, $x \leq y$ implies that y is in F too;
- a **proper subset** U of P is called an ultrafilter on P if U is a filter on P , and there is no **proper filter** F on P that properly extends U (that is, such that U is a **proper subset** of F).

<https://en.wikipedia.org/wiki/Ultrafilter>

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Filter Examples (1)

- Let $X = 1, 2, 3$
Choose some element from X say $F = 1, 1, 2, 1, 3, 1, 2, 3$
- Then every **intersection** of an element of F with another element in F is in F again.
Examples: $1 \cap 1, 2, 3 = 1$ $1, 2 \cap 1, 2, 3 = 1, 2$
 $1, 3 \cap 1, 2, 3 = 1, 3$ $1, 2, 3 \cap 1, 2, 3 = 1, 2, 3$
- Also the original $X = 1, 2, 3$ is also in F .
Here $F = 1, 1, 2, 1, 3, 1, 2, 3$ is called the **filter** on $X = 1, 2, 3$

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

Filter Examples (2)

- Suppose we have the collection $G = \{1, 1, 2, 1, 3, 2, 3, 1, 2, 3\}$
- Then we have $1, 3 \cap 2, 3 = 3$ but 3 isn't in G .
So this G is not called a filter.
- Now with $F = \{1, 1, 2, 1, 3, 1, 2, 3\}$
can we put as any other element in it
so that after placing the extra element it is still a filter?
Probably not in this case.
So on $X = \{1, 2, 3\}$, $F = \{1, 1, 2, 1, 3, 1, 2, 3\}$ is an **Ultrafilter**.

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

Filter Examples (3)

- If we have started say with $H = 1, 1, 2, 1, 2, 3$
this is still a **filter** on $X = 1, 2, 3$
but we can still add $1, 3$
and it will still be classified as **filter**.
- So on $X = 1, 2, 3$
 $F = 1, 1, 2, 1, 3, 1, 2, 3$ is an **Ultrafilter**
but $H = 1, 1, 2, 1, 2, 3$ is a filter but not an **Ultrafilter**.

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

Filter Examples (4)

- Now suppose we have $X = 1, 2, 3, 4$
Let $F = 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$
- Every in intersection of element of F is in F again.
We have as examples $1, 4 \cap 1, 4 = 1, 4$ $1, 4 \cap 1, 2, 4 = 1, 4$
 $1, 4 \cap 1, 3, 4 = 1, 4$ $1, 2, 4 \cap 1, 2, 4 = 1, 2, 4$ $1, 2, 4 \cap 1, 3, 4 = 1, 4$
 $1, 3, 4 \cap 1, 3, 4 = 1, 3, 4$ $1, 2, 3, 4 \cap 1, 2, 3, 4 = 1, 2, 3, 4$
- Also $X = 1, 2, 3, 4$ is also in $F = 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$
and the null element $\emptyset =$ is not in F .

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

Filter Examples (5)

- We call F a **filter** but not an **Ultrafilter** on $X = 1, 2, 3, 4$
- We can still add element in it and it will still be a **filter** for instance by adding the element 1 from $X = 1, 2, 3, 4$ we can have the filter $F = 1, 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$
- This is an **Ultrafilter** on $X = 1, 2, 3, 4$ as we cannot add any further element from $X = 1, 2, 3, 4$ that satisfies **closures** on **intersection**.

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

Filter Examples (6)

- There is another collection of sets taken from $X = 1, 2, 3, 4$
- which is the powerset
 $P = \{1, 2, 3, 4, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$
- This contain the **null element** $\emptyset =$ so we cannot call this as **Ultrafilter**.
- This is not a **proper filter** according to the article in Wikipedia.
- In the **powerset** every **intersection** of element is again in the **powerset** again but it contains t
- he **null element** $\emptyset =$ and isn't classified as proper filter.

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

Filter Examples (7)

- There is another collection of sets taken from $X = 1, 2, 3, 4$
- which is the powerset
 $P = \{1, 2, 3, 4, \{1, 2, 1, 3, 1, 4, 2, 3, 2, 4, 3, 4, 1, 2, 3, 1, 2, 4, 1, 3, 4, 2, 3, 4, 1, 2, 3, 4\}$
- This contain the **null element** $\emptyset =$ so we cannot call this as **Ultrafilter**.
- This is not a **proper filter** according to the article in Wikipedia.
- In the **powerset** every **intersection** of element is again in the **powerset** again but it contains t
- he **null element** $\emptyset =$ and isn't classified as proper filter.

