Binary Arithmetic (4A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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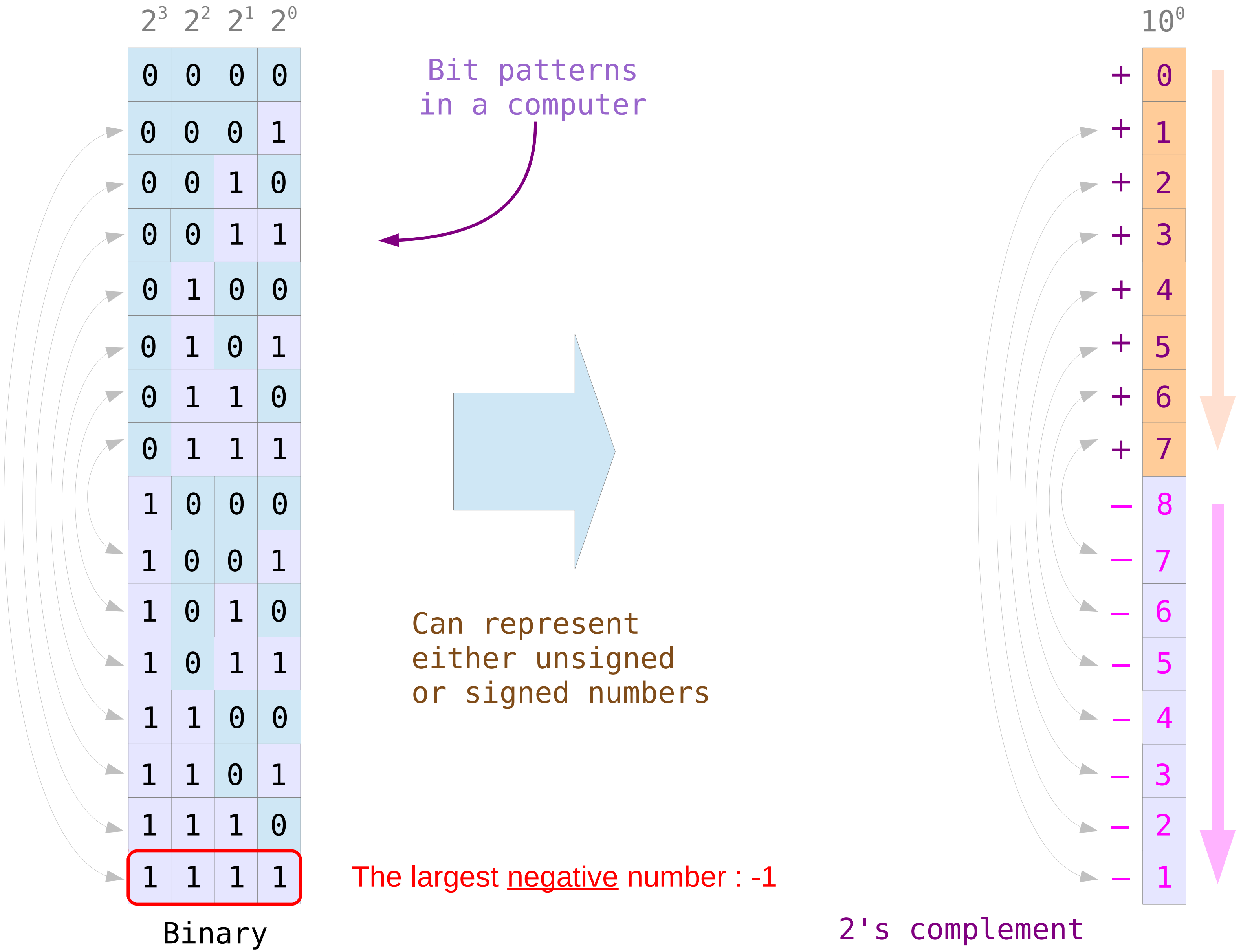
4-bit 2's Complement Bit Pattern



