Stationary Random Processes - Examples

Young W Lim

Jun 27, 2022

Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Outline

$\sin(t)$, $A\sin(t)$

- $\sin(t)$
  - not **random process**.
- $x(t) = A\sin(t)$
  - can be a **random process** if A is a **random variable**
  - However, $x(t)$ is not **stationary**, but it is **cyclostationary**,
  - its statistical properties vary periodically.

https://dsp.stackexchange.com/questions/32000/why-is-sint-a-stationary-process

$A\sin(t+\phi)$

- $x(t) = A\sin(t+\phi)$
  - the $x(t)$ process is **stationary** because of the added **random phase**
  - the random phase $\phi \in [0, 2\pi]$ is a **uniformly distributed random variable** which is independent of A.
  - its statistical properties are independent of t, and hence, the process is **stationary**.

https://dsp.stackexchange.com/questions/32000/why-is-sint-a-stationary-process

Signals in an oscilloscope

When analyzing a signal with an oscilloscope,
it can be observed that

the signal's **amplitude spectrum**
does not vary over moving windows

so a sinusoidal wave is sort of **stationary** in frequency.

Additionally, the signal is itself **stationary** in envelope

(modulus 1 for the analytic version of the signal).

https://dsp.stackexchange.com/questions/32000/why-is-sint-a-stationary-process

Window function (1)

In signal processing and statistics, a **window function** is a mathematical function that is

- zero-valued outside of some chosen interval
- normally symmetric around the middle of the interval
- usually near a maximum in the middle
- usually tapering away from the middle.

https://en.wikipedia.org/wiki/Window_function

Window function (2)

when another function or waveform is
"multiplied" by a **window function**,

the product is also zero-valued outside the interval:
all that is left is the part where they overlap,
the "*view through the window*".

https://en.wikipedia.org/wiki/Window_function

Envelope

- the **envelope** of an oscillating signal is a smooth curve outlining its extremes.
- the **envelope** thus generalizes the concept of a constant amplitude into an instantaneous amplitude.
- a modulated sine wave varying between an upper **envelope** and a lower **envelope**.
- the **envelope function** may be a function of time, space, angle, or indeed of any variable

https://en.wikipedia.org/wiki/Envelope_(waves)

Random Variable Definition

A random variable

a real **function** over a **sample space** $S = \{s_1, s_2, s_3, \ldots, s_n\}$

$$s \rightarrow X(s)$$

$$x = X(s)$$

a random variable : a capital letter X
a particular value : a lowercase letter x

a **sample space** $S = \{s_1, s_2, s_3, \ldots, s_n\}$
an element of S : s

Random Variable Example

Example

$X(s_1) = x_1 \qquad s_1 \longrightarrow x_1$

$X(s_2) = x_2 \qquad s_2 \longrightarrow x_2$

$\cdots \qquad \cdots$

$X(s_n) = x_n \qquad s_n \longrightarrow x_n$

$S = \{ s_1, s_2, s_3, \ldots, s_n \}$ a sample space

$X = \{ x_1, x_2, x_3, \ldots, x_n \}$ a random variable

Random Process (1)

A random process

a function of both **time** $t$and **outcome** θ

$$X(t, \theta)$$

assigning a time function to every **outcome** θ_i

$$\theta_i \rightarrow x_i(t)$$

where $x_i(t) = x(t,\theta_i)$

the family of such time functions
is called a random process
and denoted by $X(t, \theta)$

Random Process (2)

A random process

a **random process** $X(t, \theta)$
assigns a **time function** for a every **outcome** θ

$$x(t,\theta) = X(t, \theta)$$

a short notation

$$x(t) = X(t)$$

Ensemble of time functions

Time functions

A random process $X(t, \theta)$ represents
a family or ensemble of time functions

$$X(t, \theta_1) = x_1(t) \qquad \theta_1 \longrightarrow x_1(t) = \cos(\omega t + \theta_1)$$
$$X(t, \theta_2) = x_2(t) \qquad \theta_2 \longrightarrow x_2(t) = \cos(\omega t + \theta_2)$$
$$\cdots \qquad \cdots$$
$$X(t, \theta_n) = x_n(t) \qquad \theta_n \longrightarrow x_n(t) = \cos(\omega t + \theta_n)$$

$S = \{ \theta_1, \theta_2, \theta_3, \ldots, \theta_n \}$ a sample space
$X(t) = \{ x_1(t), x_2(t), x_3(t), \ldots, x_n(t) \}$ a random process

A sample function $x(t, \theta)$

A random process $X(t, \theta)$ represents
a family or ensemble of time functions

$$\theta \rightarrow x(t, \theta) = \cos(\omega t + \theta)$$

$x(t, \theta)$ represents

- a sample function
- an ensemble member
- a realization of the process

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Random process $X(t, \theta)$

A random process $X(t, \theta)$ represents
a family or ensemble of time functions

$$\theta \rightarrow x(t, \theta) = \cos(\omega t + \theta)$$
$$x(t) = X(t, \theta)$$

$X(t, \theta)$ becomes

- a single time function $x(t, \theta)$
- when t is a variable and θ is fixed at an **outcome**