[https://en.wikipedia.org/wiki/Filter_\(set_theory\)#filter_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)

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Topology

- **topology**
from the Greek words
τόπος, 'place, location',
and λόγος, 'study'

<https://en.wikipedia.org/wiki/Topology>

Topology (2)

- **topology** is concerned with the properties of a **geometric object** that are preserved
 - under **continuous deformations** such as
 - stretching
 - twisting
 - crumpling
 - bending
 - that is, without
 - closing holes
 - opening holes
 - tearing
 - gluing
 - passing through itself

<https://en.wikipedia.org/wiki/Topology>

Topological space (1)

- a **topological space** is, roughly speaking,
a **geometrical space**
in which **closeness** is defined
but cannot necessarily be **measured**
by a **numeric distance**.

https://en.wikipedia.org/wiki/Borel_set

Topological space (2)

- More specifically, a **topological space** is
 - a set whose elements are called points,
 - along with an additional structure called a topology,
- which can be defined as
 - a set of neighbourhoods for each point
 - that satisfy some axioms
formalizing the concept of closeness.

https://en.wikipedia.org/wiki/Borel_set

Topological space (3)

- There are several *equivalent definitions* of a **topology**, the most commonly used of which is the **definition** through **open sets**, which is easier than the others to manipulate.

https://en.wikipedia.org/wiki/Borel_set

Topological space (4)

- A **topological space** is the most **general type** of a **mathematical space** that allows for the definition of
 - **limits**
 - **continuity**
 - **connectedness**
- Although very **general**, the concept of **topological spaces** is **fundamental**, and used in virtually every branch of modern mathematics.
- The study of **topological spaces** in their own right is called **point-set topology** or **general topology**.

https://en.wikipedia.org/wiki/Topological_space

Topological space (5)

- Common types of **topological spaces** include
 - **Euclidean spaces** : a **set** of **points** satisfying certain **relationships**, expressible in terms of **distance** and **angles**.
 - **metric spaces** : a **set** together with a notion of **distance** between **points**. The **distance** is measured by a function called a **metric** or **distance function**.
 - **manifolds** : a topological space that *locally* resembles **Euclidean space** near each point. More precisely, an **n-manifold** is a topological space with the property that each **point** has a **neighborhood** that is **homeomorphic** to an **open subset** of **n-dimensional Euclidean space**.

https://en.wikipedia.org/wiki/Topological_space

Discrete Topology

- a **discrete space** is a **topological space**,
in which the **points** form a **discontinuous sequence**,
meaning they are isolated from each other in a certain sense.
- The **discrete topology** is
the finest **topology** that can be given on a **set**.
 - every **subset** is **open**
 - every **singleton subset** is an **open set**

https://en.wikipedia.org/wiki/Discrete_space

Singleton

- a **singleton**, also known as a **unit set** or **one-point set**, is a **set** with exactly one element.
- for example, the **set** $\{0\}$ is a **singleton** whose single element is 0

https://en.wikipedia.org/wiki/Discrete_space

Indiscrete Space (1)

- a **topological space** with the **trivial topology** is one where the only **open sets** are the **empty set** and the **entire space**.
- Such spaces are commonly called **indiscrete**, **anti-discrete**, **concrete** or **codiscrete**.
 - every **subset** can be **open** (the **discrete topology**), or
 - no **subset** can be **open** (the **indiscrete topology**) except the space itself and the empty set .

https://en.wikipedia.org/wiki/Discrete_space

Indiscrete Space (2)

- Intuitively, this has the consequence that all **points** of the **space** are "**lumped together**" and cannot be **distinguished** by topological means (not topologically **distinguishable** points)
- Every **indiscrete space** is a **pseudometric space** in which the **distance** between any two **points** is zero.

https://en.wikipedia.org/wiki/Discrete_space

T_0 Space

- a **topological space** X is a T_0 **space** or **if** for every **pair** of distinct points of X , at least one of them has a **neighborhood** not containing the other.
- In a T_0 **space**, all **points** are topologically distinguishable.
- This condition, called the T_0 **condition**, is the weakest of the **separation axioms**.
- Nearly all topological spaces *normally* studied in mathematics are T_0 **space**.