2^3	2^2	2^1	2^0		10^0
0	0	0	0	+	0
0	0	0	1	+	1
0	0	1	0	+	2
0	0	1	1	+	3
0	1	0	0	+	4
0	1	0	1	+	5
0	1	1	0	+	6
0	1	1	1	+	7
1	0	0	0	–	8
1	0	0	1	–	7
1	0	1	0	–	6
1	0	1	1	–	5
1	1	0	0	–	4
1	1	0	1	–	3
1	1	1	0	–	2
1	1	1	1	–	1

Bit patterns
in a computer

Can represent
either unsigned
or signed numbers

The largest negative number : -1

Binary

2's complement

The Sum of 4-bit 2's Complement Numbers

$$\begin{array}{ccccc} & & & & \\ & 0&0&0&0\\ & 1&0&0&0\\ \hline 1&1&0&0&0 \end{array} \quad \begin{array}{cc} & 0\\ - & 8\\ \hline & 8 \end{array} \qquad \begin{array}{ccccc} 1&1& & & \\ & 0&1&0&0\\ & 1&1&0&0\\ \hline 1&0&0&0&0 \end{array} \quad \begin{array}{cc} + & 4\\ - & 4\\ \hline - & 0 \end{array}$$

$$\begin{array}{c} +X\\ -X\\ \hline 2^n \end{array}$$

$\longleftrightarrow$

(n+1)-bit

$$\begin{array}{ccccc} 1&1&1&1& \\ & 0&0&0&1\\ & 1&1&1&1\\ \hline 1&0&0&0&0 \end{array} \quad \begin{array}{cc} + & 1\\ - & 1\\ \hline & 0 \end{array} \qquad \begin{array}{ccccc} 1&1&1&1& \\ & 0&1&0&1\\ & 1&0&1&1\\ \hline 1&0&0&0&0 \end{array} \quad \begin{array}{cc} + & 5\\ - & 5\\ \hline & 0 \end{array}$$

$$\begin{array}{ccccc} 1&1&1& & \\ & 0&0&1&0\\ & 1&1&1&0\\ \hline 1&0&0&0&0 \end{array} \quad \begin{array}{cc} + & 2\\ - & 2\\ \hline & 0 \end{array} \qquad \begin{array}{ccccc} 1&1&1&1& \\ & 0&1&1&0\\ & 1&0&1&0\\ \hline 1&0&0&0&0 \end{array} \quad \begin{array}{cc} + & 6\\ - & 6\\ \hline & 0 \end{array}$$

$$\begin{array}{ccccc} 1&1&1&1& \\ & 0&0&1&1\\ & 1&1&0&1\\ \hline 1&0&0&0&0 \end{array} \quad \begin{array}{cc} + & 3\\ - & 3\\ \hline & 0 \end{array} \qquad \begin{array}{ccccc} 1&1&1&1& \\ & 0&1&1&1\\ & 1&0&0&1\\ \hline 1&0&0&0&0 \end{array} \quad \begin{array}{cc} + & 7\\ - & 7\\ \hline & 0 \end{array}$$

Subtraction with Complements

$$\begin{array}{rr} & \boxed{+X} \\ + & \boxed{-X} \\ \hline & 2^n \end{array}$$

$$\longleftrightarrow$$

(n+1)-bit

$$\begin{array}{rr} & 2^n \\ - & \boxed{+X} \\ \hline & \boxed{-X} \end{array}$$

2's complement

$$2^n - \boxed{X} = \boxed{-X}$$

2's complement of X

= -X

$$2^n - X = -X$$

$$\begin{array}{rr} & \boxed{M} \\ - & \boxed{N} \\ \hline \end{array}$$

$$\begin{array}{rr} & \boxed{M} \\ + & \boxed{2^n - N} \\ \hline & 2^n + M - N \end{array}$$