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Random variables with time

a **random process** $X(t, s)$ represents a single time function when t is a variable and s is fixed at an outcome

a random process $X(t, s)$ represents a single **random variable** when both t and s are fixed at a time and an outcome, respectively

$$X_i = X(t_i, s) = X(t_i) \qquad \textbf{\textit{random variable}}$$

$$X(t, s) = X(t) \qquad \textbf{\textit{random process}}$$

Random phase in $X(t) = \cos(\omega t + \Theta)$

Consider the output of a sinusoidal oscillator
that has a **random phase** and an **amplitude** of the form:

$$X(t) = \cos(\omega t + \Theta)$$

where the **random variable** $\Theta \sim U([0, 2\pi])$

to specify the explicit dependence
on the underlying **sample space** S
the oscillator output can be written as

$$x(t, \Theta) = \cos(\omega t + \Theta)$$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Random variable $X_t(\theta)$

Consider the **random variable**

$$X(t, \theta) = \cos(\omega t + \theta)$$

where the time t is fixed
In other words,

$$X_t(\theta) = \cos(\omega t + \theta)$$

where $\theta_0 = \omega t$ is fixed (a *non-random* quantity)
thus the time t is fixed

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Values of a time function

Consider the **random variable** for the fixed time t

$$X_t(\theta) = \cos(\omega t + \theta)$$

if the sample value θ as well as the time t is fixed,
then the values of the time function

$$x_1 = x(t_1) = \cos(\omega t_1 + \theta)$$
$$x_2 = x(t_2) = \cos(\omega t_2 + \theta)$$

where x is the **time function** for a fixed outcome θ and
let x_i denotes the value of the time function x at times t_i
(here x_i is not a sample function)

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Outline

$f_X(x)$ of $X(t) = \cos(\omega t + \Theta)$

- Uniform Random Variable Θ
- Random Process $X(t) = \cos(\omega t + \Theta)$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

first order distribution

To get the **first order distribution** of the **random process**

$$X(t) = \cos(\omega t + \Theta)$$

consider the **first order distribution** of the **random variable**

$$X_t(\Theta) = \cos(\theta_0 + \Theta)$$

where $\theta_0 = \omega t$ is fixed (a *non-random* quantity)

$f_X(x)$ can be obtained via the **derivative method**

$$\frac{d}{dx} F_X(x) = f_\Theta(\theta) \cdot \frac{d\theta}{dx}$$

$$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \qquad |x| < 1$$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

$f_X(x)$ of $X(t) = \cos(\omega t + \Theta)$

The **first order distribution** of the process $X(t) = \cos(\omega t + \Theta)$

$$f_X(x) = \frac{1}{\pi\sqrt{1 - x^2}}, \qquad |x| < 1$$

- dependent only on the set of values x
  that the process $X(t)$ takes
- independent of
  - the particular **sampling instant** t
  - the constant **phase offset** $\theta_0 = \omega t$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

random variable Θ

Let Θ be a uniform **random variable** on $[0, 2\pi]$, then

$$F_\Theta(\theta) = \frac{\theta}{2\pi}$$

Let

$$X_t(\Theta) = \cos(\theta_0 + \Theta)$$

be the **random variable** describing x in terms of Θ.

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

$F_X(x) = F_\Theta(\theta_1) - F_\Theta(\theta_2)$

$$
\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= P(\cos(\omega t + \Theta) \leq x) \\
&= P\left(\cos^{-1}(x) \leq \omega t + \Theta \leq 2\pi - \cos^{-1}(x)\right) \\
&= P\left(\cos^{-1}(x) - \omega t \leq \Theta \leq 2\pi - \cos^{-1}(x) - \omega t\right) \\
&= P\left(\Theta \leq 2\pi - \cos^{-1}(x) - \omega t\right) - P\left(\Theta \leq \cos^{-1}(x) - \omega t\right) \\
&= F_\Theta\left(2\pi - \cos^{-1}(x) - \omega t\right) - F_\Theta\left(\cos^{-1}(x) - \omega t\right) \\
&= F_\Theta(\theta_1) - F_\Theta(\theta_2)
\end{aligned}
$$

Random variable X, a particular value x

Random variable Θ, particular values θ_1 and θ_2

https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation

Chain rule

The chain rule

$$\frac{d}{dx}F_X(x) = \frac{d}{d\theta}F_\Theta(\theta) \cdot \frac{d\theta}{dx}$$

Random variable X, a particular value x

Random variable Θ, a particular value θ

$$\frac{d}{d\theta}F_\Theta(\theta) = f_\Theta(\theta) \qquad \frac{d}{d\theta}\left(\frac{\theta}{2\pi}\right) = \frac{1}{2\pi}$$

$$\frac{d}{dx}F_X(x) = \frac{d}{d\theta}F_\Theta(\theta) \cdot \frac{d\theta}{dx} = f_\Theta(\theta) \cdot \frac{d\theta}{dx} = \frac{1}{2\pi}\frac{d\theta}{dx}$$

https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation

derivative of $F_X(x)$

Differentiating both sides, we get:

$$\begin{aligned}\frac{d}{dx}F_X(x) &= \frac{d}{dx}\left\{F_\Theta\left(2\pi - \cos^{-1}(x) - \omega t\right) - F_\Theta\left(\cos^{-1}(x) - \omega t\right)\right\} \\ &= \frac{d}{d\theta}F_\Theta\left(2\pi - \cos^{-1}(x) - \omega t\right)\ \frac{d}{dx}\left(2\pi - \cos^{-1}(x) - \omega t\right) \\ &- \frac{d}{d\theta}F_\Theta\left(\cos^{-1}(x) - \omega t\right)\ \frac{d}{dx}\left(\cos^{-1}(x) - \omega t\right)\end{aligned}$$