https://en.wikipedia.org/wiki/Kolmogorov_space

Topologically distinguishable points

- Intuitively, an **open set** provides a *method* to *distinguish* two **points**.
- two points in a **topological space**, there exists an **open set**
 - containing one point but
 - not containing the other (distinct) point
 - the two points are **topologically distinguishable**.

https://en.wikipedia.org/wiki/Open_set

Topologically distinguishable points

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Metric spaces

- In this manner, one may speak of whether two **points**, or more generally two **subsets**, of a **topological space** are "**near**" without concretely defining a **distance**.
- Therefore, **topological spaces** may be seen as a generalization of **spaces** equipped with a notion of **distance**, which are called **metric spaces**.

https://en.wikipedia.org/wiki/Open_set

Outline

- 1 Open Sets and Neighborhoods
 - Open Set
 - Neighborhood
 - Class
- 2 Filters
 - Filter
 - Proper Filter and Ultra Filter
 - Filter Example
- 3 Topological Space
 - Topological Space
 - A discrete topology
 - Examples of topology

Why called a discrete topology? (1)

- the **discrete topology** is the **finest** topology
- it cannot be subdivided further.
- if you think of the elements of the set
as **indivisible "discrete"** atoms,
each one appears as a **singleton set**.
- can effectively "see" the **individual points**
in the topology itself.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-called-...>

Why called a discrete topology? (2)

- the **indiscrete topology** consists only of X itself and \emptyset .
- This topology obscures everything about *how many points* were in the original set.
- It fully agglomerates the points of the set together.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-called-...>

Why called a discrete topology? (3)

- helpful to think of **topologies** as **obscuring** or **blurring** together the *underlying points* of the set.
- **topologies** are all about **nearness relations**: points in an **open set** are in the vicinity of one another.
- **topologically indistinguishable** points points that never appear alone in an **open set**,
 - they are so **close** as to be **identical**, from the perspective of the topology,

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-called-...>

Why called a discrete topology? (4)

- the **discrete topology**
 - has no **indistinguishable points**.
 - **obscures** nothing about the underlying set.
 - each **point** in the set is
 - clearly highlighted
 - distinguishable
 - recoverable as an **open singleton set** in the topology.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-called-...>

Why called a discrete topology? (5)

- If you think of **topologies** that can arise from **metrics**, the **discrete topology** arises from **metrics** such as

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

- This metric "shatters" the **points** X , isolating each one within its own **unit ball**.
 - In such a space, the only **convergent sequences** are the ones that are eventually constant;
 - you can't find points **arbitrarily close** to any other points.
 - because points are **isolated** in this way,
 - it makes sense to call the space "**discrete**".

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-called-...>

Why called a discrete topology? (6-1)

- Every **function** from a **discrete space** is automatically continuous.
- for this reason, the **discrete topology** is the one that best "represents" X in **topological space**.
- Indeed, in many ways the nature of a **set** is characterized by its **functions**,
- and the nature of a **topological space** is characterized by its continuous functions.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-called-...>

Why called a discrete topology? (6-2)

- So, note that if T is any **topological space**, there's a natural bijjective correspondence between **functions** $f : X \rightarrow \text{set}(T)$ and continuous morphisms $g : \text{discrete}(X) \rightarrow T$.
- For every **function** on X , you can find a continuous function on $\text{discrete}(X)$, and given any continuous function on $\text{discrete}(X)$, you can uniquely recover a **function** on X
- The **discrete topology** best represents the structure of the **set** X which, as you say, is discretized into individual points.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-called-...>

Why called a discrete topology? (7-1)

- Throughout **abstract algebra**, **isomorphisms** describe which structures are "the same".
- A **topological isomorphism** (a **homeomorphism**) between two **topologies** says that they are essentially the same topology.
- An **isomorphism** of **sets** is just a **bijection**;
- it says that the **sets** contain the same number of **elements**.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-called-...>

Why called a discrete topology? (7-3)

- Continuing the discussion of functions above, two discrete topologies are topologically isomorphic (homeomorphic) if and only if their underlying sets are isomorphic as sets (bijective).
- Put casually, this means that the discrete-topology-creating process maintains the similarity and differences between the underlying sets: discrete topologies are the same if and only if their underlying sets are.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-called-...>

Why called a discrete topology? (8)

- This is all the more important when we realize that sets are the same when they have the same number of points.
- Hence discrete topologies are the same when (and only when) their underlying sets have "discrete points" in the same quantity.
- You can count the points in a discrete topology through isomorphisms, and the discrete topology is the only topology for which this is possible.