$$2^n + M - N$$

$$\longleftrightarrow$$

(n+1)-bit

when M ≥ N

an end carry

positive number

$$2^n + M - N$$

$$\longleftrightarrow$$

(n)-bit

when M < N

no end carry

negative number

Subtraction with Complements

$$\begin{array}{rr} & M \\ - & N \\ \hline \end{array}$$

$$\begin{array}{rr} & M \\ + & 2^n - N \\ \hline & 2^n + M - N \end{array}$$

$2^n + M - N$ ⟷

(n+1)-bit

when M ≥ N

an end carry

positive number

$2^n + M - N$ ⟷

(n)-bit

when M < N

no end carry

negative number

$4-3$

$$\begin{array}{ccccc|c} 1 & 1 & & & & \\ & 0 & 1 & 0 & 0 & +4 \\ & 1 & 1 & 0 & 1 & -3 \\ \hline 1 & 0 & 0 & 0 & 1 & +1 \end{array}$$

$3-4$

$$\begin{array}{cccc|c} 0 & 0 & 1 & 1 & +3 \\ 1 & 1 & 0 & 0 & -4 \\ \hline 1 & 1 & 1 & 1 & -1 \end{array}$$

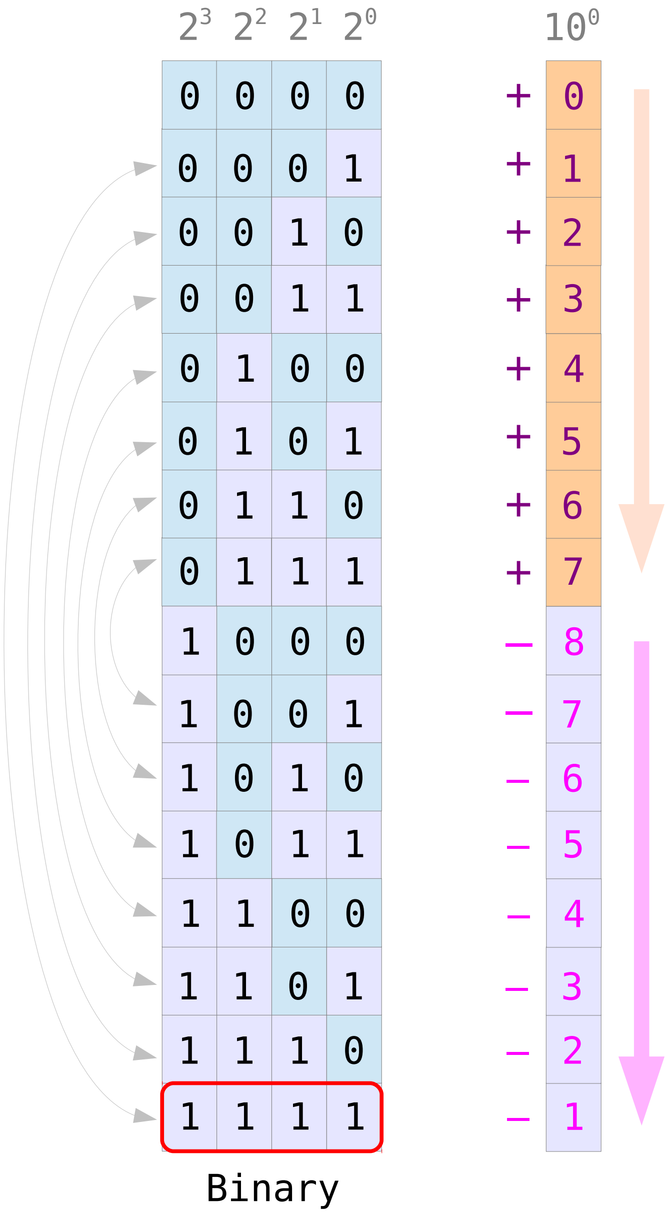
$-3-(-4)$

$$\begin{array}{ccccc|c} 1 & 1 & & & & \\ & 1 & 1 & 0 & 1 & -3 \\ & 0 & 1 & 0 & 0 & +4 \\ \hline 1 & 0 & 0 & 0 & 1 & +1 \end{array}$$