note

$$\theta_1 = 2\pi - \cos^{-1}(x) - \omega t \qquad \frac{d\theta_1}{dx} = -\frac{d}{dx}\cos^{-1}(x)$$

$$\theta_2 = \cos^{-1}(x) - \omega t \qquad \frac{d\theta_2}{dx} = +\frac{d}{dx}\cos^{-1}(x)$$

https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation

$f_X(x)$ of $X(t) = \cos(\omega t + \Theta)$

$$X_t(\Theta) = \cos(\omega t + \Theta)$$

$$\cos^{-1}(x) \le \omega t + \Theta \le 2\pi - \cos^{-1}(x)$$

$$F_X(x) = F_\Theta\left(2\pi - \cos^{-1}(x) - \omega t\right) - F_\Theta\left(\cos^{-1}(x) - \omega t\right)$$

using the chain rule

$$\frac{d}{dx}F_X(x) = \frac{d}{d\theta}F_\Theta(\theta)\frac{d\theta}{dx} = f_\Theta(\theta)\frac{d\theta}{dx} = \frac{1}{2\pi}\frac{d\theta}{dx}$$

$$f_X(x) = f_\Theta\left(2\pi - \cos^{-1}(x) - \omega t\right)\ \frac{d}{dx}\left(-\cos^{-1}(x)\right)$$
$$- f_\Theta\left(\cos^{-1}(x) - \omega t\right)\ \frac{d}{dx}\left(\cos^{-1}(x)\right)$$

https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation

$f_X(x)$ of $X(t) = \cos(\omega t + \Theta)$

$$f_X(x) = f_\Theta\left(2\pi - \cos^{-1}(x) - \omega t\right) \frac{d}{dx}\left(-\cos^{-1}(x)\right) - f_\Theta\left(\cos^{-1}(x) - \omega t\right) \frac{d}{dx}\left(\cos^{-1}(x)\right)$$

Now, since $f_\Theta(\theta) = \frac{1}{2\pi}$ and $\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$, we have:

$$f_X(x) = \frac{1}{2\pi}\left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right) = \frac{1}{\pi\sqrt{1-x^2}}$$

https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation

$f_X(x)$ of $X(t) = \cos(\omega t + \Theta)$ (10)

Consider the output of a sinusoidal oscillator
that has a random phase and an amplitude of the form:

$$X(t) = cos(\omega t + \Theta)$$

where Θ is a uniform random variable on $[0, 2\pi]$
then the **first order pdf** of $X(t)$ is

$$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \qquad x \in (-1, 1)$$

Note that the probability is unaffected by angular velocity
and initial phase (ω, θ_0), which is, intuitively, expected.

https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation

$f_X(x)$ of $X = \cos(\omega T + \phi)$ (1)

- Uniform Random Variable T
- Random Variable $X = \cos(\omega T + \phi)$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

$f_X(x)$ of $X = \cos(\omega T + \phi)$ (2)

Let T be a uniform **random variable** on $[0, \frac{2\pi}{\omega}]$ that describes time. Then

$$F_T(t) = \frac{\omega}{2\pi} \cdot t = ft$$

where f is the oscilation's frequency. Now, let:

$$X = \cos(\omega T + \phi)$$

be the **random variable** describing x in terms of T.
it is not a time function

$$X(t) \neq \cos(\omega T + \phi)$$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

$f_X(x)$ of $X = \cos(\omega T + \phi)$ (3)

$$
\begin{aligned}
F_X(x) &= P(X \le x) \\
&= P(\cos(\omega T + \phi) \le x) \\
&= P\left(\cos^{-1}(x) \le \omega T + \phi \le 2\pi - \cos^{-1}(x)\right) \\
&= P\left(\frac{\cos^{-1}(x) - \phi}{\omega} \le T \le \frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right) \\
&= P\left(T \le \frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right) - P\left(T \le \frac{\cos^{-1}(x) - \phi}{\omega}\right) \\
&= F_T\left(\frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right) - F_T\left(\frac{\cos^{-1}(x) - \phi}{\omega}\right) \\
&= F_T(t_1) - F_T(t_2)
\end{aligned}
$$

Random variable X, a particular value x

Random variable T, particular values t_1 and t_2

https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation

$f_X(x)$ of $X(t) = \cos(\omega T + \phi)$ (4)

The chain rule

$$\frac{d}{dt} F_T(t) = \frac{d}{d\theta} F_\Theta(\theta) \cdot \frac{dt}{dx}$$

Random variable T, a particular value t

Random variable Θ, a particular value θ

$$\frac{d}{dt} F_T(t) = f_T(t) \qquad \frac{d}{dt}\left(\frac{\omega}{2\pi} \cdot t\right) = \frac{\omega}{2\pi}$$

$$\frac{d}{dt} F_T(t) = \frac{d}{dt} F_T(t) \cdot \frac{dt}{dx} = f_T(t) \cdot \frac{dt}{dx} = \frac{\omega}{2\pi} \frac{dt}{dx}$$

https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation

$f_X(x)$ of $X(t) = \cos(\omega T + \phi)$ (5)

