

# CTFT (2A)

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- Continuous Time Fourier Transform

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# CTFS and CTFT

## Continuous Time Fourier Series

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-j\omega_0 k t} dt \quad \longleftrightarrow \quad x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+j\omega_0 k t}$$

Discrete Frequency - Aperiodic

Periodic Continuous Time Signal

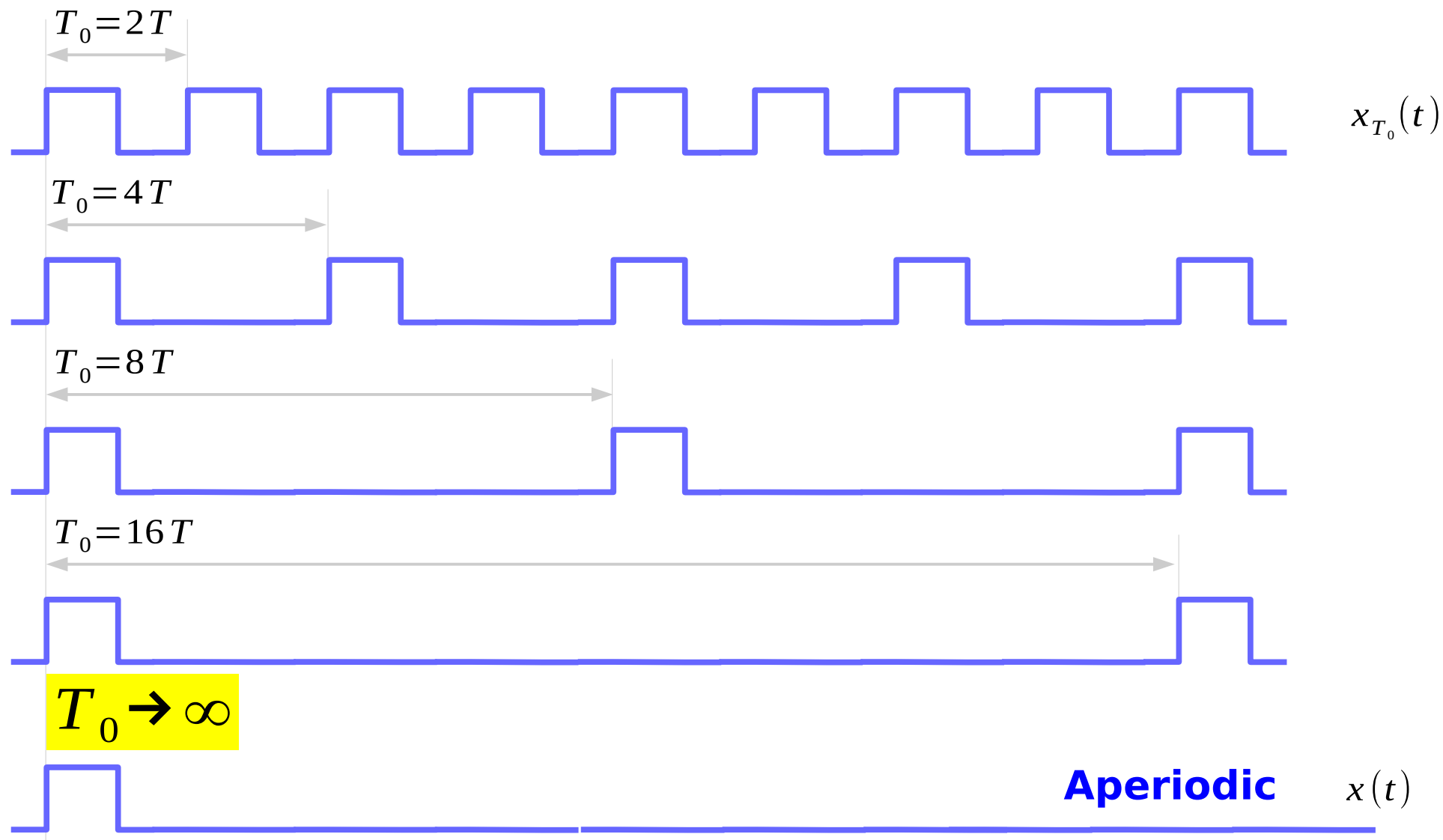
## Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