$-4-(-3)$

$$\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 1 & +3 \\ \hline 1 & 1 & 1 & 1 & -1 \end{array}$$

Overflow in the 4-bit 2's Complement System



2^3	2^2	2^1	2^0		10^0
0	0	0	0	+	0
0	0	0	1	+	1
0	0	1	0	+	2
0	0	1	1	+	3
0	1	0	0	+	4
0	1	0	1	+	5
0	1	1	0	+	6
0	1	1	1	+	7
1	0	0	0	–	8
1	0	0	1	–	7
1	0	1	0	–	6
1	0	1	1	–	5
1	1	0	0	–	4
1	1	0	1	–	3
1	1	1	0	–	2
1	1	1	1	–	1

Binary

$$\begin{array}{ccccc|c} 0 & 1 & 1 & & & \\ & 0 & 1 & 1 & 0 & +6 \\ & 0 & 1 & 1 & 1 & +7 \\ \hline & 1 & 1 & 0 & 1 & \cancel{-3} \end{array}$$

$$\begin{array}{ccc} & M & (+) \\ + & N & (+) \\ \hline & (>2^n-1) & (-) \end{array}$$

$$\begin{array}{ccccc|c} 1 & & & & & \\ & 1 & 0 & 0 & 0 & -8 \\ & 1 & 0 & 0 & 1 & -7 \\ \hline 1 & 0 & 0 & 0 & 1 & \cancel{+1} \end{array}$$

$$\begin{array}{ccc} & M & (-) \\ + & N & (-) \\ \hline & (<-2^n) & (+) \end{array}$$

2's complement

Unsigned Addition

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & \\ & 1 & 0 & 1 & 0 \\ & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 & 1 \end{array} \qquad \begin{array}{r} +\ 10 \\ +\ 5 \\ \hline +\ 15 \end{array}$$

$$\begin{array}{ccccc} 1 & 1 & 1 & 0 & \\ & 1 & 0 & 1 & 0 \\ & 0 & 1 & 1 & 1 \\ \hline \textcircled{1} & 0 & 0 & 0 & 1 \end{array} \qquad \begin{array}{r} +\ 10 \\ +\ 7 \\ \hline \cancel{+\ 1} \end{array}$$

overflow

N-bit

N-bit

N-bit

2's complement

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & \\ & 1 & 0 & 1 & 0 \\ & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 1 & 1 \end{array} \qquad \begin{array}{r} +\ 10 \\ +\ 5 \\ \hline +\ 15 \end{array}$$

$$\begin{array}{ccccc} 1 & 1 & 1 & 0 & \\ & 1 & 0 & 1 & 0 \\ & 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 \end{array} \qquad \begin{array}{r} +\ 10 \\ +\ 7 \\ \hline +\ 17 \end{array}$$