Differentiating both sides, we get:

$$\begin{aligned}
\frac{d}{dx}F_X(x) &= \frac{d}{dx}\left\{F_T\left(\frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right) - F_T\left(\frac{\cos^{-1}(x) - \phi}{\omega}\right)\right\} \\
&= \frac{d}{dt}F_T\left(\frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right)\ \frac{d}{dx}\left(\frac{\pi - \cos^{-1}(x) - \phi}{\omega}\right) \\
&- \frac{d}{dt}F_T\left(\frac{\cos^{-1}(x) - \phi}{\omega}\right)\ \frac{d}{dx}\left(\frac{\cos^{-1}(x) - \phi}{\omega}\right)
\end{aligned}$$

note

$$t_1 = \frac{2\pi - \cos^{-1}(x) - \phi}{\omega} \qquad \frac{dt_1}{dx} = \frac{-\cos^{-1}(x)}{\omega}$$

$$t_2 = \frac{\cos^{-1}(x) - \phi}{\omega} \qquad \frac{dt_2}{dx} = \frac{+\cos^{-1}(x)}{\omega}$$

https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation

$f_X(x)$ of $X(t) = \cos(\omega T + \phi)$ (6)

$$X(t) = \cos(\omega T + \phi)$$

$$\cos^{-1}(x) \le \omega T + \phi \le 2\pi - \cos^{-1}(x)$$

$$F_X(x) = F_T\left(\frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right) - F_T\left(\frac{\cos^{-1}(x) - \phi}{\omega}\right)$$

using the chain rule

$$\frac{d}{dt}F_T(t) = \frac{d}{dt}F_T(t)\cdot\frac{dt}{dx} = f_T(t)\cdot\frac{dt}{dx} = \frac{\omega}{2\pi}\frac{dt}{dx}$$

$$f_X(x) = f_T\left(\frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right)\ \frac{d}{dx}\left(-\frac{\cos^{-1}(x)}{\omega}\right)$$
$$- f_T\left(\frac{\cos^{-1}(x) - \phi}{\omega}\right)\ \frac{d}{dx}\left(\frac{\cos^{-1}(x)}{\omega}\right)$$

https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation

$f_X(x)$ of $X = \cos(\omega T + \phi)$ (7)

Differentiating both sides, we get:

$$
\begin{aligned}
f_X(x) = f_T\left(\frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right) \frac{d}{dx}\left(-\frac{\cos^{-1}(x)}{\omega}\right) \\
- f_T\left(\frac{\cos^{-1}(x) - \phi}{\omega}\right) \frac{d}{dx}\left(\frac{\cos^{-1}(x)}{\omega}\right)
\end{aligned}
$$

Now, since $f_T(t) = \frac{\omega}{2\pi}$ and $\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$, we have:

$$
\begin{aligned}
f_X(x) &= \frac{1}{2\pi}\left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right) \\
&= \frac{1}{\pi\sqrt{1-x^2}}
\end{aligned}
$$

https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation

$f_X(x)$ of $X = \cos(\omega T + \phi)$ (8)

$$f_X(x) = \frac{1}{\pi\sqrt{1^2 - x^2}}, \quad x \in (-1, 1)$$

the probability is unaffected by angular velocity (ω) and initial phase (ϕ), which is, intuitively, expected.

https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation

Outline

Second order distribution (1)

to get the **second-order distribution**
use the **conditional distribution** $f_{X(t_1)|X(t_2)}(x_1|x_2)$
as in :

$$f_{X(t_1),X(t_2)}(x_1,x_2) = f_{X(t_2)}(x_2) f_{X(t_1)|X(t_2)}(x_1|x_2)$$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Second order distribution (2)

$x_2 = x(t_2) = \cos(\omega t_2 + \theta)$
This can happen only when :

$$(\omega t_2 + \theta) = \cos^{-1}(x_2)$$
$$(\omega t_2 + \theta) = 2\pi - \cos^{-1}(x_2)$$

$$\theta = \cos^{-1}(x_2) - \omega t_2$$
$$\theta = 2\pi - \cos^{-1}(x_2) - \omega t_2$$

where $0 \leq \cos^{-1}(x_2) \leq \pi$ and $0 \leq \theta \leq 2\pi$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Second order distribution (3)

given that $x_2 = x(t_2) = \cos(\omega t_2 + \theta)$:
find θ,

$$\theta = \begin{cases} +\left(\cos^{-1}(x_2) - \omega t_2\right) \\ -\left(\cos^{-1}(x_2) + \omega t_2\right) \end{cases}$$

then $x_1 = x(t_1) = \cos(\omega t_1 + \theta)$ have two values

$$x(t_1) = \begin{cases} \cos\left(\omega t_1 + \left(\cos^{-1}(x(t_2)) - \omega t_2\right)\right) = x_{11} \\ \cos\left(\omega t_1 - \left(\cos^{-1}(x(t_2)) + \omega t_2\right)\right) = x_{12} \end{cases}$$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Second order distribution (4)

given that $x_2 = x(t_2) = \cos(\omega t_2 + \theta)$
find θ, then $x_1 = x(t_1) = \cos(\omega t_1 + \theta)$
has only two values with an equal probability 0.5

$$x(t_1) = \begin{cases} \cos\left(\omega t_1 + \left(\cos^{-1}(x(t_2)) - \omega t_2\right)\right) = x_{11} \\ \cos\left(\omega t_1 - \left(\cos^{-1}(x(t_2)) + \omega t_2\right)\right) = x_{12} \end{cases}$$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Second order distribution (5)

the **conditional distribution** of $x(t_1) = x_1$ given that $x(t_2) = x_2$:

$$f_{X(t_1)|X(t_2)}(x_1|x_2) = \left(\frac{1}{2}\delta(x_1 - x_{11}) + \frac{1}{2}\delta(x_1 - x_{12}) \right)$$
$$= \frac{1}{2}\delta\left(x_1 - \cos\left[\omega t_1 + \left(\cos^{-1}(x_2) - \omega t_2\right)\right]\right)$$
$$+ \frac{1}{2}\delta\left(x_1 - \cos\left[\omega t_1 - \left(\cos^{-1}(x_2) + \omega t_2\right)\right]\right)$$

$$f_{X(t_1)|X(t_2)}(x(t_1)|x(t_2)) = \left(\frac{1}{2}\delta(x(t_1) - x_{11}) + \frac{1}{2}\delta(x(t_1) - x_{12}) \right)$$
$$= \frac{1}{2}\delta\left(x(t_1) - \cos\left[\omega t_1 + \left(\cos^{-1}(x(t_2)) - \omega t_2\right)\right]\right)$$
$$+ \frac{1}{2}\delta\left(x(t_1) - \cos\left[\omega t_1 - \left(\cos^{-1}(x(t_2)) + \omega t_2\right)\right]\right)$$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

First order distribution $f_X(x)$ (1)

the **first order distribution** of $x(t_2) = x_2 = \cos(\omega t_2 + \theta)$:

$$f_{X(t_2)}(x_2) = \frac{1}{2\pi\sqrt{1 - x_2^2}}$$

$$f_{X(t_2)}(x(t_2)) = \frac{1}{2\pi\sqrt{1 - x^2(t_2)}}$$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

First order distribution $f_X(x)$ (2)

the **first order distribution** $f_X(x)$ of $X(t,\theta) = \cos(\omega t + \theta)$

- dependent only on the set of values x $(-1 \leq x \leq 1)$ that the process $X(t,\theta)$ takes
- independent of
  - the particular sampling instant t
  - the constant phase offset $\theta_0 = \omega t$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Second order distribution (7)

The **second order pdf** of the process $X(t) = cos(\omega t + \Theta)$

$$f_{X(t_1),X(t_2)}(x_1, x_2) = f_{X(t_1)}(x_1) f_{X(t_2)|X(t_1)}(x_2|x_1)$$
$$= f_{X(t_1)}(x_1) \left(\frac{1}{2}\delta(x_2 - x_{21}) + \frac{1}{2}\delta(x_2 - x_{22}) \right)$$
$$f_{X(t_1),X(t_2)}(x_1, x_2) = f_{X(t_2)}(x_2) f_{X(t_1)|X(t_2)}(x_1|x_2)$$
$$= f_{X(t_1)}(x_2) \left(\frac{1}{2}\delta(x_1 - x_{11}) + \frac{1}{2}\delta(x_1 - x_{12}) \right)$$

where $x(t_1) = x_1$ and $x(t_2) = x_2$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Second order distribution (8)

$$f_{X(t_1),X(t_2)}(x_1,x_2) = f_{X(t_2)}(x_2) f_{X(t_1)|X(t_2)}(x_1|x_2)$$

$$= \left\{ \frac{1}{2\pi\sqrt{1-x_2^2}} \right\} \delta\left(x_1 - \cos\left[\omega t_1 + \left(\cos^{-1}(x_2) - \omega t_2\right)\right]\right)$$

$$+ \left\{ \frac{1}{2\pi\sqrt{1-x_2^2}} \right\} \delta\left(x_1 - \cos\left[\omega t_1 - \left(\cos^{-1}(x_2) + \omega t_2\right)\right]\right)$$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Second order distribution (9)

$$f_{X(t_1),X(t_2)}(x(t_1),x(t_2)) = f_{X(t_2)}(x(t_2))f_{X(t_1)|X(t_2)}(x(t_1)|x(t_2))$$

$$= \left\{ \frac{1}{2\pi\sqrt{1-x^2(t_2)}} \right\} \delta\left(x(t_1) - \cos\left[\omega t_1 + \left(\cos^{-1}(x(t_2)) - \omega t_2\right)\right]\right)$$
$$+ \left\{ \frac{1}{2\pi\sqrt{1-x^2(t_2)}} \right\} \delta\left(x(t_1) - \cos\left[\omega t_1 - \left(\cos^{-1}(x(t_2)) + \omega t_2\right)\right]\right)$$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Second order distribution (10)

The **second order pdf** can thus be written as

$$\begin{aligned} f_{X(t_1),X(t_2)}(x_1,x_2) &= f_{X(t_2)}(x_2) f_{X(t_2)|X(t_1)}(x_1|x_2) \\ &= f_{X(t_2)}(x_2)\left(\frac{1}{2}\delta(x_1 - x_{11}) + \frac{1}{2}\delta(x_1 - x_{12})\right) \end{aligned}$$

$$\begin{aligned} f_{X(t_1),X(t_2)}(x(t_1),x(t_2)) &= f_{X(t_2)}(x(t_2)) f_{X(t_2)|X(t_1)}(x(t_1)|x(t_2)) \\ &= f_{X(t_2)}(x(t_2))\left(\frac{1}{2}\delta(x(t_1) - x_{11}) + \frac{1}{2}\delta(x(t_1) - x_{12})\right) \end{aligned}$$

These depend only on $t_2 - t_1$,
and thus **the second order pdf** is **stationary**

