

# Stationarity

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles, Jr. and B. Shi

# Outline

- 1 First-Order Stationary Processes
- 2 Correlation and Covariance Functions

# First Order Stationary

$N$  Gaussian random variables

## Definition

if the first order density function does not change with a shift in time origin

$$f_X(x_1; t_1) = f_X(x_1; t_1 + \Delta)$$

must be true for any time  $t_1$  and any real number  $\Delta$  if  $X(t)$  is to be a first-order stationary

# Consequences of stationarity

## $N$ Gaussian random variables

### Definition

$f_X(x, t_1)$  is independent of  $t_1$   
the process mean value is a constant

$$m_X(t) = \bar{X} = \text{constant}$$

the process mean value  
N Gaussian random variables

## Definition

$$m_X(t) = \bar{X} = \text{constant}$$

$$m_X(t_1) = \int_{-\infty}^{\infty} x f_X(x; t_1) dx$$

$$m_X(t_2) = \int_{-\infty}^{\infty} x f_X(x; t_2) dx$$

$$m_X(t_1) = m_X(t_1 + \Delta)$$

# Second-Order Stationary Process

$N$  Gaussian random variables

## Definition

if the second order density function does not change with a shift in time origin

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

must be true for any time  $t_1, t_2$  and any real number  $\Delta$  if  $X(t)$  is to be a second-order stationary

Auto-correlation function

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] = R_{XX}(\tau)$$

# $N^{\text{th}}$ -order Stationary Processes

$N$  Gaussian random variables

## Definition

if the second order density function does not change with a shift in time origin

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N) = f_X(x_1, \dots, x_N; t_1 + \Delta, \dots, t_N + \Delta)$$

must be true for any time  $t_1, \dots, t_N$  and any real number  $\Delta$  if  $X(t)$  is to be a second-order stationary



# Wide Sense Stationary Process

$N$  Gaussian random variables

## Definition

$$m_X(t) = \bar{X} = \text{constant}$$

$$E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

## The properties of autocorrelation functions (1)

 $N$  Gaussian random variables

## Definition

$$|R_{XX}(\tau)| \leq R_{XX}(0)$$

$$R_{XX}(-\tau) = R_{XX}(\tau)$$

$$R_{XX}(0) = E[X^2(t)]$$

$$P[|X(t+\tau) - X(t)| > \varepsilon] = \frac{2}{\varepsilon^2} (R_{XX}(0) - R_{XX}(\tau))$$

# The properties of autocorrelation functions (2)

$N$  Gaussian random variables

## Definition

if  $X(t) = \bar{X} + N(t)$  where  $N(t)$  is WSS, is zero-mean, and has autocorrelation function  $R_{NN}(\tau) \rightarrow 0$  as  $|\tau| \rightarrow \infty$ , then

$$\lim_{|\tau| \rightarrow \infty} R_{XX}(\tau) = \bar{X}^2$$

if  $X(t)$  is mean square periodic, i.e, there exists a  $T \neq 0$  such that  $E[(X(t+T) - X(t))^2] = 0$  for all  $t$ , then  $R_{XX}(t)$  will have a periodic component with the same period

$R_{XX}(\tau)$  cannot have an arbitrary shape

# Crosscorrelation functions (1)

$N$  Gaussian random variables

## Definition

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

$$R_{XY}(t, t + \tau) = E[X(t)Y(t + \tau)] = R_{XY}(\tau)$$

if

$$R_{XY}(t, t + \tau) = 0$$

then  $X(t)$  and  $Y(t)$  are called orthogonal processes

# Crosscorrelation functions (2)

$N$  Gaussian random variables

## Definition

if  $X(t)$  and  $Y(t)$  are statistically independent

$$R_{XY}(t, t + \tau) = E[X(t)Y(t + \tau)] = m_X(t)m_X(t + \tau)$$

if  $X(t)$  and  $Y(t)$  are stistically independent and are at least WSS,

$$R_{XY}(\tau) = \overline{XY}$$

which is constant

## The properties of crosscorrelation functions (1)

 $N$  Gaussian random variables

## Definition

$$R_{XY}(\tau) = R_{XY}(-\tau)$$

$$|R_{XY}(\tau)| = \sqrt{R_{XX}(0)R_{YY}(0)}$$

$$|R_{XY}(\tau)| \leq \frac{1}{2} [R_{XX}(0) + R_{YY}(0)]$$

## The properties of crosscorrelation functions (2)

 $N$  Gaussian random variables

## Definition

$$R_{YX}(-\tau) = E[Y(t)X(t-\tau)] = E[Y(s+t)X(s)] = R_{XY}(\tau)$$

$$E\left[\{Y(t+\tau) + \alpha X(t)\}^2\right] \geq 0$$

the geometric mean of two positive numbers cannot exceed their arithmetic mean