N-bit

N-bit

(N+1)-bit

2's complement

Signed Addition

0	0	0	0			
	0	0	1	1	+	3
	0	1	0	0	+	4
0	0	1	1	1	+	7

0	1	1	0			
	0	0	1	1	+	3
	0	1	1	0	+	6
0	1	0	0	1	~~−~~	~~7~~

overflow

N-bit

N-bit

N-bit

2's complement

1	1	0	0			
	1	1	0	1	−	3
	1	1	0	0	−	4
1	1	0	0	1	−	7

1	0	0	0			
	1	1	0	1	−	3
	1	0	1	0	−	6
1	0	1	1	1	~~+~~	~~7~~

overflow

N-bit

N-bit

N-bit

2's complement

1	1	0	0			
	1	1	0	1	−	3
	0	1	1	0	+	6
1	0	0	1	1	+	3

0	0	1	0			
	0	0	1	1	+	3
	1	0	1	0	−	6
0	1	1	0	1	−	3

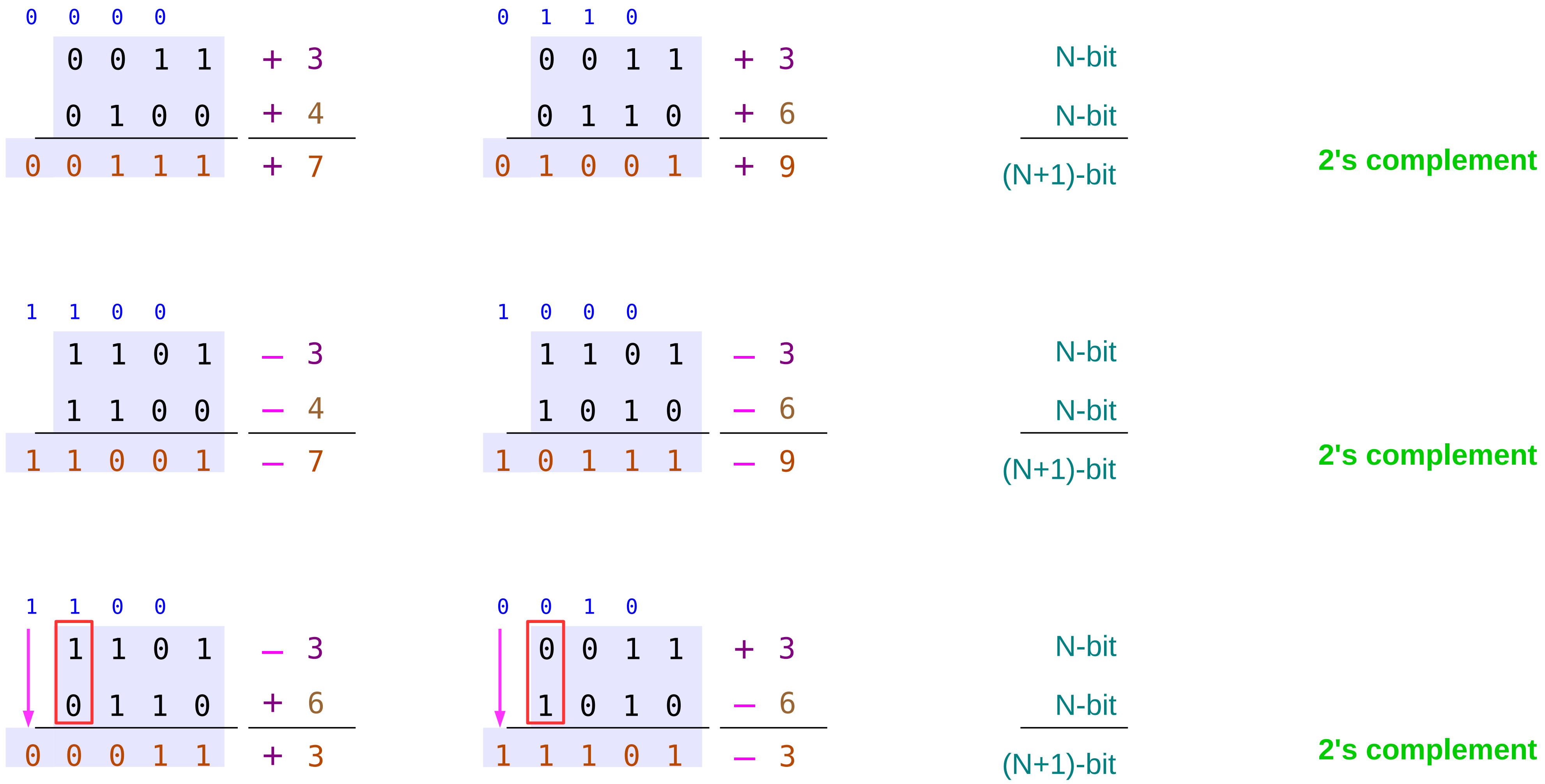
N-bit

N-bit

N-bit

2's complement

Signed Addition : (N+1)-bit Result



0 0 0 0

0 0 1 1 + 3

0 1 0 0 + 4

0 0 1 1 1 + 7

0 1 1 0

0 0 1 1 + 3

0 1 1 0 + 6

0 1 0 0 1 + 9

N-bit

N-bit

(N+1)-bit

2's complement

1 1 0 0

1 1 0 1 − 3

1 1 0 0 − 4

1 1 0 0 1 − 7