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Second order distribution (11)

given that $x(t_2) = x_2 = \cos(\omega t_2 + \theta)$
find θ, then $x(t_1) = x_1 = \cos(\omega t_1 + \theta)$
has only two values with an equal probability 0.5

$$x(t_1) = \begin{cases} x_{11} = \cos\left(\omega t_1 + \left(\cos^{-1}(x(t_2)) - \omega t_2\right)\right) \\ x_{12} = \cos\left(\omega t_1 - \left(\cos^{-1}(x(t_2)) + \omega t_2\right)\right) \end{cases}$$

$$\begin{aligned} f_{X(t_1),X(t_2)}(x(t_1),x(t_2)) &= f_{X(t_2)}(x(t_2)) f_{X(t_2)|X(t_1)}(x(t_1)|x(t_2)) \\ &= f_{X(t_2)}(x(t_2)) \left(\frac{1}{2}\delta(x(t_1) - x_{11}) + \frac{1}{2}\delta(x(t_1) - x_{12}) \right) \end{aligned}$$

These depend only on $t_2 - t_1$,
and thus **the second order pdf** is **stationary**

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Second order distribution (12)

$\delta(x(t_1) - x_{11})$ when $x(t_1)$ is equal to $x_{11} = \cos(\omega t_1 + \theta_1)$
$\delta(x(t_1) - x_{12})$ when $x(t_1)$ is equal to $x_{12} = \cos(\omega t_1 + \theta_2)$

$$f_{X(t_2)}(x(t_2)) = \frac{1}{2\pi\sqrt{1 - x^2(t_2)}}$$

These depend only on $t_2 - t_1$,
and thus **the second order pdf** is **stationary**

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Second-Order Stationary Process

$f_X(x_1, x_2; t_1, t_2)$

if $X(t)$ is to be a **second-order stationary**

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

must be true for any time t_1, t_2 and any real number Δ

the second order density function
does not change with a shift in time origin

Second-Order Stationary Process

$f_X(x_1, x_2; t_1, t_2)$

- $f_X(x_1, x_2; t_1, t_2)$ is independent of t_1 and t_2
  the second order density function
  does not change with a shift in time origin

- the **autocorrelation function**

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

Outline

Example: $X(t) = \cos(\omega t + \Theta)$

- the random process $X(t)$
- the **first-order** moments μ_X
- the **second-order moments** σ_X^2

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Example: $X(t) = \cos(\omega t + \Theta)$

The **mean** of the process is obtained
by taking the **expectation** operator
with respect to the **random** parameter Θ on both sides

$$X_t(\Theta) = \cos(\omega t + \Theta)$$
$$E_\Theta\left[X_t(\Theta)\right] = E_\Theta[\cos(\omega t + \Theta)]$$

note that the **expectation** integral is a linear operation:

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Example: $X(t) = \cos(\omega t + \Theta)$

$$\begin{aligned}\mu_X = E_\Theta\left[X_t(\Theta)\right] &= E_\Theta[\cos(\omega t + \Theta)] \\ &= E_\Theta[\cos(\omega t)\cos(\Theta) - \sin(\omega t)\sin(\Theta)] \\ &= E_\Theta[\cos(\Theta)]\cos(\omega t) - E_\Theta[\sin(\Theta)]\sin(\omega t)\end{aligned}$$

Since the random parameter Θ is uniformly distributed

$$\begin{aligned}\mu_X &= E_\Theta[\cos(\Theta)]\cos(\omega t) - E_\Theta[\sin(\Theta)]\sin(\omega t) \\ &= \cos(\omega t)\left(\frac{1}{2\pi}\right)\int_0^{2\pi}\cos(\theta)d\theta - \sin(\omega t)\left(\frac{1}{2\pi}\right)\int_0^{2\pi}\sin(\theta)d\theta \\ &= 0\end{aligned}$$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Example: $X(t) = \cos(\omega t + \Theta)$

The variance of the random process X(t)

$$\sigma_X^2 = E_\Theta[(x_t(\Theta) - \mu_X)^2] = E_\Theta\left[[x_t(\Theta)]^2\right] - \mu_X^2$$

Substituting the mean of the process

$$\begin{aligned}\sigma_X^2 &= \left(\frac{1}{2\pi}\right)\int_0^{2\pi}\cos^2(\omega t + \theta)d\theta \\ &= \left(\frac{1}{2\pi}\right)\int_0^{2\pi}[1 + \cos(2\omega t + 2\theta)2]d\theta \\ &= \frac{1}{2}\end{aligned}$$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Example: $X(t) = \cos(\omega t + \Theta)$

the average power of the random sinusoidal signal X(t)

$$P_{ave}^{X} = \sigma_X^2 = \frac{1}{2}$$

.

the same as the average power of a sinusoid the phase is not random

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Example: $X(t) = \cos(\omega t + \Theta)$

the correlation between the R.Vs $x(t_1)$ and $x(t_2)$ denoted as $R_{XX}(t_1, t_2)$

$$
\begin{aligned}
R_{XX}(t_1, t_2) = E_\Theta[x(t_1)x(t_2)] &= \int_0^{2\pi} \cos[\omega t_1 + \theta] \cos[\omega t_2 + \theta] d\theta \\
&= \left(\frac{1}{4\pi}\right) \int_0^{2\pi} \cos[\omega(t_1 + t_2) + 2\theta] d\theta \\
&+ \left(\frac{1}{4\pi}\right) \int_0^{2\pi} \cos[\omega(t_1 - t_2)] d\theta \\
&= \left(\frac{1}{2}\right) cos[\omega(t_1 - t_2)]
\end{aligned}
$$

.