<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-called-...>

Outline

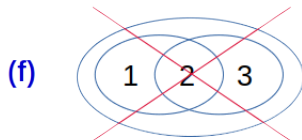
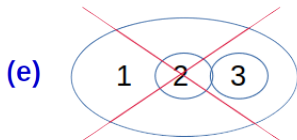
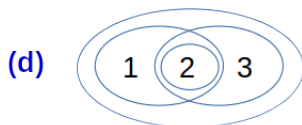
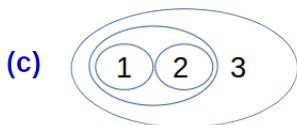
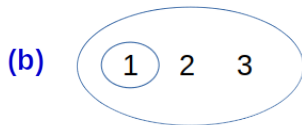
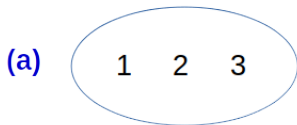
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Examples of topology (1)

- Let τ be denoted with the circles, here are four examples **(a)**, **(b)**, **(c)**, **(d)**, and two non-examples **(e)**, **(f)** of topologies on the three-point set $\{1, 2, 3\}$.
- **(e)** is not a topology because the union of $\{2\}$ and $\{3\}$ [i.e. $\{2, 3\}$] is missing;
- **(f)** is not a topology because the intersection of $\{1, 2\}$ and $\{2, 3\}$ [i.e. $\{2\}$], is missing.

https://en.wikipedia.org/wiki/Topological_space

Examples of topology (2)



Every union of (c)

(c) is a topology $\{\{\}, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$
every union of (c)

\cup	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{2\}$	$\{2\}$	$\{1,2\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$

https://en.wikipedia.org/wiki/Topological_space

Every intersection of (c)

(c) is a topology $\{\{\}, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$
every intersection of (c)

\cap	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$
$\{1\}$	$\{\}$	$\{1\}$	$\{\}$	$\{1\}$	$\{1\}$
$\{2\}$	$\{\}$	$\{\}$	$\{2\}$	$\{2\}$	$\{2\}$
$\{1,2\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2\}$
$\{1,2,3\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$

https://en.wikipedia.org/wiki/Topological_space

Every union of (f)

(f) is not a topology $\{\{\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$
every union of (f)

\cup	$\{\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3\}$
$\{2,3\}$	$\{2,3\}$	$\{1,2,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$

https://en.wikipedia.org/wiki/Topological_space

Every intersection of (f)

(f) is not a topology $\{\{\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$
every intersection of (f)

\cap	$\{\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$
$\{1,2\}$	$\{\}$	$\{1,2\}$	$\{2\}$	$\{1,2\}$
$\{2,3\}$	$\{\}$	$\{2\}$	$\{2,3\}$	$\{2,3\}$
$\{1,2,3\}$	$\{\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$

https://en.wikipedia.org/wiki/Topological_space

Examples of topology (3)

- Given $X = \{1, 2, 3, 4\}$,
the *trivial* or *indiscrete topology* on X is
the family $\tau = \{\{\}, \{1, 2, 3, 4\}\} = \{\emptyset, X\}$
consisting of only the two subsets of X
required by the axioms
forms a topology of X .

https://en.wikipedia.org/wiki/Topological_space

Examples of topology (4)

- Given $X = \{1, 2, 3, 4\}$,
the family $\tau = \{\{\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$
 $= \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, X\}$
of six **subsets** of X forms another **topology** of X .

https://en.wikipedia.org/wiki/Topological_space

Examples of topology (5)

- Given $X = \{1, 2, 3, 4\}$,
the *discrete topology* on X is
the *power set* of X , which is the family $\tau = \wp(X)$
consisting of *all possible subsets* of X .
the family

$$\begin{aligned}\tau = & \{\{\}, \{1\}, \{2\}, \{3\}, \{4\} \\ & \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \\ & \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\end{aligned}$$

- In this case the topological space (X, τ)
is called a *discrete space*.

https://en.wikipedia.org/wiki/Topological_space

Examples of topology (6)

- Given $X = \mathbb{Z}$, the set of integers, the family τ of all finite subsets of the integers plus \mathbb{Z} itself is not a topology, because (for example) the union of all finite sets not containing zero is not finite but is also not all of \mathbb{Z} , and so it cannot be in τ .

https://en.wikipedia.org/wiki/Topological_space