1 0 0 0

1 1 0 1 − 3

1 0 1 0 − 6

1 0 1 1 1 − 9

N-bit

N-bit

(N+1)-bit

2's complement

1 1 0 0

1 1 0 1 − 3

0 1 1 0 + 6

0 0 0 1 1 + 3

0 0 1 0

0 0 1 1 + 3

1 0 1 0 − 6

1 1 1 0 1 − 3

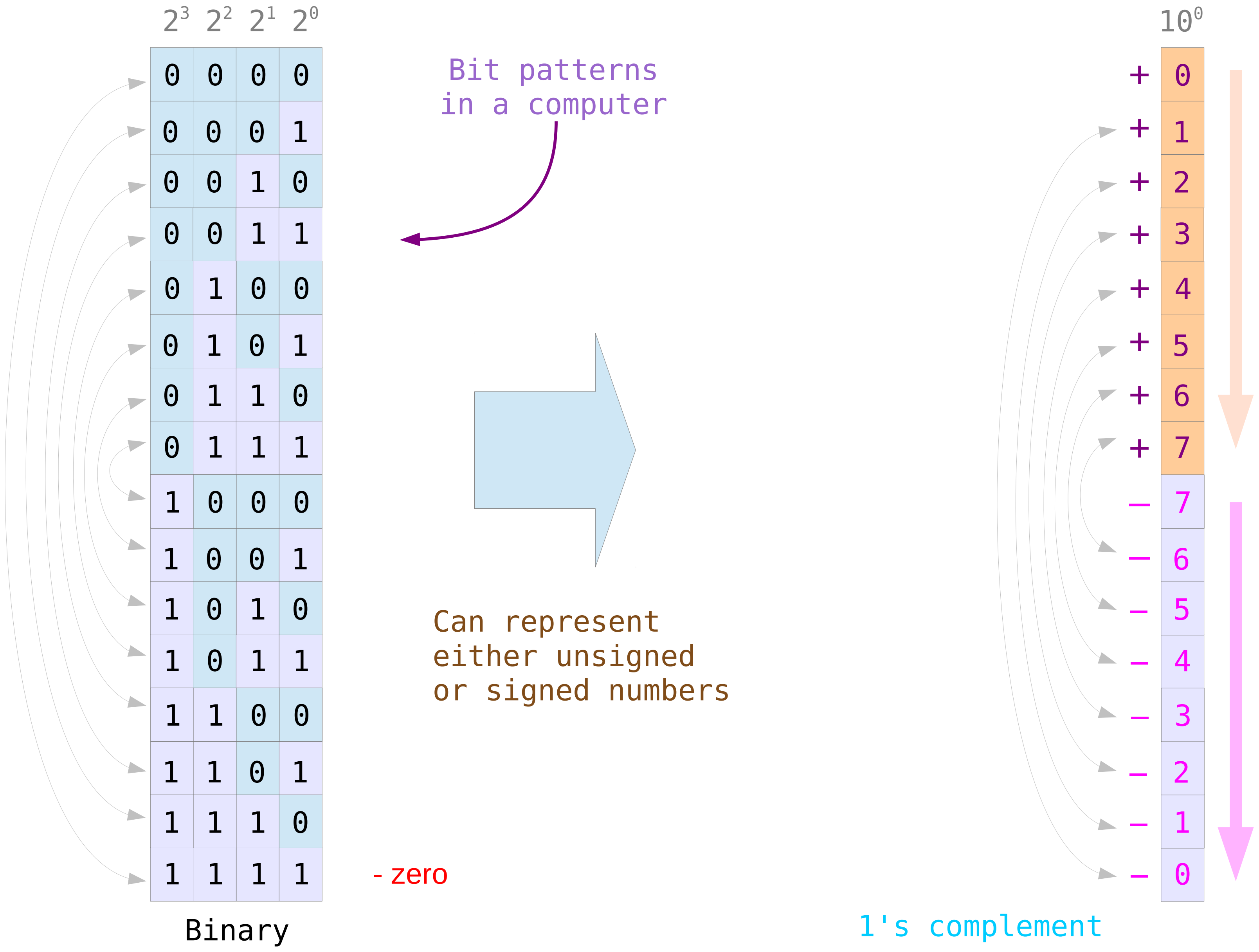
N-bit

N-bit

(N+1)-bit

2's complement

4-bit 1's Complement Bit Pattern



2^3	2^2	2^1	2^0		10^0
0	0	0	0	+	0
0	0	0	1	+	1
0	0	1	0	+	2
0	0	1	1	+	3
0	1	0	0	+	4
0	1	0	1	+	5
0	1	1	0	+	6
0	1	1	1	+	7
1	0	0	0	−	7
1	0	0	1	−	6
1	0	1	0	−	5
1	0	1	1	−	4
1	1	0	0	−	3
1	1	0	1	−	2
1	1	1	0	−	1
1	1	1	1	−	0

Binary

1's complement

Bit patterns
in a computer

Can represent
either unsigned
or signed numbers

- zero

The Sum of 4-bit 1's Complement Numbers

$$\begin{array}{ccccccc} & 0&0&0&0 & & 0\\ & 1&0&0&0 & - & 7\\ \hline 0&1&0&0&0 & - & 7\end{array}$$