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Example: $X(t) = \cos(\omega t + \Theta)$

The covariance of R.Vs $X(t_1)$ and $X(t_2)$ denoted $C_{XX}(t_1, t_2)$

$$C_{XX}(t_1, t_2) = R_{xx}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) = \left(\frac{1}{2}\right)\cos[\omega(t_1 - t_2)]$$

The correlation coefficient of the R.Vs $X(t_1)$ and $X(t_2)$ denoted $\rho_{XX}(t_1, t_2)$

$$\rho_{XX}(t_1, t_2) = \cos[\omega(t_1 - t_2)]$$

.

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Example: $X(t) = \cos(\omega t + \Theta)$

Looking at the **mean** and the **variance**
of the random process $X(t)$
we can see that they are shift-invariant and
consequently the process is **first-order stationary**.
The ACF and other second-order statistics of the process are
dependent only on the variable $\tau = t_1 - t_2$.
The random process $X(t)$ is therefore a **WSS** process also.
The ACF can then expressed in terms of the variable $\tau = t_1 - t_2$ as:

$$R_{XX}(\tau) = \left(\frac{1}{2}\right)\cos(\omega\tau)$$

http://ece-research.unm.edu/bsanthan/ece541/examp.pdf

Outline

Example A.1: $X(t) = \cos(\omega t)$

A **white noise** is not necessarily **strictly stationary**.

Let ω be a **random variable uniformly distributed** in the interval $(0, 2\pi)$

define the time series $\{X(t)\}$

$$X(t) = \cos(\omega t) \quad (t = 1, 2, \ldots)$$

https://en.wikipedia.org/wiki/Stationary_process

Example A.1: $X(t) = \cos(\omega t)$

Then

$$E[X(t)] = \frac{1}{2\pi}\int_0^{2\pi} \cos(t\omega)d\omega = 0$$
$$Var(X(t)) = \frac{1}{2\pi}\int_0^{2\pi} \cos^2(t\omega)d\omega = 1/2$$
$$Cov(x(t), x(s)) = \frac{1}{2\pi}\int_0^{2\pi} \cos(t\omega)\cos(s\omega)d\omega = 0 \quad \forall t \neq s$$

So $\{X(t)\}$ is a **white noise**,
however it is not **strictly stationary**.

https://en.wikipedia.org/wiki/Stationary_process

Example A.2: $X(t) = \cos(t + U)$

a **stationary process** example
for which any single realisation has
an apparently noise-free structure,

Let U have a uniform distribution on $(0, 2\pi]$ and
define the time series $\{X(t)\}$ by

$$X(t) = \cos(t + U) \quad \text{for } t \in \mathbb{R}$$

then $\{X(t)\}$ is **strictly stationary** (**SSS**).

https://en.wikipedia.org/wiki/Stationary_process

Example A.2: $X(t) = \cos(t + U)$

Show that $X(t)$ is a **WSS** process.
We need to check two conditions:

$$\mu_X(t) = \mu_X \quad \text{for } t \in \mathbb{R}$$

$$R_X(t_1, t_2) = R_X(t_1 - t_2) \quad \text{for } t_1, t_2 \in \mathbb{R}$$

https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php

Example A.2: $X(t) = \cos(t + U)$

$$
\begin{aligned}
\mu_X(t) &= E[X(t)] \\
&= E[\cos(t+U)] \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(t+u)\,du \\
&= 0, \qquad \text{for all } t \in \mathbb{R}.
\end{aligned}
$$

https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php

Example A.2: $X(t) = \cos(t + U)$

$$
\begin{aligned}
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= E[\cos(t_1 + U)\cos(t_2 + U)] \\
&= E\left[\frac{1}{2}\cos(t_1 + t_2 + 2U) + \frac{1}{2}\cos(t_1 - t_2)\right] \\
&= E\left[\frac{1}{2}\cos(t_1 + t_2 + 2U)\right] + E\left[\frac{1}{2}\cos(t_1 - t_2)\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(t_1 + t_2 + u)\, du + \frac{1}{2}\cos(t_1 - t_2) \\
&= 0 + \frac{1}{2}\cos(t_1 - t_2) = \frac{1}{2}\cos(t_1 - t_2), \quad \text{for all } t_1, t_2 \in \mathbb{R}.
\end{aligned}
$$

https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php

Example A.3: $X(t) = \alpha \cos(\omega t + \Theta)$

The random phase signal $X(t) = \alpha cos(\omega t + \Theta)$
where $\Theta \in U[0, 2\pi]$ is **SSS**

it is known that the **first order pdf** is

$$f_{X(t)}(x) = \frac{1}{\pi\alpha\sqrt{1-(x/\alpha)^2}}, \quad -\alpha < x < +\alpha$$

which is independent of t, and is therefore **stationary**

http://isl.stanford.edu/~abbas/ee278/lect07.pdf

Example A.3: $X(t) = \alpha \cos(\omega t + \Theta)$

To find the **second order pdf**,
note that if we are given the value of $X(t)$ at one point, say t_1,
there are (at most) two possible **sample functions**