Continuous Frequency - Aperiodic

Aperiodic Continuous Time Signal

# Aperiodic Signal Conversion



# From Summation to Integration

CTFS

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-j\omega_0 k t} dt$$



$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+j\omega_0 k t}$$

$$T_0 \rightarrow \infty \quad \omega = \left( \frac{2\pi}{T_0} \right) \rightarrow 0$$

$$\omega_0 \rightarrow d\omega, \quad k\omega_0 \rightarrow \omega$$

$$x_{T_0}(t) \rightarrow x(t), \quad C_k T_0 \rightarrow X(j\omega)$$

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} C_k e^{+jk\omega_0 t} \cdot \frac{2\pi}{2\pi} \cdot \frac{T_0}{T_0}$$
$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} C_k T_0 e^{+jk\omega_0 t} \cdot \frac{2\pi}{T_0}$$

CTFT

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$



$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

# From CTFS to CTFT

$$\begin{aligned}x_{T_0}(t) &= \sum_{k=-\infty}^{+\infty} c_k e^{+j\omega_0 kt} \cdot 1 \\&= \sum_{k=-\infty}^{+\infty} c_k e^{+j\omega_0 kt} \cdot \left(\frac{T_0}{2\pi}\right) \cdot \left(\frac{2\pi}{T_0}\right) \\&= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} c_k T_0 e^{+j\omega_0 kt} \cdot \left(\frac{2\pi}{T_0}\right)\end{aligned}$$

$$\begin{aligned}x_{T_0}(t) &= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} c_k T_0 e^{+j\omega_0 kt} \cdot \omega_0 \\x(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega\end{aligned}$$

$$\begin{aligned}T_0 \rightarrow \infty \quad \omega &= \left(\frac{2\pi}{T_0}\right) \rightarrow 0 \\ \omega_0 \rightarrow d\omega, \quad k\omega_0 &\rightarrow \omega \\ x_{T_0}(t) \rightarrow x(t), \quad C_k T_0 &\rightarrow X(j\omega)\end{aligned}$$

# CTFT of Time Domain Impulse

## Continuous Time Fourier Transform

$$x(t) = A\delta(t) \quad \longleftrightarrow \quad X(j\omega) = A$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{+\infty} A\delta(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{+\infty} A\delta(t) e^0 dt \\ &= A \int_{-\infty}^{+\infty} \delta(t) dt = A \end{aligned}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} A e^{+j\omega t} d\omega \\ &= \frac{A}{2\pi} \int_{-\infty}^{+\infty} e^{+j\omega t} d\omega = A\delta(t) \end{aligned}$$

# CTFT of Frequency Domain Impulse

## Continuous Time Fourier Transform

$$X(j\omega) = 2\pi \delta(\omega) \iff x(t) = 1$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2\pi \delta(\omega) e^{+j\omega t} d\omega \\ &= \int_{-\infty}^{+\infty} \delta(\omega) e^0 d\omega = 1 \end{aligned}$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} dt = 2\pi \delta(\omega)$$



# CTFS of Impulse Train

## Continuous Time Fourier Series

$$C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$x(t) = \sum_{n=0}^{\infty} C_n e^{+jn\omega_0 t}$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$\begin{aligned} C_n &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jn\omega_s t} dt \\ &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) dt = \frac{1}{T_s} \end{aligned}$$

$$\begin{aligned} p(t) &= \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_s t} \\ &= \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t} \end{aligned}$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

# CTFT of Impulse Train

## Continuous Time Fourier Transform

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$P(j\omega) = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) e^{-jn\omega t} dt = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t} e^{-jn\omega t} dt$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} \int_{-\infty}^{+\infty} e^{-j(\omega - n\omega_s)t} dt$$

$$= \sum_{n=-\infty}^{+\infty} \left( \frac{2\pi}{T_s} \right) \delta(\omega - n\omega_s)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \left( \frac{2\pi}{T_s} \right) \delta(\omega - n\omega_s) e^{+j\omega t} d\omega = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} \int_{-\infty}^{+\infty} \delta(\omega - n\omega_s) e^{+j\omega t} d\omega$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t} = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

# Other Convention

## Continuous Time Fourier Transform {unitary, angular frequency}

$$X(j\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

## Continuous Time Fourier Transform {non-unitary, angular frequency}

$$X(j\omega) = 1 \cdot \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

## References

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