$$\begin{array}{ccccccc} & 0&0&0&1 & + & 1\\ & 1&1&1&0 & - & 1\\ \hline 0&1&1&1&1 & - & 0\end{array}$$

$$\begin{array}{ccccccc} & 0&0&1&0 & + & 2\\ & 1&1&0&1 & - & 2\\ \hline 0&1&1&1&1 & - & 0\end{array}$$

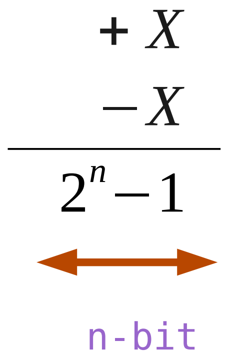
$$\begin{array}{ccccccc} 1&1&1&1& & & \\ & 0&0&1&1 & + & 3\\ & 1&1&0&0 & - & 3\\ \hline 0&1&1&1&1 & - & 0\end{array}$$

$$\begin{array}{ccccccc} & 0&1&0&0 & + & 4\\ & 1&0&1&1 & - & 4\\ \hline 0&1&1&1&1 & - & 0\end{array}$$

$$\begin{array}{ccccccc} & 0&1&0&1 & + & 5\\ & 1&0&1&0 & - & 5\\ \hline 0&1&1&1&1 & - & 0\end{array}$$

$$\begin{array}{ccccccc} & 0&1&1&0 & + & 6\\ & 1&0&0&1 & - & 6\\ \hline 0&1&1&1&1 & - & 0\end{array}$$

$$\begin{array}{ccccccc} 1&1&1&1& & & \\ & 0&1&1&1 & + & 7\\ & 1&0&0&0 & - & 7\\ \hline 0&1&1&1&1 & - & 0\end{array}$$