- $X(t_1) = x_1$
  - at t_1, two sinusoid waves intersect with each other
- $X(t_2) = x_{21}$ or x_{22}
  - at t_2, two sinusoid waves do not intersect with each other

http://isl.stanford.edu/~abbas/ee278/lect07.pdf

Example A.3: $X(t) = \alpha \cos(\omega t + \Theta)$

The **second order pdf** can thus be written as

$$f_{X(t_1),X(t_2)}(x_1, x_2) = f_{X(t_1)}(x_1) f_{X(t_2)|X(t_1)}(x_2|x_1)$$
$$= f_{X(t_1)}(x_1) \left(\frac{1}{2}\delta(x_2 - x_{21}) + \frac{1}{2}\delta(x_2 - x_{22}) \right)$$

which depends only on $t_2 - t_1$,
and thus the second order pdf is **stationary**

http://isl.stanford.edu/~abbas/ee278/lect07.pdf

Example A.3: $X(t) = \alpha \cos(\omega t + \Theta)$

- if we know that $X(t_1) = x_1$ and $X(t_2) = x_2$, the sample path is totally determined except when $x_1 = x_2 = 0$,
- when $x_1 = x_2 = 0$, two paths may be possible
- thus all **n-th order pdf**s are **stationary**

http://isl.stanford.edu/~abbas/ee278/lect07.pdf

Outline

Example B.1: $X(t) = Y$

Let Y be any scalar **random variable**,
and define a time-series $\{X(t)\}$, by

$$X(t) = Y \qquad \text{for all } t.$$

Then $\{X(t)\}$ is a **stationary** time series

- **realisations** consist of a series of **constant** values,
- a different **constant** value for each **realisation**.

https://en.wikipedia.org/wiki/Stationary_process

Example B.1: $X(t) = Y$

$$X(t) = Y \qquad \text{for all } t.$$

$X(t)$ is a **first-order stationary**

$$f_X(x_1; t_1) = f_X(x_1; t_1 + \Delta) = const$$

$X(t)$ is a **second-order stationary**

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta) = const$$

$X(t)$ is to be a $\boldsymbol{N^{th}}$**-order stationary**

$$f_X(x_1, \cdots, x_N; t_1, \cdots, t_N) = f_X(x_1, \cdots, x_N; t_1 + \Delta, \cdots, t_N + \Delta) = const$$

https://en.wikipedia.org/wiki/Stationary_process

Example B.2: $Z(t) = X(t) + Y(t)$

Let $X(t)$ and $Y(t)$ be
two jointly **WSS** random processes.

Consider the random process $Z(t)$

$$Z(t) = X(t) + Y(t)$$

.
Show that $Z(t)$ is **WSS**.

https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php

Example B.2: $Z(t) = X(t) + Y(t)$

Since $X(t)$ and $Y(t)$ are jointly WSS, we conclude

$$\mu_{X(t)} = \mu_X$$
$$\mu_{Y(t)} = \mu_Y$$
$$R_X(t_1, t_2) = R_X(t_1 - t_2)$$
$$R_Y(t_1, t_2) = R_Y(t_1 - t_2)$$
$$R_{XY}(t_1, t_2) = R_{XY}(t_1 - t_2)$$

https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php

Example B.2: $Z(t) = X(t) + Y(t)$

Since X(t) and Y(t) are jointly WSS, we conclude

$$\begin{aligned}\mu_Z(t) &= E[X(t) + Y(t)] \\ &= E[X(t)] + E[Y(t)] \\ &= \mu_X + \mu_Y.\end{aligned}$$

https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php

Example B.2: $Z(t) = X(t) + Y(t)$

Since X(t) and Y(t) are jointly WSS, we conclude

$$\begin{aligned} R_Z(t_1, t_2) &= E\left[\left(X(t_1) + Y(t_1)\right)\left(X(t_2) + Y(t_2)\right)\right] \\ &= E[X(t_1)X(t_2)] + E[X(t_1)Y(t_2)] \\ &+ E[Y(t_1)X(t_2)]E[Y(t_1)Y(t_2)] \\ &= R_X(t_1 - t_2) + R_{XY}(t_1 - t_2) \\ &+ R_{YX}(t_1 - t_2) + R_Y(t_1 - t_2). \end{aligned}$$

https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php

Example B.3: $X(t) = \pm \sin t, \pm \cos t$

Let

$$X(t) = \begin{cases} +\sin t & p_0 = \frac{1}{4} \\ -\sin t & p_1 = \frac{1}{4} \\ +\cos t & p_2 = \frac{1}{4} \\ -\cos t & p_3 = \frac{1}{4} \end{cases}$$

$$E[X(t)] = 0$$

$$R_X(t_1, t_2) = \frac{1}{2} cos(t_2 - t_1)$$

thus $X(t)$ is WSS

http://isl.stanford.edu/~abbas/ee278/lect07.pdf

Example B.3: $X(t) = \pm\sin t, \pm\cos t$

Let

$$X(t) = \begin{cases} +\sin t & p_0 = \frac{1}{4} \\ -\sin t & p_1 = \frac{1}{4} \\ +\cos t & p_2 = \frac{1}{4} \\ -\cos t & p_3 = \frac{1}{4} \end{cases}$$

But $X(0)$ and $X(\frac{\pi}{4})$ do not have the same pmf (different ranges), so the first order pmf is not stationary, and the process is not **SSS**

http://isl.stanford.edu/~abbas/ee278/lect07.pdf

Outline

Cyclostationary Process

- A cyclostationary process is one whose distribution is periodic in time.
- Cyclostationarity is a characteristic of a **probability distribution**.
- A **random variable** whose **distribution** changes periodically with time, or is perodically time-varying, is referred to as a **cyclostationary process**.
- Let's return to the example of flipping a coin but with a caveat: the coin will only be flipped on Mondays and not flipped all other days of the week.
- However, the distribution is periodically time-varying when focusing on each week day individually. For example, the distribution of the coin flip on Monday is defined as
- (2) \begin{equation*}p_{c,weekly}\left(x | \text{day} = \text{Monday}\right) =\begin{cases}0.5, & x =