$$\begin{array}{r} +X\\ -X\\ \hline 2^n-1\end{array}$$

1's complement

Subtraction with Complements

$$\begin{array}{r} M \\ -\ N \\ \hline \end{array}$$

$$\begin{array}{r} M \\ +\ 2^n - N - 1 \\ \hline 2^n + M - N - 1 \end{array}$$

$$2^n + M - N - 1 \quad \longleftrightarrow$$

(n+1)-bit

when M ≥ N

an end carry

positive number

wrap the carry back to the LSB

$$M - N$$

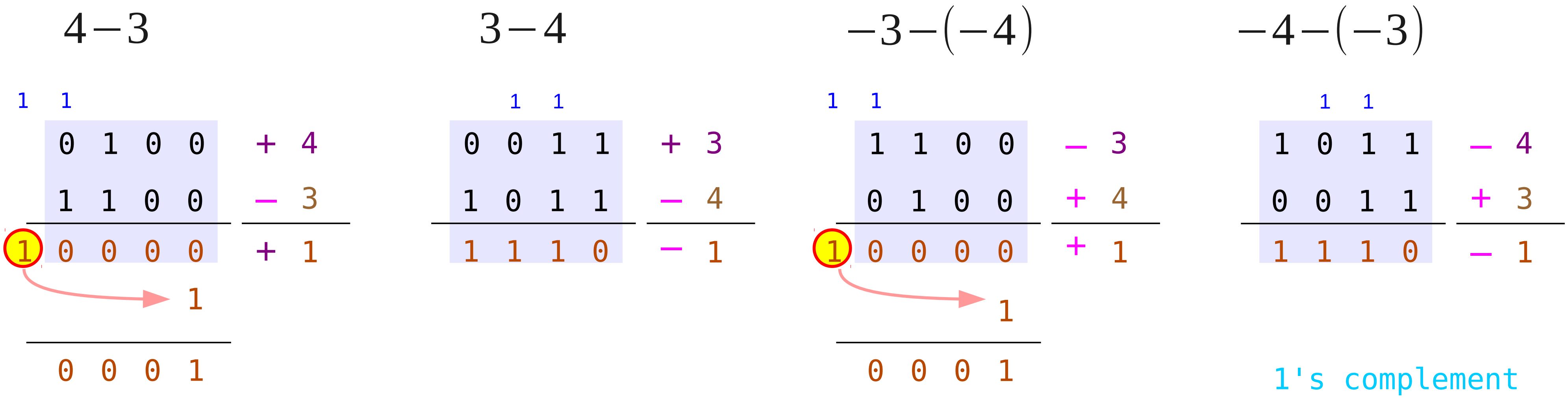
$$2^n + M - N - 1 \quad \longleftrightarrow$$

(n)-bit

when M < N

no end carry

negative number



$$4-3$$

```
 1 1
   0 1 0 0    + 4
   1 1 0 0    - 3
 ---------   -----
 1 0 0 0 0    + 1
         1
 ---------
   0 0 0 1
```

$$3-4$$

```
     1 1
 0 0 1 1    + 3
 1 0 1 1    - 4
 -------   -----
 1 1 1 0    - 1
```

$$-3-(-4)$$

```
 1 1
   1 1 0 0    - 3
   0 1 0 0    + 4
 ---------   -----
 1 0 0 0 0    + 1
         1
 ---------
   0 0 0 1
```

$$-4-(-3)$$

```
     1 1
 1 0 1 1    - 4
 0 0 1 1    + 3
 -------   -----
 1 1 1 0    - 1
```

1's complement

Unsigned Addition

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & \\ & 0 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array} \quad \begin{array}{cc} + & 1 \\ - & 1 \\ \hline & 0 \end{array}$$

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & \\ & 0 & 1 & 0 & 1 \\ & 1 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array} \quad \begin{array}{cc} + & 5 \\ - & 5 \\ \hline & 0 \end{array}$$

Handling Overflow Flag

Software Interrupt	**INTO**
Jump if overflow	**JO**
Jump if not overflow	**JNO**

References

[1] http://en.wikipedia.org/
[2] M. M. Mano, C. R. Kime, “Logic and Computer Design Fundamentals”, 4th ed.
[3] M. M. Mano, M. D. Ciletti, “Digital Design”, 5th ed.
[4] D. M. Harris, S. L. Harris, “Digital Design and Computer Architecture”